DEMONSTRATIO MATHEMATICA
Vol. XXXVII No 4 2004

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EQUIPOWER AND EQUICHORDAL EXTENSION

Abstract. In this paper we consider the possibility of extension of a concave function $f : [0, a] \to [0, +\infty)$ to equipower convex curve or equichordal convex curve with axis of symmetry. The extension is possible if and only if f satisfies a differential inequality of the second degree.

1. Introduction

Let C be a convex curve bounding a convex region D. The following definitions hold:

DEFINITION 1. C is called an *equipower curve* if there exists a point O in the region D with the following property:

if a chord PQ of C passes through O, then

$$|OP|\,|OQ| = c = const.$$

and the product does not depend on the choice of a chord.

The point O is called the *equipower point* of C and c is called the *equipower constant.*

DEFINITION 2. A point O of D is called an *equichordal point* if all chords passing through this point have the same length.

DEFINITION 3. C is called an *equichordal curve* if D contains an equichordal point.

E. J. Rosenbaum [6] asked how many distinct interior power points in a convex body D have ensured that it is a disk. J.B. Kelly [4] showed that if the boundary C of D has a unique tangent line at each of its points then two interior power points are sufficient to ensure that a convex region D is a disk. One interior power point may not be sufficient in this problem. A counterexample was already given by K.Yanagihara in [10], [11]. Further, L. Zuccheri in [12] proved that any convex region with two interior power points is a disk, without further assumption on its boundary, and extended the question to exterior points.

The well-known equichordal problem from geometry asks whether there exists an equichordal curve in the plane that admits two distinct equichordal points. The problem was posed by Fujiwara in 1916 [3] and independently by Blaschke, Rothe and Weitzenböck in 1917 [1]. Many papers are devoted to equichordal curves, e.g.: A.Dirac [2], E.Wirsing [9],V.Klee [5], R.Schäfke and H.Volkmer [8]. M.Rychlik [7] solved the equichordal problem. Namely, he proved that the equichordal curve in the plane with two distinct equichordal points does not exist.

2. Preliminaries for equipower extension

Let $\mathcal{E}$ denote the family of all equipower curves with the axis of symmetry and with the equipower point lying on that axis of symmetry. We may assume that the x-axis is the axis of symmetry and the origin O is the equipower point.

Denote by $\mathcal{E}(c)$ the subfamily of $\mathcal{E}$ such that

1° the part of graph of a curve $\Gamma \in \mathcal{E}$ lying in the upper-plane is a graph of some function,

2°the equipower constant is equal to c,

3° all curves are of the class C^2, possibly without points of intersection with the axis of symmetry,

4° all curves are strictly convex.

Let $\Gamma \in \mathcal{E}$. We assume that the graph of a function $f : \left[\frac{-c}{a}, a\right] \to [0, +\infty]$, where a, c are fixed positive numbers, is the graph of the curve Γ in the upper-plane. Obviously, the complete graph $gr\Gamma$ of Γ is formed by graphs of the f and $-f$. We consider a chord γ of Γ passing through the origin and a point $(x, f(x)) \in gr\Gamma, x \neq 0$. We denote by $(\varphi(x), -f(\varphi(x)))$ the coordinates of end-point of γ.

Then we have the following relations

$$\frac{f(\varphi(x))}{-\varphi(x)} = \frac{f(x)}{x} \quad \text{for} \quad x \in [\frac{-c}{a}, 0) \cup (0, a] \tag{1}$$

and

$$\sqrt{x^2 + f(x)^2}\sqrt{\varphi^2(x) + f(\varphi(x))^2} = c. \tag{2}$$

Hence we get immediately

$$\varphi(x) = \frac{-cx}{x^2 + f(x)^2} \quad \text{for } x \in \left[\frac{-c}{a}, a\right]. \tag{3}$$

It is easy to see that φ is a strictly decreasing function.

3. Equipower extension

We fix two positive numbers c and a. Let $\mathcal{F}$ denote the family of all functions $f : [0, a] \to [0, +\infty)$ satisfying the following conditions:

(4) $\quad f(a) = 0, \qquad f(0) = \sqrt{c},$

(5) $\quad f(x) > 0 \qquad \text{for } x \in (0, a),$

(6) $\quad f \in C^2(0, a),$

(7) $\quad \text{there exist} \qquad \lim_{x \to 0_+} f'(x), \qquad \lim_{x \to 0_+} f''(x),$

(8) $\quad \lim_{x \to a_-} \lim f'(x) = -\infty,$

(9) $\quad 2xf(x) f'(x) < f(x)^2 - x^2 \qquad \text{for } x \in (0, a),$

(10) $\quad f''(x) < 0 \qquad \text{for } x \in (0, a).$

Considerations provided below justify above conditions.

With each function $f \in \mathcal{F}$ we associate the function $\varphi : [0, a] \to [0, +\infty)$ given by (3). Since

$$\varphi'(x) = c\frac{x^2 + 2xf(x) f'(x) - f(x)^2}{(x^2 + f(x)^2)^2} \qquad \text{for } x \in (0, a), \tag{11}$$

so the condition (9) means that φ is a strictly decreasing function.We have

$$\varphi(0) = 0 \qquad \text{and} \qquad \varphi(a) = \frac{-c}{a}, \tag{12}$$

so φ transforms the interval $[0, a]$ onto $\left[\frac{-c}{a}, 0\right]$ and

$$\varphi'(0_+) = \lim_{x \to 0_+} \varphi'(x) = -1. \tag{13}$$

The monotonity of the function φ allow us to extend the function $f : [0.a] \to [0, +\infty)$ to the function $F : \left[\frac{-c}{a}, a\right] \to [0, +\infty)$ by the rule

$$F(x) = \begin{cases} f(x) & \text{for } x \in [0, a] \\ \frac{-x}{\varphi^{-1}(x)} f\left(\varphi^{-1}(x)\right) & \text{for } x \in [\frac{-c}{a}, 0). \end{cases} \tag{14}$$

The main result in this part says that a curve Γ obtained from the graphs of the functions F and $-F$ is an element of the family $\mathcal{E}(c)$.

We start with an explanation that $F \in C^1\left(\frac{-c}{a}, a\right)$. Let

$$\Phi(x) = \frac{\varphi(x)}{x} = \frac{-c}{x^2 + f(x)^2} \qquad \text{for } x \in (0, a) \tag{15}$$

and

$$\rho = f'(0_+) = \lim_{x \to 0_+} f't(x). \tag{16}$$

Then we have

(17) $$\Phi'(x) = 2c\frac{x + f(x)f'(x)}{(x^2 + f(x)^2)^2},$$

(18) $$\Phi''(x) = 2c\frac{(1 + f'(x)^2 + f(x)f''(x))(x^2 + f(x)^2) - 4(x + f(x)f'(x))^2}{(x^2 + f(x)^2)^3}$$

for $x \in (0, a)$

and

(19) $$\Phi'(0_+) = \lim_{x \to 0_+} \Phi'(x) = 2c\frac{\rho}{\sqrt{c}^3},$$

(20) $$\Phi''(0_+) = \lim_{x \to 0_+} \Phi''(x) = \frac{2}{c}\left[1 + \sqrt{c}f''(0_+) - 3\rho^2\right].$$

From (14) we have

$$-F(\varphi(x)) = \Phi(x)f(x) \qquad \text{for } x \in (0, a), \text{ i.e. } -F \circ \varphi = \Phi f.$$

Hence we obtain

(21) $$-\varphi' F' \circ \varphi = \Phi' f + \Phi f'.$$

Making use of (13) and (19) we get

(22) $$F'(0_-) = f'(0_+).$$

It means that $F \in C^1\left(\frac{-c}{a}, a\right)$. With respect to the condition (8) it is clear that the curve Γ obtained from the graphs of the F and $-F$ is a C^1-curve.

4. Main theorem for equipower curves

Differentiating (11) we get

(23) $$\varphi''(x) = 2cx\frac{1 + f'(x)^2 + f(x)f''(x)}{(x^2 + f(x)^2)^2} - 4\frac{x + f(x)f'(x)}{x^2 + f(x)^2}\varphi'(x)$$

for $x \in \left(\frac{-c}{a}, a\right)$.

Employing (13) and (16) we have

(24) $$\varphi''(0_+) = \frac{4}{\sqrt{c}}\sqrt{\rho}.$$

We want to show

(25) $$F''(0_-) = f''(0_+).$$

For this purpose we differentiate (21) obtaining

(26) $$-\varphi'' F' \circ \varphi - (\varphi')^2 F'' \circ \varphi = \Phi'' f + 2\Phi' f' + \Phi f''.$$

If $x \to 0_+$, then making use of (16), (19), (20), (22) and (24) we get

$$f''(0_+) = \frac{-1}{\sqrt{c}}\left(1+\rho^2\right). \tag{27}$$

The above condition guarantees that $F \in C^2\left(\frac{-c}{a}, a\right)$.

The conditions: Γ is strictly convex curve and F'' is negative are equivalent, so with respect to (26) we have to verify that

$$F' \circ \varphi\varphi'' + \frac{x^2\varphi'' - 2x\varphi' + 2\varphi}{x^3} f + 2\frac{x\varphi' - \varphi}{x^2} f' + \frac{\varphi}{x} f'' > 0. \tag{28}$$

Let

$$L = F' \circ \varphi\varphi'' \tag{29}$$

and

$$P = \frac{x^2\varphi'' - 2x\varphi' + 2\varphi}{x^3} f + 2\frac{x\varphi' - \varphi}{x^2} f' + \frac{\varphi}{x} f''. \tag{30}$$

Substituting φ, φ' and φ'' into (30) we obtain

$$\begin{aligned} &\frac{1}{c}\left(x^2+f^2\right) P \\ &= 6x^2 f\left(f'\right)^2 - 4x^3 f' - 12xf^2 f' - 2f^3\left(f'\right)^2 - x^4 f'' + f^4 f'' - 6x^2 f + 2f^3. \end{aligned} \tag{31}$$

Now we find the form of L with the help of (21)

$$F' \circ \varphi\varphi'' = F' \circ \varphi\varphi' \frac{\varphi''}{\varphi'}$$

and

$$\begin{aligned} \frac{1}{c}\left(x^2+f^2\right) L = \frac{x^2 f' - 2xf - f^2 f'}{x^2 + 2xff' - f^2}&(-2x^3 + 6xf^2 - 12x^2 ff' \\ &+ 4f^3 f' + 2x^3\left(f'\right)^2 - 6xf^2(f')^2 + 2x^3 ff'' + 2xf^3 f''). \end{aligned} \tag{32}$$

The inequality (28) can be written in the form

$$\frac{1}{c}\left(x^2+f^2\right)(L+P) > 0,$$

i.e.

$$\begin{aligned} &\frac{x^2 f' - 2xf - f^2 f'}{x^2 + 2xff' - f^2}(-2x^3 + 6xf^2 - 12x^2 ff' + 4f^3 f' + 2x^3\left(f'\right)^2 \\ &\qquad - 6xf^2\left(f'\right)^2 + 2x^3 ff'' + 2xf^3 f'') + 6x^2 f\left(f'\right)^2 - 4x^3 f' - 12xf^2 f' \\ &\qquad - 2f^3\left(f'\right)^2 - x^4 f'' + f^4 f'' - 6x^2 f + 2f^3 > 0. \end{aligned}$$

The denominator $x^2 + 2xff' - f^2$ is negative, so we have

$$\left(x^2 f' - 2xf - f^2 f'\right)\left(-2x^3 + 6xf^2 - 12x^2 ff' + 4f^3 f' + 2x^3\left(f'\right)^2\right.$$

$$- 6xf^2 (f')^2 + 2x^3 ff'' + 2xf^3 f'') + (6x^2 f(f')^2 - 4x^3 f' - 12xf^2 f'$$
$$- 2f^3(f')^2 - x^4 f'' + f^4 f'' - 6x^2 f + 2f^3)(x^2 + 2xff' - f^2) < 0.$$

Collecting terms with f'', f', $(f')^2$ and $(f')^3$ we obtain the following inequality

$$f''\left(-x^6 - 3x^4 f^2 - 3x^2 f^4 - f^6\right) + f'\left(2x^5 + 4x^3 f^2 + 2xf^4\right)$$
$$+ (f')^2(-2x^4 f - 4x^2 f^3 - 2f^5)$$
$$+ (f')^3(2x^5 + 4x^3 f^2 + 2xf^4) - (2x^4 f + 4x^2 f^3 + 2f^5) < 0,$$

i.e.

$$-(x^2 + f^2)^3 f'' + 2x(x^2 + f^2)^2 (f')^3 - 2f(x^2 + f^2)^2 (f')^2$$
$$+ 2x(x^2 + f^2)^2 f' - 2f(x^2 + f^2)^2 < 0.$$

The last expression has the form

$$\frac{1}{2}(x^2 + f^2) f'' > x(f')^3 - f(f')^2 + xf' - f$$

or

$$\frac{1}{2}(x^2 + f^2) f'' > (xf' - f)(1 + (f')^2).$$

Thus we proved the following theorem:

THEOREM 1. *The curve Γ obtained from the graphs of the F and $-F$ is an element of $\mathcal{E}(c)$ if and only if the function $f \in \mathcal{F}$ satisfies the differential inequality*

$$f''(x) > 2\frac{xf'(x) - f(x)}{x^2 + f^2(x)}\left[1 + f'(x)^2\right] \qquad \textit{for} \qquad x \in (0, a) \tag{33}$$

and (27)

$$f''(0_+) = \frac{-1}{\sqrt{c}}\left(1 + f'(0_+)^2\right).$$

5. Example

Let $f(x) = \left(1 - \frac{4}{3}x^2\right)^{\frac{3}{8}}$ for $x \in \left[0, \frac{\sqrt{3}}{2}\right]$, Fig. 1.

Notice that

$$f(0) = 1 = \sqrt{c} \qquad \text{so } c = 1,$$

$$f\left(\frac{\sqrt{3}}{2}\right) = 0,$$

$$f'(x) = \frac{-x}{\left(1 - \frac{4}{3}x^2\right)^{\frac{5}{8}}} < 0 \qquad \text{for } x \in \left[0, \frac{\sqrt{3}}{2}\right),$$

$$f'(0) = 0,$$

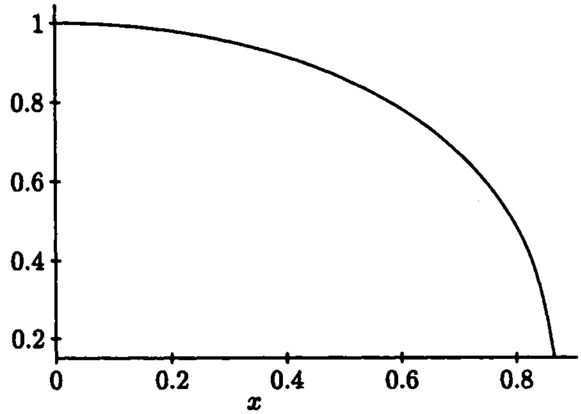

Fig. 1

$$f''(x) = \frac{-\left(1+\frac{1}{3}x^2\right)}{\left(1-\frac{4}{3}x^2\right)^{\frac{13}{8}}} < 0 \qquad \text{for } x \in \left[0, \frac{\sqrt{3}}{2}\right).$$

Since $f''(0) = -1$, then the condition $f''(0_+) = \frac{-1}{\sqrt{c}}\left(1 + f'(0_+)^2\right)$ is satisfied.

We verify the condition $2xf(x)f'(x) < f(x)^2 - x^2$ for $x \in [0, \frac{\sqrt{3}}{2})$. Equivalently it means that

$$42x\left(1-\frac{4}{3}x^2\right)^{\frac{3}{8}}\left(\frac{-x}{\left(1-\frac{4}{3}x^2\right)^{\frac{5}{8}}}\right) < \left(1-\frac{4}{3}x^2\right)^{\frac{3}{4}} - x^2$$

or

$$x^2\left(1-\frac{4}{3}x^2\right)^{\frac{1}{4}} < 1 + \frac{2}{3}x^2.$$

To show the latest inequality we denote by l and p the left and the right side of the inequality, respectively . The range of the function $l(x) = x^2\left(1-\frac{4}{3}x^2\right)^{\frac{1}{4}}$ defined on $[0, \frac{\sqrt{3}}{2})$ is the interval $\left[0, \frac{3}{5^{\frac{5}{4}}}\right]$. The function $p(x) = 1 + \frac{2}{3}x^2$ considered for $x \in [0, \frac{\sqrt{3}}{2})$ attains values in the interval $[1, \frac{3}{2})$ and it is strictly increasing. Thus the condition is satisfied.

We note that

$$\varphi(x) = \frac{-x}{x^2 + \left(1-\frac{4}{3}x^2\right)^{\frac{3}{4}}} < 0 \qquad \text{for } x \in [0, \frac{\sqrt{3}}{2}).$$

With respect to the previous considerations we have

$$\varphi'(x) = \frac{x^2\left(1-\frac{4}{3}x^2\right)^{\frac{1}{4}} - \left(1+\frac{2}{3}x^2\right)}{\left[x^2+\left(1-\frac{4}{3}x^2\right)^{\frac{3}{4}}\right]^2\left(1-\frac{4}{3}x^2\right)^{\frac{1}{4}}} < 0 \qquad \text{for } x \in [0, \frac{\sqrt{3}}{2}).$$

Now we verify the condition $f''(x) > 2\frac{xf'(x)-f(x)}{x^2+f^2(x)}\left[1+f'(x)^2\right]$ for $x \in [0, \frac{\sqrt{3}}{2})$, i.e.

$$\frac{-\left(1+\frac{1}{3}x^2\right)}{\left(1-\frac{4}{3}x^2\right)^{\frac{13}{8}}} > 2\frac{\frac{-x^2}{\left(1-\frac{4}{3}x^2\right)^{\frac{5}{8}}} - \left(1-\frac{4}{3}x^2\right)^{\frac{3}{8}}}{x^2+\left(1-\frac{4}{3}x^2\right)^{\frac{3}{4}}}\left(1+\frac{x^2}{\left(1-\frac{4}{3}x^2\right)^{\frac{5}{4}}}\right).$$

Simplifying we get the form

$$\left(1+\frac{1}{3}x^2\right)x^2\left(1-\frac{4}{3}x^2\right)^{\frac{1}{4}} + \left(1-\frac{4}{3}x^2\right)\left(1+\frac{1}{3}x^2\right) < 2\left(1-\frac{1}{3}x^2\right)x^2 + 2\left(1-\frac{1}{3}x^2\right)\left(1-\frac{4}{3}x^2\right)^{\frac{5}{4}}$$

or the equivalent form

$$\left(1-\frac{4}{3}x^2\right)^{\frac{1}{4}}\left(\frac{5}{9}x^4 - \frac{13}{3}x^2 + 2\right) - \frac{2}{9}x^4 + 3x^2 - 1 > 0 \qquad \text{for } x \in \left[0, \frac{\sqrt{3}}{2}\right).$$

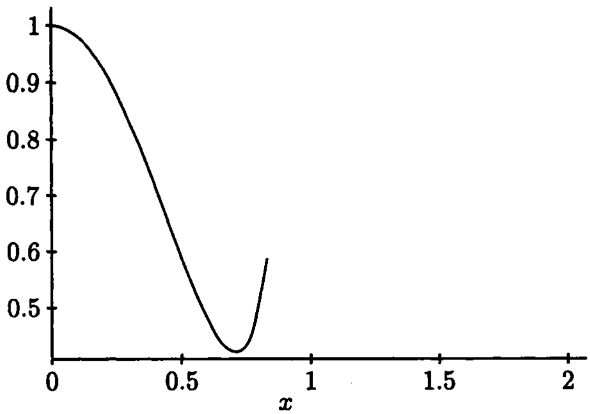

Fig. 2

Denote by g the left side of the inequality. The function g is a positive-valued for $x \in [0, \frac{\sqrt{3}}{2})$, Fig. 2.

6. Preliminaries for equichordal extension

Let $\mathcal{B}$ denote the family of all equichordal curves with the axis of symmetry and with the equichordal point lying on that axis of symmetry. We may assume that the x-axis is the axis of symmetry and the origin O is the equichordal point.

Denote by $\mathcal{B}(c)$ the subfamily of $\mathcal{B}$ such that

1° the part of graph of a curve $\Gamma \in \mathcal{B}$ lying in the upper-plane is a graph of some function,

2° the equichordal constant is equal to c,

3° all curves are of the class C^2, possibly without points of intersection with the axes of symmetry,

4° all curves are strictly convex.

Let $\Gamma \in \mathcal{B}$. We assume that the graph of a function $f : [a-c, a] \to [0, +\infty]$, where a, c are fixed positive numbers, is the graph of the curve Γ in the upper-plane. Obviously, the complete graph $gr\Gamma$ of Γ is formed by graphs of the f and $-f$. We consider a chord γ of Γ passing through the origin and a point $(x, f(x)) \in gr\Gamma, x \neq 0$. We denote by $(\psi(x), -f(\psi(x)))$ the coordinates of end-point of γ.

Then we have the following relations

$$\frac{f(\psi(x))}{-\psi(x)} = \frac{f(x)}{x} \qquad \text{for} \qquad x \in [a-c, 0) \cup (0, a] \tag{34}$$

and

$$\sqrt{x^2 + f(x)^2} + \sqrt{\psi^2(x) + f(\psi(x))^2} = c. \tag{35}$$

Hence we get immediately

$$\psi(x) = x - c\frac{x}{\sqrt{x^2 + f(x)^2}} \qquad \text{for} \qquad x \in [a-c, a]. \tag{36}$$

It is easy to see that ψ is a strictly decreasing function.

7. Equichordal extension

We fix two positive numbers c and a. Let $\mathcal{F}$ denote the family of all functions $f : [0, a] \to [0, +\infty)$ satisfying the following conditions:

$$f(a) = 0, \qquad f(0) = \frac{c}{2}, \tag{37}$$

$$f(x) > 0 \qquad \text{for } x \in (0, a), \tag{38}$$

$$f \in C^2(0, a), \tag{39}$$

$$\text{there exist } \lim_{x \to 0_+} \lim f'(x), \qquad \lim_{x \to 0_+} f''(x), \tag{40}$$

$$\lim_{x \to a_-} f'(x) = -\infty, \tag{41}$$

$$cxf(x)\, f'(x) < cf^2(x) - \sqrt{x^2 + f(x)^2}^3 \qquad \text{for } x \in (0, a), \tag{42}$$

$$f''(x) < 0 \qquad \text{for } x \in (0, a). \tag{43}$$

Considerations provided below justify above conditions. With each function $f \in \mathcal{F}$ we associate the function $\psi : [0, a] \to [0, +\infty)$ given by (36). Since

$$\psi'(x) = 1 + cf(x)\, \frac{xf'(x) - f(x)}{\sqrt{x^2 + f(x)^2}^3} \qquad \text{for } x \in (0, a), \tag{44}$$

so the condition (42) means that ψ is a strictly decreasing function. We have

$$\psi(0) = 0 \qquad \text{and} \qquad \psi(a) = a - c, \tag{45}$$

so ψ transforms the interval $[0, a]$ onto $[a - c, 0]$ and

$$\psi'(0_+) = \lim_{x \to 0_+} \psi'(x) = -1. \tag{46}$$

The monotonity of the function ψ allow us to extend the function $f : [0.a] \to [0, +\infty)$ to the function $F : [a - c, a] \to [0, +\infty)$ by the rule

$$F(x) = \begin{cases} f(x) & \text{for } x \in [0, a] \\ \frac{-x}{\psi^{-1}(x)} f\left(\psi^{-1}(x)\right) & \text{for } x \in [a - c, 0). \end{cases} \tag{47}$$

The main result in this part says that a curve Γ obtained from the graphs of the functions F and $-F$ is an element of the family $\mathcal{B}(c)$.

We start with an explanation that $F \in C^1(a - c, a)$. Let

$$\Psi(x) = \frac{\psi(x)}{x} = 1 - \frac{c}{\sqrt{x^2 + f(x)^2}} \qquad \text{for } x \in (0, a) \tag{48}$$

and

$$\rho = f'(0_+) = \lim_{x \to 0_+} f'(x). \tag{49}$$

Then we have

$$\Psi'(x) = c\frac{x + f(x)f'(x)}{\sqrt{x^2 + f(x)^2}^3}, \tag{50}$$

$$\Psi''(x) = c\frac{(1 + f'(x)^2 + f(x)f''(x))(x^2 + f(x)^2) - 3(x + f(x)f'(x))^2}{\sqrt{x^2 + f(x)^2}^5} \qquad \text{for } x \in (0, a) \tag{51}$$

and

$$\Psi'(0_+) = \lim_{x \to 0_+} \Psi'(x) = \frac{4}{c}\rho, \tag{52}$$

$$\Psi''(0_+) = \lim_{x \to 0_+} \Psi''(x) = \frac{4}{c^2}\left[2 - 4\rho^2 + cf''(0_+)\right]. \tag{53}$$

From (47) we have

$$-F(\psi(x)) = \Psi(x) f(x) \qquad \text{for } x \in (0,a), \text{ i.e. } -F \circ \psi = \Psi f.$$

Hence we obtain

$$-\psi' F' \circ \psi = \Psi' f + \Psi f'. \tag{54}$$

Making use of (46) and (52) we get

$$F'(0_-) = f'(0_+). \tag{55}$$

It means that $F \in C^1(a-c,a)$. With respect to the condition (41) it is clear that the curve Γ obtained from the graphs of the F and $-F$ is a C^1-curve.

8. Main theorem for equichordal curves

Differentiating (44) we get

$$\psi''(x) = \frac{c}{\sqrt{x^2 + f(x)^2}^5} \times \left[3xf^2 + f'\left(2f^3 - 4x^2 f\right) + (f')^2\left(x^3 - 2xf^2\right) + f''xf\left(x^2 + f^2\right)\right] \quad \text{for } x \in (0,a). \tag{56}$$

Employing (46) and (49) we have

$$\psi''(0_+) = \frac{8}{c}\rho. \tag{57}$$

We want to show

$$F''(0_-) = f''(0_+). \tag{58}$$

For this purpose we differentiate (54) obtaining

$$-\psi'' F' \circ \psi - (\psi')^2 F'' \circ \psi = \Psi'' f + 2\Psi' f' + \Psi f''. \tag{59}$$

If $x \to 0_+$, then making use of $(49), (52), (53), (55)$ and (57) we get

$$f''(0_+) = \frac{-2}{c}\left(1 + 2\rho^2\right). \tag{60}$$

The above condition guarantees that $F \in C^2(a-c,a)$.

The conditions: Γ is strictly convex curve and F'' is negative are equivalent, so with respect to (59) we have to verify that

$$F' \circ \psi \psi'' + \frac{x^2\psi'' - 2x\psi' + 2\psi}{x^3} f + 2\frac{x\psi' - \psi}{x^2} f' + \frac{\psi}{x} f'' > 0. \tag{61}$$

Let

$$L = F' \circ \psi \psi'' \tag{62}$$

and

$$P = \frac{x^2\psi'' - 2x\psi' + 2\psi}{x^3} f + 2\frac{x\psi' - \psi}{x^2} f' + \frac{\psi}{x} f''. \tag{63}$$

Substituting ψ, ψ' and ψ'' into (63) we obtain

$$\begin{aligned}\sqrt{x^2+f^2}^5 P = c\left[-2x^2 f + f^3 + 2x^3 f' - 4xf^2 f' + 3x^2 f\,(f')^2\right] \\ + f''\left[cf^2 + \left(\sqrt{x^2+f^2} - c\right)\left(x^2+f^2\right)\right]\left(x^2+f^2\right).\end{aligned} \tag{64}$$

Now we find the form of L with the help of (54)

$$L = F' \circ \psi \psi'' = F' \circ \psi \psi' \frac{\psi''}{\psi'}$$

and

$$\begin{aligned}&\sqrt{x^2+f^2}^5 L \\ &= \frac{-cxf + \left(cx^2 - \sqrt{x^2+f^2}^3\right) f'}{\sqrt{x^2+f^2}^3 + cf\,(xf' - f)} \\ &\quad \times c\left[3xf^2 + f'\left(2f^3 - 4x^2 f\right) + (f')^2\left(x^3 - 2xf^2\right) + xff''\left(x^2+f^2\right)\right].\end{aligned} \tag{65}$$

The inequality (61) can be written in the form $\sqrt{x^2+f^2}^5 (L+P) > 0$, i.e.

$$\begin{aligned}&\frac{-cxf + \left(cx^2 - \sqrt{x^2+f^2}^3\right) f'}{\sqrt{x^2+f^2}^3 + cf\,(xf' - f)} \\ &\quad \times c\left[3xf^2 + f'\left(2f^3 - 4x^2 f\right) + (f')^2\left(x^3 - 2xf^2\right) + xff''\left(x^2+f^2\right)\right] \\ &\quad + c\left[-2x^2 f + f^3 + 2x^3 f' - 4xf^2 f' + 3x^2 f\,(f')^2\right] \\ &\quad + f''\left[cf^2 + \left(\sqrt{x^2+f^2} - c\right)\left(x^2+f^2\right)\right]\left(x^2+f^2\right) > 0.\end{aligned}$$

The denominator $\sqrt{x^2+f^2}^3 + cf\,(xf' - f)$ is negative, so we have

$$\begin{aligned}&\left[-cxf + \left(cx^2 - \sqrt{x^2+f^2}^3\right) f'\right] \\ &\quad \times \left[3xf^2 + f'\left(2f^3 - 4x^2 f\right) + (f')^2\left(x^3 - 2xf^2\right) + xff''\left(x^2+f^2\right)\right] \\ &\quad + \left[\sqrt{x^2+f^2}^3 + cf\,(xf' - f)\right]\end{aligned}$$

$$\times\left[-2x^2 f + f^3 + 2x^3 f' - 4xf^2 f' + 3x^2 f\left(f'\right)^2 + f^2 f''\left(x^2+f^2\right)\right]$$

$$+\left[\sqrt{x^2+f^2}^3 + cf\left(xf'-f\right)\right]\left(\frac{1}{c}\sqrt{x^2+f^2}-1\right)\left(x^2+f^2\right)^2 f'' < 0.$$

Collecting terms with $f'', f', (f')^2$ and $(f')^3$ we obtain the following inequality

$$\left(f - xf'\right)^3\left(\sqrt{x^2+f^2}-c\right)$$
$$+2\sqrt{x^2+f^2}\left(xf'-f\right)\left(x+ff'\right)^2 + \frac{1}{c}\left(\sqrt{x^2+f^2}-c\right)\sqrt{x^2+f^2}^5 f'' < 0$$

i.e.

$$\frac{1}{c}\sqrt{x^2+f^2}^5 f'' > \left(xf'-f\right)\left[\left(xf'-f\right)^2 + 2\left(x+ff'\right)^2\frac{\sqrt{x^2+f^2}}{c-\sqrt{x^2+f^2}}\right].$$

The last expression has the form

$$f'' > c\frac{xf'-f}{\left(x^2+f^2\right)^2}\left[\frac{\left(xf'-f\right)^2}{\sqrt{x^2+f^2}} + 2\frac{\left(x+ff'\right)^2}{c-\sqrt{x^2+f^2}}\right].$$

Thus we proved the following theorem:

THEOREM 2. *The curve Γ obtained from the graphs of the F and $-F$ is an element of $\mathcal{B}(c)$ if and only if the function $f \in \mathcal{F}$ satisfies the differential inequality*

$$(66)\quad f''(x) > c\frac{xf'-f}{\left(x^2+f^2\right)^2}\left[\frac{\left(xf'-f\right)^2}{\sqrt{x^2+f^2}} + 2\frac{\left(x+ff'\right)^2}{c-\sqrt{x^2+f^2}}\right] \quad \text{for } x \in (0,a)$$

and

$$(67)\qquad f''(0_+) = \frac{-2}{c}\left(1 + 2f'(0_+)^2\right).$$

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Received May 9, 2003; revised version March, 2004.