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A PROOF OF THE PROJECTIVE DESARGUES AXIOM IN THE DESARGUESIAN AFFINE PLANE

Abstract. We give a short proof that the projective Desargues axiom is valid in the Desarguesian affine planes.

1. Introduction

In the paper [3] Kusak and Prażmowski showed, without the use of a construction of a (coordinate) field, that the projective Desargues axiom PDes is valid in plane Desarguesian affine geometry. Recall that PDes states that if the three pairs of corresponding sides of two perspective triangles intersect each other then the three points of intersection lie on a line (the so-called *axis*). In the paper we shall give a shorter proof of this fact.

Therefore, it makes sense to consider a (weak) affine plane with PDes assumed as an extra axiom. Such an approach can be justified if we consider an affine plane completed with its improper points (in fact – a projective plane, see [1]); in such a case an axis of a Desargues configuration can be an arbitrary line, not only the one distinguished – the line in infinity – as we assume in the classical affine Desargues axiom.

2. Desarguesian affine plane

DEFINITION 2.1. An affine plane is a structure $\mathfrak{A} = \langle A, \mathcal{L} \rangle$, where A is a nonempty set (*set of points*) and $\mathcal{L}$ is a family of subsets of A (called *lines*) which satisfy the following conditions (see [1]):

A1 For any two distinct points there exists exactly one line passing through them.

A2 If the point p is not on the line L, then there is a unique line on p missing L.

A3 There exist three non collinear points.

NOTATION 2.2. From now, small Latin letters will be used for points. Points on one line are called *collinear.*

By $\overline{ab}$ we denote the line which contains two distinct points a, b.

We write $L \parallel M$ if $L, M \in \mathcal{L}$ and either $L \cap M = \emptyset$ or $L = M$,

$$ab \parallel cd \quad \text{iff} \quad \overline{ab} \parallel \overline{cd} \quad \text{or} \quad a = b \quad \text{or} \quad c = d.$$

The following sentence is referred to as the Affine Desargues Axiom (cf. [4]).

ADes: Let $o \neq a, b, c, a', b', c'$ and let three pairwise distinct lines $\overline{aa'}$, $\overline{bb'}$, $\overline{cc'}$ pass through o. If $ab \parallel a'b'$ and $ac \parallel a'c'$, then $bc \parallel b'c'$.

DEFINITION 2.3. *An affine plane which satisfies the affine Desargues axiom is called* Desarguesian.

3. Results

The following statement is called the Projective Desargues Axiom (cf. [2]).

PDes: Let $o \neq a, b, c, a', b', c'$ and let three pairwise distinct lines $\overline{aa'}$, $\overline{bb'}$, $\overline{cc'}$ pass through o. If $\overline{ab} \cap \overline{a'b'} = \{p\}$, $\overline{ac} \cap \overline{a'c'} = \{q\}$, and $\overline{bc} \cap \overline{b'c'} = \{r\}$, then the points p, q, r are collinear .

In the sequel we shall consider a Desarguesian affine plane (see 2.3). We shall prove that such plane satisfies the projective Desargues axiom. To this aim we begin with a preparatory lemma.

LEMMA 3.1. *Let $o \neq a, b, c, a', b', c'$, and let the three pairwise distinct lines $\overline{aa'}$, $\overline{bb'}$, $\overline{cc'}$ pass through o. If $ab \parallel a'b'$, $\overline{ac} \cap \overline{a'c'} = \{q\}$, and $\overline{bc} \cap \overline{b'c'} = \{r\}$, then $ab \parallel qr$.*

Proof. If a, b, c or a', b', c' are collinear, then the claim is evident. Consider the line through a parallel to $a'c'$ – this line shares a point c'' with $\overline{oc'}$ (since $\overline{a'c'}$ intersects $\overline{oc'}$). We apply ADes twice. First, we apply ADes to triangles abc'' and $a'b'c'$ and we get $bc'' \parallel b'c'$. Secondly, we apply ADes to triangles abc'' and qrc, which have c as a perspective center. Finally, we conclude with $ab \parallel rq$.

THEOREM 3.2. *The projective Desargues axiom is true in the affine Desarguesian plane.*

Proof. We adopt the premises of PDes. If a, b, c or a', b', c' are collinear, then the claim is evident. So, assume that a, b, c and a', b', c' are not collinear. Suppose that p does not belong to $\overline{qr}$. Draw a line L through a' parallel to $\overline{ac}$; then this line intersects $\overline{oc}$ in a point c''. If the points a', b', c'' are collinear, then we consider the line L through b' parallel to $\overline{bc}$, which intersects $\overline{oc}$ in c''' such that a', b', c''' are not collinear. We obtain two triangles a', b', c'' and q, r, c which has c' as their perspective center; note that one pair of their sides is parallel ($a'c'' \parallel cq$).

First, we note that $\overline{b'c''} \cap \overline{cr} \neq \emptyset$. Indeed, otherwise $\overline{bc} \parallel \overline{b'c''}$ and we would have two triangles a, b, c and a', b', c'' perspective from o with two pairs of corresponding sides parallel. By ADes we infer that $\overline{ab} \cap \overline{a'b'} = \emptyset$, which contradicts our assumptions. Let $\{z\} := \overline{b'c''} \cap \overline{cr}$.

Next, we shall prove that $\overline{a'b'} \cap \overline{qr} \neq \emptyset$. Assume the opposite holds. By ADes, $\overline{b'c''} \cap \overline{cr} = \emptyset$, which is inconsistent with the existence of the point z. Let $\{p'\} = \overline{a'b'} \cap \overline{qr}$.

By 3.1, we can now infer that $\overline{a'c''} \parallel \overline{zp'}$; so $\overline{ac} \parallel \overline{zp'}$.

Let us consider two triangles a, b, c and a', b', c'' – they have o as their perspective center, and one pair of their sides is parallel ($ac \parallel a'c''$). The other two pairs of their sides intersect at p and z. By 3.1, $\overline{ac} \parallel \overline{zp}$, so $\overline{zp} \parallel \overline{zp'}$ and, consequently, the points z, p, p' are collinear. Clearly, a', p, p' are also collinear. Since $\overline{zp} \neq \overline{a'p}$ (because the points a', b', c'' are not collinear), we conclude with $p = p'$ and thus $p \in \overline{qr}$. ∎

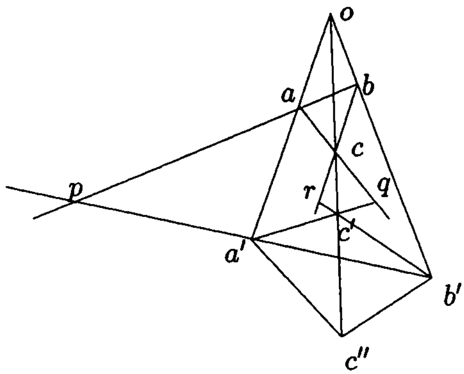

Fig. 1. A schema of the proof of 3.2

References

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[2] M. Kordos, *Foundations of Projective and Metric Projective Geometry* (in Polish), PWN, Warsaw 1984.

[3] E. Kusak, K. Prażmowski, *On affine reducts of Desargues axiom*, Bull. PAS 1-2(1987), 77–86.

[4] W. Szmielew, *From Affine to Euclidean Geometry*, PWN – Polish Scientific Publishers, Warszawa-Poland, D. Reidel Publishing Company Dordrecht-Holland 1983.

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