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ON SOME RIGHT INVERTIBLE OPERATORS IN DIFFERENTIAL SPACES

Abstract. In this paper we consider the right invertibility problem of some linear operators defined on the algebra of smooth functions on a differential space.

1. Introduction

Since the beginning of the sixties one can observe the increase of number of works dealing with many different generalizations of a differentiable manifold concept. Some natural generalizations appeared independently in works of several authors. The motivation for such generalizations has become evident in many mathematical problems [1]–[8], [10]–[12]. Also in physics there is an urgent necessity to model some physical phenomena on a sufficiently "non-smooth" arena [30]–[32]. The smooth manifold structure breaks down especially in the quantum gravity regime of the very early Universe. Cosmological singularities require generalized geometric concepts in which they could be described and investigated as points inside the space-time. However, in the frame of a suitable generalization there should still remain some well defined basic geometric concepts. The standard formulation of the physical laws is based on differential equations and this seems to be inevitable. From this point of view, within a suitable generalization we need, certain kind of differential equations should be possible to define. Naturally, an important question will arise then about methods of finding their solutions. The main geometric object considered in this paper is the so-called differential space introduced by R. Sikorski [10]. A differential space in the sense of Sikorski is a natural generalization of a real manifold concept and it turnes out to be a geometric version of the algebraic concept of a ringed space [23]. Such an approach brings into prominence the fundamental fusion of geometry with algebra. Properties of geometric objects are recognized from its differential structures which are defined as commutative algebras of the so-called smooth functions or—more generally—as noncommutative algebras which are very attractive in the modern theoretical physics.

As it was mentioned above, in differential geometry as well as in physics one needs notions and methods which have been developed in mathematical analysis, for example differential equations and methods telling us how to solve them. By a partial differential equation in differential geometry one can understand a subbundle of the bundle of jets. This approach in differential spaces is proposed in [28]. But the formulation itself does not automatically lead to methods of solving such equations since, in particular, in such general spaces there can be well defined non-trivial differential operators whereas simultaneously it can be no smooth non-constant curves [29]. Therefore the hope seems to be in purely algebraic formulation of the necessary concepts of mathematical analysis together with the efficient methods for solving problems. Then differential geometry as well as the necessary analytic methods would be organized in a uniform way. Fortunately, since over thirty years we observe dynamic development of the algebraic analysis founded by D. Przeworska-Rolewicz [13]–[17]. This elegant algebraic approach to differential equations and integration can naturally be applied in those generalized geometric objects. An important and interesting application of algebraic analysis in differential geometry of manifolds has been shown by G. Virsik [19].

2. Preliminaries

2.1. Differential spaces

Let M be a non-empty set and C be a set of real functions defined on M. The weakest topology on M with respect to which all functions of C are continues will be denoted by τ_C. The continuity of the real functions is considered with respect to the usual topology of the one dimensional manifold $(\mathbb{R}, \mathcal{E}_1)$. In this paper we will use the notation $\mathcal{E}_n \equiv C^\infty(\mathbb{R}^n)$, for any $n \in \mathbb{N}$.

For any subset $A \subset M$, let C_A denote the set of all real functions h on A such that, for any $p \in A$, there exist an open neighbourhood $U \in \tau_C$ of p and a function $f \in C$ satisfying the condition $h|_{A\cap U} = f|_{A\cap U}$.

In turn, let scC denote the family of all real functions on M which are of the form $\omega \circ (f_1, \ldots, f_n)$ for some $\omega \in \mathcal{E}_n$, $f_1, \ldots, f_n \in C$, and $n \in \mathbb{N}$.

A family C is called a differential structure in the sense of Sikorski on M iff $C = C_M = scC$. The elements of a differential structure C are called smooth real functions on M.

The pair (M, C) is said to be a differential space in the sense of Sikorski.

In the natural way, a differential structure C is a real comutative algebra (linear ring) with the pointwise definition of addition and multiplication of its elements.

For an arbitrary family $\{(M_j, C_j)\}_{j\in J}$ of differential spaces, by their Cartesian product we mean the differential space $(\bigtimes_{j\in J} M_j, \prod_{j\in J} C_j)$, where $\prod_{j\in J} C_j = (sc\{\pi_j : j \in J\})_M$ is the initial differential structure induced by the natural projections $\pi_j : M \to M_j$, for $j \in J$.

Any linear operator $D : C \to C$ satisfying the Leibniz condition $D(fg) = D(f)g + fD(g)$, for any $f, g \in C$, is known as the so-called derivative of C. The collection of all derivatives defined on a given algebra C will be denoted by $Der(C)$ and naturally it is a C-module.

In differential geometry one defines a tangent vector to a given differential space or manifold (M, C) at a point $p \in M$ as a linear mapping $v_p : C \to \mathbb{R}$ satisfying the Leibniz condition $v_p(fg) = v_p(f)g(p) + f(p)v_p$, for all $f, g \in C$.

Equivalently, a mapping $v_p : C \to \mathbb{R}$ is a tangent vector to a differential space (M, C) at a point $p \in M$, if for any natural $n \in \mathbb{N}$, $\omega \in \mathcal{E}_n$ and $f_1, \ldots, f_n \in C$, the following chain rule

$$v_p(\omega \circ (f_1, \ldots, f_n)) = \sum_{i=1}^{n} \omega'_{|i}(f_1(p), \ldots, f_n(p)) \cdot v_p(f_i)$$

is satisfied [26].

The family of all tangent vectors to (M, C) at $p \in M$ is a linear space over $\mathbb{R}$ denoted by T_pM. Of course, the addition of tangent vectors and the multiplication by scalars are defined in the pointwise manner.

Then by a tangent vector field to (M, C) we mean any mapping $V : M \to \bigcup_{p\in M} T_pM$, such that $V(p) \in T_pM$, for any $p \in M$.

A tangent vector field V is said to be smooth tangent vector field to (M, C) if, for any $f \in C$, the real function $g : M \to \mathbb{R}$, defined by $g(p) = V(p)f$, for any $p \in M$, is a smooth function, i.e. $g \in C$.

The family of all smooth tangent vector fields to a given differential space (M, C) will be denoted by $Vec(M, C)$ and in a natural way it also is a C-module.

The mapping $\Phi : Der(C) \ni D \mapsto V_D \in Vec(M, C)$ defined by the formula $V_D(p)f = (Df)(p)$, for any $f \in C$ and $p \in M$ is an isomorphism of C-modules. Its inverse $\Phi^{-1} : Vec(M, C) \ni V \mapsto D_V \in Der(C)$ is given by $(D_V f)(p) = V(p)f$, for any $f \in C$ and $p \in M$. On the strength of this isomorphism the derivatives and smooth vector fields are often identified.

For any differential space (M, C), a derivative $D \in Der(C)$ can be equivalently characterized by the chain rule

$$D(\omega \circ (f_1, \ldots, f_n)) = \sum_{i=1}^{n} \omega'_{|i} \circ (f_1, \ldots, f_n) \cdot D(f_i),$$

where $\omega \in \mathcal{E}_n$, $f_1, \ldots, f_n \in C$ and $n \in \mathbb{N}$.

If a differential space (M, C) is a differentiable manifold, the derivation D_V associated with a smooth vector field V can be identified with its Lie derivative $L_V : C \to C$.

The operators mentioned above are known as vector fields or derivatives of the first order. Their higher order counterparts will be defined below.

For a differential space (M, C), given point $p \in M$ and a natural number $m \in \mathbb{N}$, let us take out the ideal α_p^m from the ring C, namely

$$\alpha_p^m = \{(f_1 - f_1(p)) \cdot \ldots \cdot (f_m - f_m(p)) : f_1, \ldots, f_m \in C\} .$$

Then, for any $k \in \mathbb{N}$, by a k-th order tangent vector to (M, C) at a point $p \in M$ we mean any linear mapping $v_p : C \to \mathbb{R}$ satisfying the following conditions:

$$v_p(f) = 0 ,$$

whenever $f \in C$ is a constant function and

$$v_p|\alpha_p^{k+1} = 0 .$$

It is an easy task to show that in the case when $k = 1$ the concept of a 1-st order tangent vector coincides with the previous definition of a tangent vector to (M, C) at a given point $p \in M$, based on the Leibniz condition.

In general, for the k-th order tangent vectors we have the generalized chain rule [26]

$$v_p^{(k)}(\omega \circ (f_1, \ldots, f_n)) = \sum_{m=1}^{k} \sum_{1 \leq i_1 \leq \ldots \leq i_m \leq n} \omega_{|i_1 \ldots i_m}^{(m)}(f_1(p), \ldots, f_n(p)) \cdot$$
$$\cdot v_p^{(k)}((f_{i_1} - f_{i_1}(p)) \cdot \ldots \cdot (f_{i_m} - f_{i_m}(p))).$$

The collection $T_p^{(k)} M$ of all tangent vectors to (M, C) at $p \in M$, for any $k \in \mathbb{N}$, is naturally a linear space over $\mathbb{R}$, and it is called the k-th order tangent space to (M, C) at $p \in M$.

If (M, C) is a differentiable space of constant dimension (a differentiable manifold, in particular) of the dimension $dim(M, C) = n$, $n \in \mathbb{N}$, such that for any point $p \in M$ there exists a vector basis $V_1, \ldots, V_n \in Vec(U)$ on an open neighbourhood U of p satisfying the condition $[V_i, V_j] = 0$, for $i, j = 1, \ldots, n$, then the k-th order tangent space to (M, C) has the dimension [26, 20]

$$dim\, T_p^{(k)} M = \sum_{m=1}^{k} \binom{n + m - 1}{m} .$$

In a general case of a differential space (M, C) the above formula gives only the upper bound of the dimension, namely

$$dim\, T_p^{(k)} M \leq \sum_{m=1}^{k} \binom{n+m-1}{m}.$$

EXAMPLE 2.1.1. Let us consider the differential space (M, C), being the differential subspace of the plane $(\mathbb{R}^2, \mathcal{E}_2)$, such that $M = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$ and $C = \mathcal{E}_{2M}$. At the point $(0,0)$ we have $dim\ T_{(0,0)}M = 2$ since any tangent vector to (M, C) at $(0,0)$ is defined by its values on the restrictions $\pi_{1|M}$ and $\pi_{1|M}$, where π_1,π_1 are the usual projections. So we get $n = 2$. In turn, any tangent vector of the 2-nd order must be defined on $\pi_{1|M}$, $\pi_{1|M}$ and additionally on their products $\pi_{i|M} \cdot \pi_{j|M}$, for $i, j \in 1, 2$ (see [26]). Since in our case one of the products vanishes, namely $\pi_{1|M} \cdot \pi_{2|M} \equiv 0$, we obtain only four linearly independent tangent vectors of the 2-nd order and consequently $dim\ T^{(2)}_{(0,0)}M = 4$, whereas for the whole plane we have $dim\, T^{(2)}_{(0,0)}\mathbb{R}^2 = 5$.

Now, a mapping $V : M \to \bigcup_{p \in M} T_p^{(k)} M$, where $k \in \mathbb{N}$, is said to be a k-th order tangent vector field to (M, C) if $V(p) \in T_p^{(k)}M$, for any $p \in M$. A k-th order tangent vector field to (M, C) is said to be smooth if, for any $f \in C$, the function $g : M \to \mathbb{R}$, defined by $g(p) = X(p)f$, is smooth, i.e. $g \in C$. Naturally, the collection of all smooth k-th order vector fields to (M, C) is a C-module, and will be denoted by $Vec^{(k)}(M, C)$.

Every smooth k-th order tangent vector field $V \in Vec^{(k)}(M, C)$ gives rise to the so-called k-th order derivative $D_V : C \to C$ of the ring C, defined by $(D_V f)(p) = V(p)f$, for any $f \in C$. The collection of all k-th order derivatives of the algebra C will be denoted by $Der^{(k)}(C)$. On the other hand, for any $D \in Der^{(k)}(C)$ we define $V_D \in Vec^{(k)}(M, C)$ by $V_D(p)f = (Df)(p)$, for any $f \in C$ and $p \in M$.

EXAMPLE 2.1.2. Let $D_1, \ldots, D_k \in Der(C)$, $k \in \mathbb{N}$. Then $D \equiv D_1 \circ \ldots \circ D_k \in Der^{(k)}(C)$ and $V_D \in Vec^{(k)}(M, C)$.

2.2. Right invertible operators

For the reader's convenience we give here a short survey of the basic concepts concerning the right invertible operators but the comprehensive treatment of the topic one will find in references [13]–[17].

Let X be a real linear space and $L(X)$ be the family of all linear operators in X with the domains being linear subspaces of X. Then, for any $A \in L(X)$, let $\mathcal{D}_A$ denote the domain of A and let $L_0(X) = \{A \in L(X) : \mathcal{D}_A = X\}$.

A linear operator $D : \mathcal{D}_D \to X$, where $\mathcal{D}_D \subset X$ is a linear subspace (the domain of D), is said to be a right invertible operator if there exists a linear operator $R \in L_0(X)$, the so-called right inverse of D, such that the composition of the mappings is the identity on X, i.e. $DR = I$.

The collection of all right invertible operators will be denoted by $\mathcal{R}(X)$. In turn, the set of all right inverses of a given $D \in \mathcal{R}(X)$ will be denoted by $\mathcal{R}_D$ or as an indexed family, i.e. $\mathcal{R}_D = \{R_\gamma\}_{\gamma\in\Gamma}$.

If $R \in \mathcal{R}_D$ is a given right inverse of $D \in \mathcal{R}(X)$, then the family $\mathcal{R}_D$ can be characterized by $\mathcal{R}_D = \{R + (I - RD)A : A \in L_0(X)\}$.

For any $x \in X$ and $y \in X$ such that $Dy = x$ we say that y is a primitive element of x. Hence, for any $x \in X$ and $R \in \mathcal{R}_D$ the element Rx is a primitive element of x.

The set of all primitive elements of a given $x \in X$ is called the indefinite integral of $x \in X$ and will be denoted by $\mathcal{I}(x)$.

The kernel $Z_D = ker(D)$ is called the space of constant elements of D. It is easy to see that $\mathcal{R}_D x = \mathcal{R}_D x + Z_D = \{Rx + (I - RD)Ax : A \in L_0(X)\} = Rx + Z_D$, for any given $R \in \mathcal{R}_D$ and any non-zero element $x \in X$.

So we get $\mathcal{I}(x) = \mathcal{R}_D x + Z_D = Rx + Z_D$, for any $x \in X$ and $R \in \mathcal{R}_D$.

Any projection F of X onto Z_D, i.e. $F^2 = F$ and $Im\ F = Z_D$, is said to be an initial operator for D. Let us denote by $\mathcal{F}_D$ the family of all initial operators for a given $D \in \mathcal{R}(X)$. We say that an initial operator $F \in \mathcal{F}_D$ corresponds to $R \in \mathcal{R}_D$ if additionally $FR = 0$.

The two families $\mathcal{R}_D$ and $F \in \mathcal{F}_D$ uniquely determine each other. Indeed, for any $R \in \mathcal{R}_D$ we define $F = I - RD \in \mathcal{F}_D$. The converse is given by the formula $R = R_1 - FR_1$, where $R_1 \in \mathcal{R}_D$ can be any since the result does not depend on the choice of R_1. Thus, for any $\gamma \in \Gamma$ we have $F_\gamma = I - R_\gamma D \in \mathcal{F}_D$ and consequently we can write $\mathcal{F}_D = \{F_\gamma\}_{\gamma\in\Gamma}$. By a simple calculation one can show that $F_\alpha F_\beta = F_\beta$ and $F_\beta R_\alpha = R_\alpha - R_\beta$, for any $\alpha, \beta \in \Gamma$.

For any indices $\alpha, \beta, \gamma \in \Gamma$, one can also prove that $F_\beta R_\gamma - F_\alpha R_\gamma = F_\beta R_\alpha$, which means that in fact the left side of this equation does not depend on γ. Hence one can define the operator of definite integration $I_\alpha^\beta = F_\beta R_\gamma - F_\alpha R_\gamma$, for any $\alpha, \beta, \gamma \in \Gamma$.

Amongst many properties of the operator I_α^β there is $I_\alpha^\beta D = F_\beta - F_\alpha$ or equivalently $I_\alpha^\beta D\,x = F_\beta\,x - F_\alpha\,x$, for any $x \in \mathcal{D}_D$.

For any initial operator F and $x \in X$, the element $Fx \in \mathcal{D}_D$ is called the initial value of x. Hence, for any $x \in X$ and its arbitrary primitive element $y \in X$, i.e. $Dy = x$, we get $I_\alpha^\beta x = F_\beta y - F_\alpha y \in \mathcal{D}_D$ which is called the definite integral of x.

3. Linear operators and their integral mappings

Let (M, C) be a real differentiable manifold and V be a smooth vector field on (M, C). Then a smooth mapping $\gamma : (\alpha, \beta) \to M$ is said to be an integral curve of V if

$$(Vf)(\gamma(t)) = (f \circ \gamma)'(t),$$

for any function $f \in C$ and $t \in (\alpha, \beta)$. The problem however can appear when (M, C) is a general geometric object, as for example a Sikorski differential space or a Frölicher smooth space [10, 4]. Namely, there exist "rough" differential spaces in which there is no nonconstant smooth curve in the above sense and consequently no candidates for an integral curve of a nonzero vector field. Such a situation illustrates the following example.

EXAMPLE 3.1. Let us take $(\mathbb{Q}, \mathcal{E}_{1\mathbb{Q}})$. Then all smooth mappings

$$\gamma : ((\alpha, \beta), C^{\infty}((\alpha, \beta)) \to (\mathbb{Q}, \mathcal{E}_{1\mathbb{Q}})$$

are constant mappings. In turn, if we take the set of rational numbers $\mathbb{Q}$ as the subspace of the real line $(\mathbb{R}, \mathcal{E}_1)$ in the Frölicher category [4], we obtain even worse result: all structure curves are constant and all real functions defined on $\mathbb{Q}$ become smooth.

Therefore it seems reasonable to consider a generalized concept of an integral curve for a linear operator defined on the algebra of smooth functions of some differential space.

Let (A, Λ) and (M, C) be two Sikorski differential spaces, $\gamma : (A, \Lambda) \to (M, C)$ be a smooth mapping and let D be a right invertible linear operator defined on Λ. Then, we will say that the mapping γ is a D-integral mapping of a linear operator V defined on C if

$$(Vf) \circ \gamma = D(f \circ \gamma).$$

This concept we can illustrate with the well known example.

EXAMPLE 3.2. Let us take $\gamma : ((\alpha, \beta), C^{\infty}(\alpha, \beta)) \to (M, C)$, where (M, C) is a differentiable manifold and let $D \equiv \frac{d}{dt}$ be the chosen right invertible operator defined on $C^{\infty}(< \alpha, \beta >)$. Then every integral curve γ of a vector field V is automatically a $\frac{d}{dt}$-integral mapping of this vector field since

$$(Vf) \circ \gamma = \frac{d}{dt}(f \circ \gamma),$$

which we can write equivalently as $(Vf)(\gamma(t)) = (f \circ \gamma)'(t)$, for any $t \in (\alpha, \beta)$.

Another example illustrates an integral mapping of a smooth vector field defined in a differential space in which there are no non-constant smooth curves in the usual sense.

EXAMPLE 3.3. Obviously, every smooth mapping in differential spaces is continuous. It is also known that every continuous mapping from $\mathbb{R}$ into $\mathbb{Q}$ with respect to the natural topologies is a constant mapping. Therefore we conclude that every usual smooth curve in $\mathbb{Q}^n$, i.e. a smooth mapping

$\gamma : (\mathbb{R}, \mathcal{E}_1) \to (\mathbb{Q}^n, \mathcal{E}_{n\mathbb{Q}^n})$ is necessarily a constant mapping. On the other hand in the differential space $(\mathbb{Q}^n, \mathcal{E}_{n\mathbb{Q}^n})$ there are nontrivial smooth vector fields as for example the partial derivation $D_1 : \mathcal{E}_{1\mathbb{Q}^n} \to \mathcal{E}_{1\mathbb{Q}^n}$ defined on the structure generators $\pi_{i|\mathbb{Q}^n}$ by $D_1\pi_{i|\mathbb{Q}^n} = \delta_{1i}$ where $\pi_i : \mathbb{R}^n \ni (x_1, \ldots, x_n) \mapsto x_i \in \mathbb{R}$, $i = 1, \ldots, n$. In particular, for $n = 1$, we get the differential space $(\mathbb{Q}, \mathcal{E}_{1\mathbb{Q}})$ and the non-zero derivative $D \equiv D_1$. Thus, the embedding $\gamma : \mathbb{Q} \ni t \mapsto (t, q) \in \mathbb{Q}^n$, for any $q \in \mathbb{Q}^{n-1}$, is a D-integral mapping of a linear operator D_1 defined on $\mathcal{E}_{n\mathbb{Q}^n}$.

If an operator V defined on the algebra C of smooth functions of a given differential space (M, C) is right invertible and γ is a D-integral mapping of V, we can calculate its right inverse in the following way. Denote by R_V a right inverse of V and by R_D a right inverse of D. Then by definition we write the formula $(VR_Vf) \circ \gamma = D((R_Vf) \circ \gamma)$ from which we obtain

$$(R_Vf) \circ \gamma = R_D(f \circ \gamma).$$

Now we need to eliminate γ from the left side of the above formula. In the differentiable manifold case this problem was investigated and solved by G.Virsik [19].

4. Examples and applications

Many basic examples of right invertible operators one can find in Ref.[13]. Here we consider mainly certain cases which have some geometric or physical interpretation.

4.1. Discrete differential spaces

In the framework of differential spaces there are geometric objets with a discrete structure. Some discrete spaces (so called lattices) have also been considered in physics. Therefore the study of discrete cases is also motivated. Before we pass to examples we propose here a special case of the concept of a d-dimensional discrete differential space.

According to the Sikorski definition of a differential subspace, for any differential space (M, C) and a subset $A \subset M$ one defines a differential subspace (A, C_A), where C_A is the initial differential structure on A induced by the inclusion mapping $\iota : A \to M$.

Let $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ be the set of integers. Then, for any fixed $d \in \{1, 2, \ldots\}$, on the subset $\mathbb{Z}^d \subset \mathbb{R}^d$ we assume the initial differential structure $C(\mathbb{Z}^d) = \mathcal{E}_{d\mathbb{Z}^d}$. In fact, the elements of $C(\mathbb{Z}^d)$ are real generalized sequences, so we can write $C(\mathbb{Z}^d) = \mathbb{R}^{\mathbb{Z}^d}$. Thus we obtain $(\mathbb{Z}^d, C(\mathbb{Z}^d))$ as a differential subspace of the Euclidean manifold $(\mathbb{R}^d, \mathcal{E}_d)$. We will refer to $(\mathbb{Z}^d, C(\mathbb{Z}^d))$ as to the discrete d-dimensional Euclidean space since it is the discrete analog of the Euclidean d-dimensional manifold $(\mathbb{R}^d, \mathcal{E}_d)$.

The usual concept of a tangent vector becomes trivial in this discrete case since all tangent vectors are reduced to the single zero valued functional. Consequently, in the usual framework applied to the discrete case there is no nontrivial tangent objects. Hence, in a discrete case it seems natural to replace the usual notion of a tangent vector by some discrete substitute. However, the discrete substitutes of tangent vectors and the tangent vector fields do not fulfill the Leibniz condition. Moreover, they are even not the local operators. But, on the other hand, they are very much in a "logic" relation with their usual counterparts and due to D. Przeworska-Rolewicz we have the well developed calculus on such operators, which realizes maximum possible analogy with their smooth version. Thus, by discrete partial derivatives in the above space $(\mathbb{Z}^d, C(\mathbb{Z}^d))$ we shall mean the linear difference operators

$$D_i : C(\mathbb{Z}^d) \ni x \mapsto D_i x \in C(\mathbb{Z}^d),$$

$$(D_i x)_n = x_{n+\hat{1}_i} - x_n,$$

where $\hat{1}_i \in \mathbb{Z}^d$ is defined as $(\hat{1}_i)_k = \delta_{ik}$ (Kronecker delta), $i, k = 1, \ldots, d$ and $n \in \mathbb{Z}^d$. As the $C(\mathbb{Z}^d)$-module $Der(C(\mathbb{Z}^d))$ of discrete derivatives we take all linear combinations of the discrete partial derivatives D_i, $i = 1, \ldots, d$, with coefficients from $C(\mathbb{Z}^d)$.

By a tangent vector to $(\mathbb{Z}^d, C(\mathbb{Z}^d))$ at a point $n \in \mathbb{Z}^d$ we shall mean any real functional $D_i(n)$ defined by $D_i(n)x = (D_i x)_n$. Then, the the tangent space to $(\mathbb{Z}^d, C(\mathbb{Z}^d))$ at a point $n \in \mathbb{Z}^d$ is assumed to be the linear space $T_n\mathbb{Z}^d$ of all linear combinations of the functionals $D_i(n)$, $i = 1, \ldots, d$, with coefficients from $\mathbb{R}$. Therefore $d = dim\ T_n\mathbb{Z}^d$, for any $n \in \mathbb{Z}^d$, and by definition we will refer to d as to the discrete dimension of $(\mathbb{Z}^d, C(\mathbb{Z}^d))$.

Correspondingly, by a tangent vector field we understand here any mapping

$$V : \mathbb{Z}^d \to \bigcup_{m \in \mathbb{Z}^d} T_m\mathbb{Z}^d,$$

such that $V(n) \in T_n\mathbb{Z}^d$, for any $n \in \mathbb{Z}^d$. Vector fields and derivatives defined above uniquely determine each other via formulae $V_D(n)x = (Dx)_n$ and $(D_V x)_n = V(n)x$. Following the smooth case, we will identify the two concepts.

By a discrete curve in $(\mathbb{Z}^d, C(\mathbb{Z}^d))$ we shall understand here any mapping $\gamma : \mathbb{Z} \ni t \mapsto \gamma(t) \in \mathbb{Z}^d$.

In turn, we consider here a discrete curve γ to be a Δ-integral curve of a derivative D if

$$(Dx) \circ \gamma = \Delta(x \circ \gamma),$$

which we can write equivalently in the standard sequence notation as follows

$$(Dx)_{\gamma(t)} = x_{\gamma(t+1)} - x_{\gamma(t)},$$

where Δ, is given by $(\Delta\xi)_t = \xi_{t+1} - \xi_t$, for any $\xi \in C(\mathbb{Z})$ and $t \in \mathbb{Z}$ (see Sec.3).

Before passing to higher dimension d let us consider the example which is a slight modification of the example given in Ref.[13].

EXAMPLE 4.1.1. In the discrete differential space $(\mathbb{Z}, \mathbb{R}^{\mathbb{Z}})$ consider the operator D of the discrete derivation, $(Dx)_n = x_{n+1} - x_n$, for $x \in \mathbb{R}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$. Then one of its right inverses is given by

$$(Rx)_n = \begin{cases} -\sum\limits_{m=n}^{0} x_m & n \le 0 \\ 0 \quad \text{for} & n = 1 \\ \sum\limits_{m=1}^{n-1} x_m & n > 1 \end{cases} .$$

The initial operator F corresponding to this inverse R assigns to each sequence x the constant sequence of the value x_1. Other initial operators, corresponding to the other right inverses, assign to each sequence x the constant sequence of the value x_2, $x_3, \ldots$, etc. In particular, the above formula describes the right inverse of the restriction of D to $\mathbb{R}^{\mathbb{N}}$ which coincides with the example given in [13].

Let us assume the notation for some functions which will be used in the examples below. At first, by the brackets $[\ \cdot\ ]$ we denote the integer value function ("valeurs entiéres" function). Then, let

$$\chi(t) = \begin{cases} 0 \; for \; t \le 0 \\ 1 \qquad t > 0 \end{cases},$$

and, for any $T > 0$, let $\eta_T(t) = \chi(t-T) + \chi(t) - 1$, $\lambda_T^+(t) = -\chi(t)[\frac{T-t}{T}]$, $\lambda_T^-(t) = 1 - (1 - \chi(t))[\frac{T-t}{T}]$.

EXAMPLE 4.1.2. Consider the discrete space $(\mathbb{Z}, \mathbb{R}^{\mathbb{Z}})$. For a fixed $k \in \mathbb{N}$ let us define the following difference operator $D^{(k)}$ by $(D^{(k)}x)_n = x_{n+k} - x_n$, for any $x \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$.

Naturally, $D^{(1)} = D$ is the usual discrete derivation as in Ex.3.1.1, and we have the relation $(D^{(k)}x)_n = \sum\limits_{m=0}^{k-1} (Dx)_{n+m}$.

We see that the space of constants $\mathcal{Z}_{D^{(k)}}$ consists of all periodic sequences with the period $k \in \mathbb{N}$.

The operator $D^{(k)}$, for any $k \in \mathbb{N}$, is right invertible and one of its right inverses is given by the formula

$$(R^{(k)}x)_n = \eta_k(n) \sum_{m=\lambda_k^-(n)}^{\lambda_k^+(n)} x_{n-mk}.$$

Thus, the family of all right inverses of $D^{(k)}$ is given by

$$\mathcal{R}_{D^{(k)}} = \{R^{(k)} + (I - R^{(k)}D^{(k)})A : A \in L_0(\mathbb{R}^{\mathbb{Z}})\}.$$

The initial operator $F^{(k)}$ of $D^{(k)}$ corresponding to $R^{(k)}$ is given by $(F^{(k)}x)_n = x_{n+[\frac{k-n}{k}]\cdot k}$.

If we restrict ourselves to the subspace $(\mathbb{N}, \mathbb{R}^{\mathbb{N}})$ and consider the operator $D^{(k)}$ acting on $\mathbb{R}^{\mathbb{N}}$, we get its right inverse in the simpler form

$$(R^{(k)}x)_n = \sum_{m=1}^{[\frac{n-1}{k}]} x_{n-m\cdot k},$$

we assume $\sum_{m=1}^{0}$ to give always zero.

More general case we get for higher dimension d.

EXAMPLE 4.1.3. Let us consider the discrete d-dimensional Euclidean space $(\mathbb{Z}^d, C(\mathbb{Z}^d))$, for a fixed $d = 1, 2, \ldots$, and define the following partial difference operator $D_i^{(k)}$ by

$$(D_i^{(k)}x)_n = x_{n+k\hat{1}_i} - x_n,$$

for given $k \in \mathbb{N}$, $i \in \{1, \ldots, d\}$ and any $n \in \mathbb{Z}^d$. The operator $D_i^{(k)}$ is right invertible and its right inverse is given by

$$(R_i^{(k)}x)_n = \eta_k(\pi_i(n)) \sum_{m=\lambda_k^-(\pi_i(n))}^{\lambda_k^+(\pi_i(n))} x_{n-mk\hat{1}_i},$$

where $\pi_i(n) = n_i$, for any $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$.

If we put $k = 1$ in the above example we obtain the discrete partial derivations $D_i^{(k)} = D_i$, for $i = 1, \ldots, d$. In this case the initial operator F_i corresponding to the above inverse R_i assigns to each sequence $x \in C(\mathbb{Z}^d) = \mathbb{R}^{\mathbb{Z}^d}$ the sequence $F_i x$ such that $(F_i x)_n = x_{n-(\pi_i(n)+1)\hat{1}_i}$. From other initial operators we get sequences with their components given by $x_{n-(\pi_i(n)+m)\hat{1}_i}$, where $m \in \mathbb{Z}$.

Hence, the initial subspaces are given as the inverse images $\pi_i^{-1}(m)$, for $m \in \mathbb{Z}$ (for the manifold case compare [19]). If $k > 1$ the initial subspaces are "thicker" and they are given by $\pi_i^{-1}(\{m+1, m+2, \ldots, m+k\})$, for $m \in \mathbb{Z}$.

For higher order difference operators (discrete derivations, in particular) which are of the simple form $D_{i_1}^{k_1} \circ \ldots \circ D_{i_s}^{k_s}$, $s \in \mathbb{N}$, at first we obtain their right inverses as the compositions $R_{i_s}^{k_s} \circ \ldots \circ D_{i_1}^{k_1}$, where $D_{i_j}^{k_j} R_{i_j}^{k_j} = I$, for $j = 1, \ldots, s$. Then, the remaining right inverses we get by the routine manner mentioned in Sec.2.2.

Let us notice that the operators considered in this section can be uniformly interpreted within the class of all linear combinations of the translations S_k (which are invertible operators) defined on $\mathbb{Z}^d$ by $(S_k x)_n = x_{n+k}$, for $x \in C(\mathbb{Z}^d)$, $k, n \in \mathbb{Z}^d$. For example the usual discrete derivation D, its second power D^2 and the difference operator $D^{(2)}$, defined in $C(\mathbb{Z})$, can be expressed as $D = S_1 - S_0$ (of course $S_0 = I$), $D^2 = S_2 - 2S_1 - S_0$ and $D^{(2)} = S_2 - S_0$. Therefore the linear space of all linear combinations

$$c_1 S_{k_1} + \ldots + c_s S_{k_s},$$

where $c_1, \ldots, c_s \in \mathbb{R}$, $k_1, \ldots, k_s \in \mathbb{Z}^d$, $s \in \mathbb{N}$, contains the operators considered above. The linear combinations of translations are the non-local operators, which has some mathematical as well as physical consequences.

4.2. Right inverses of finite difference operators

In the present section we give the modification of the results achieved in the above discrete case. Namely, we consider differential spaces $(\mathbb{R}, C(\mathbb{R}))$ with the differential structure $C(\mathbb{R})$ invariant under certain translations. Let $T > 0$ and $S_T : C(\mathbb{R}) \to C(\mathbb{R})$, $(S_T f)(t) = S(t + T)$. We will say that a differential structure $C(\mathbb{R})$ is S_T-invariant if for any $f \in C(\mathbb{R})$ also $S_T f \in C(\mathbb{R})$. For example $\mathcal{E}_1$ is S_T-invariant, for any $T \in \mathbb{R}$, whereas the differential structure $(sc\{id_{\mathbb{R}}, |\cdot|\})_{\mathbb{R}}$, where $|\cdot|$ stands for the absolute value function, is not S_T-invariant for $T \neq 0$ since, for example, $S_T|\cdot| \notin C(\mathbb{R})$.

EXAMPLE 4.2.1. Let $C(\mathbb{R}) = \mathbb{R}^{\mathbb{R}}$ and $D^{(T)}$ be the difference operator defined on $\mathbb{R}^{\mathbb{R}}$ by the formula $(D^{(T)} f)(t) = f(t + T) - f(t)$, for a fixed $T > 0$. Its space of constants $\mathcal{Z}_{D^{(T)}}$ consists of all periodic functions with the period T. One of the right inverses of $D^{(T)}$ is given by the formula

$$(R^{(T)} f)(t) = \eta_T(t) \sum_{m=\lambda_T^-(t)}^{\lambda_T^+(t)} f(t - mT).$$

Hence, we obtain

$$\mathcal{R}_{D^{(T)}} = \{R^{(T)} + (I - R^{(T)} D^{(T)}) A : A \in L_0(\mathbb{R}^{\mathbb{R}})\}.$$

In turn, the indefinite integral of any element $f \in \mathbb{R}^{\mathbb{R}}$ is of the form

$$\mathcal{I}(f) = \mathcal{R}_{D^{(T)}} f + Z_D = \{R^{(T)} f + u : u \in Z_D\}.$$

For the operator $D^{(T)}$ defined on the space $\mathbb{R}^{\mathbb{R}^+} = \{f : (0, +\infty) \to \mathbb{R}\}$ one of its right inverses is given by

$$(R^{(T)}f)(t) = \sum_{m=1}^{-[1-\frac{t}{T}]} f(t - mT),$$

and analogously $\mathcal{I}(f) = \{R^{(T)}f + u : u \in Z_D\}$.

In this context it is natural to mention about the difference quotient operator.

EXAMPLE 4.2.2. Let $D^{(T)}$, $T > 0$, be as in Ex.4.2.1 and let us consider the difference quotient operator $\Delta^{(T)} = \frac{1}{T}D^{(T)}$ defined on $\mathbb{R}^{\mathbb{R}}$ by

$$(\Delta^{(T)}f)(t) = \frac{f(t+T) - f(t)}{T}.$$

From the Ex.4.2.1 it becomes evident that $\Delta^{(T)}$ is right invertible and $\mathcal{R}_{\Delta^{(T)}} = \{TR^{(T)} : R^{(T)} \in \mathcal{R}_{D^{(T)}}\}$.

In some applications we cut off an internal piece of a function and then bring closer or even glue together the two parts. A simple example of such an operation illustrates the following example.

EXAMPLE 4.2.3. For any fixed $a \in \mathbb{R}$ and $T > 0$, let us consider the operator D_a^T defined on $\mathbb{R}^{\mathbb{R}}$ by the formula

$$(D_a^T x)(t) = \begin{cases} f(t) & \text{for } t < a \\ f(t+T) & t \geq a \end{cases}.$$

The space $\mathcal{Z}_{D_a^T}$ of constants of D_a^T consists of all functions u such that $u_{|(-\infty,a)\cup<a+T,+\infty)} = 0$. We see that D_a^T is a right invertible operator. Indeed, a right inverse R_a^T of D_a^T can be defined by

$$(R_a^T f)(t) = \begin{cases} f(t) & t < a \\ g(t) \text{ for } a \leq t < a+T \\ f(t) & t \geq a+T \end{cases},$$

where $g :< a, a+T) \to \mathbb{R}$ is an arbitrary function.

4.3. Final remarks

Let us end this paper with some remarks concerning the right invertibility of operators defined on geometric objects which are not differentiable manifolds and then applications of the algebraic analysis in physics.

It is known that every finite dimensional differentiable manifold (M, C) can be embedded in a Euclidean manifold $(\mathbb{R}^N, \mathcal{E}_N)$ of a sufficiently high dimension $N \in \mathbb{N}$. Therefore, without loosing generality, we can assume (M, C) to be a submanifold of $(\mathbb{R}^N, \mathcal{E}_N)$. According to this assumption we can write $M \subset \mathbb{R}^N$ and $C = (\mathcal{E}_N)_M$. More general result is known for differential spaces as well as for the Frölicher spaces. Namely, every differential space (M, C) with the Hausdorff topology can be embedded in the Cartesian product $(\mathbb{R}^{C_0}, \mathcal{E})$ where C is generated by C_0 and $\mathcal{E}$ is generated by the family of projections $\{\pi_f : f \in C_0\}$. For an infinite set C_0 the Cartesian product $(\mathbb{R}^{C_0}, \mathcal{E})$ is not a differentiable manifold. Since the infinite Cartesian product, each factor being $\mathbb{R}$, is not a Banach space, the resulting differential space cannot be considered as a smooth Banach manifold. Then, let us consider the differential space$(M_{\mathbb{Q}}, C_{M_{\mathbb{Q}}})$,where $M_{\mathbb{Q}} = M \cap \mathbb{Q}^{C_0}$. It is not a differentiable manifold either, even if C_0 is finite. It is, however, a differential space with the underlying set being a dense subset in M. However, in the above cases the differential spaces are "close" to manifolds in a certain sense as they are of constant differential dimension [24, 25].

Therefore the results presented in [19] cannot be applied directly in such cases but they can be efficiently adapted after necessary modification, in particular with the help of the concept of a D-integral mapping.

Another interest is related with the right invertibility of linear operators defined in differential groups, which are groups and simultaneously differential spaces. Originally the notion of a differential group is due to K. Spallek [9] but in the sense of Sikorski they were investigated in [21, 22, 29]. Let us mention only that any Lie group is an example of a differential group. It has been proved [22, 29] that the Cartesian product of any family of differential groups is a differential group. In general, this statement does not hold for Lie groups since an infinite product of manifolds is not a manifold. Differential groups are always of constant differential dimension and they have elegant left invariant vector fields which are complete provided their integral curves in some sense are well defined. In a close relation to ifferential groups there are the so-called G-spaces, which are differential spaces on which acts a differential group. Difference operators can be investigated on their structure functions and their right invertibility is an open problem. However, one can easily notice that, for a given element g of a group G, the space of all constants of the translation $(D_g f)(x) = f(gx)$ is the family of all g invariant functions (which generalizes the concept of periodic functions). In physics, the noncommutative geometry is a big interest. Therefore applying general methods of the algebraic analysis (differential equations, in particular) to noncommutative rings, should bring interesting results.

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