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ABSTRACT VARIABLE DOMAIN HYPERBOLIC DIFFERENTIAL EQUATIONS

Abstract. An abstract problem is studied for a class of linear hyperbolic differential equations with variable domain and non-local boundary conditions. Existence and uniqueness of the strong solution are proved.

1. Introduction

This paper is devoted to the study of a boundary value problem for hyperbolic differential operator equations with variable domain and non-local boundary conditions.

Let $T > 0$, be given, H is a Hilbert space and $\{A(t)\}$ is a family of unbounded operators in H, such that for all $t \in [0,T]$, $A(t)$ are self-adjoint positive and densely definite. We suppose also that their domains $D(A(t))$ are dependent on t. We look for an H-valued function $u(t)$ which solves the following boundary value problem:

$$\text{(P)} \quad \begin{cases} \mathcal{L}u = \dfrac{d^2u}{dt^2} + A(t)u = f(t) \\ \ell_{0_\mu}u = u_{|t=0} - \mu u_{|t=T} = \varphi \; ; \; \ell_{1_\mu}u = \dfrac{du}{dt}_{|t=0} - \mu \dfrac{du}{dt}_{|t=T} = \psi, \end{cases}$$

where μ is a complex parameter, $f(t)$ is an H-valued function, φ and ψ are given in H.

In the case where the operator coefficients have constant domains, various important results were proved under different assumptions; see [1, 4, 5, 6]. The proofs were based on energy inequalities in [4, 5, 6], or on the parametric construction and a subsequent use of the Laplace transformation for the corresponding problem in the first work.

A version of the problem (P) with homogeneous Cauchy boundary conditions was studied in the works [8, 9].

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The aim of the present paper is the study of the boundary value problem (P) in a general case, with $f(t) \neq 0$ and with nonhomogeneous, non-local boundary conditions.

Summary of the paper is as follows:

In Section 1, we give notations, present main assumptions and describe functional spaces.

In Section 2, we prove the uniqueness of the strong solution. For this purpose, we use the regularizing operators.

Section 3 deals with the existence of the strong solution. We prove that the range of the operator L_μ generated by the problem (P) is dense. Then we give some examples, which illustrate the considered problem.

Let us remark that the literature for parabolic linear variable domain problem is wider than the hyperbolic one. We can mention the very recent papers [2, 3].

2. Notations, main assumptions and functional spaces

As previously mentioned, let H be a Hilbert space with the norm $|.|$ and the inner product $(,)$.

We solve the following boundary value problem:

Find a strong solution u to the equation

$$\mathcal{L}u = \frac{d^2u}{dt^2} + A(t)u = f(t) \tag{1}$$

with nonhomogeneous and non-local boundary conditions:

$$\ell_{0_\mu}u = u_{|t=0} - \mu u_{|t=T} = \varphi \ ; \ \ell_{1_\mu}u = \frac{du}{dt}_{|t=0} - \mu \frac{du}{dt}_{|t=T} = \psi. \tag{2}$$

We denote by I the interval $]0, T[\subset \mathbb{R}$, $T < \infty$.

The functions u and f are two t-variable functions from I to H; μ is a complex parameter satisfying

$$|\mu| < e^{-2aT}.$$

The constant a will be defined latter.

The linear operators $A(t), \forall t \in I$, are unbounded in H, with domains $D(A(t))$ depending on t and everywhere dense in H. We impose the following assumptions:

(a) The operators $A(t)$, for $t \in I$ are self-adjoint in H and there exists a constant c_1 not depending on v and t such that:

$$(A(t)v(t), v(t)) \geq c_1 |v(t)|^2, \ \forall v(t) \in D(A(t)), \ \forall t \in I.$$

(b) We suppose that the inverse operators to $A(t)$ exist on I and $A^{-1}(t)$ are strongly differentiable with respect to t in H with:

$$\frac{d^iA^{-1}(t)}{dt^i}, A^{\frac{1}{2}}(t)\frac{d^iA^{-1}(t)}{dt^i} \in L_\infty(I, \mathcal{L}(H)), \quad i = 1, 2.$$

(c) There exists a constant $c_2 > 0$, not depending on t and u, such that

$$-\left(\frac{dA^{-1}(t)}{dt}u(t), u(t)\right) \leq c_2\left(A^{-1}(t)u(t), u(t)\right), \quad \forall u \in H.$$

We need the following definition:

DEFINITION 1. The operator $A(t)$ is strongly differentiable with respect to t, in t_0, on $u(t_0) \in D(A(t_0))$, if there exists $t \in v(t_0)/\{t_0\}$ ($v(t_0)$ is a neighborhood of the point t_0) such that $u(t) \in D(A(t))$,

$$\frac{u(t) - u(t_0)}{t - t_0} \underset{t\to t_0}{\to} \alpha(t_0) \in D(A(t_0)),$$

and

$$\frac{A(t)u(t) - A(t_0)u(t_0)}{t - t_0} \underset{t\to t_0}{\to} \beta(t_0).$$

Then we have

$$A'(t_0)u(t_0) = \beta(t_0) - A(t_0)\alpha(t_0).$$

We remark that if the domain of the operator $A(t)$ is constant, then we find the known concept of the strong differentiability.

Now, we introduce the following functional spaces:

Let $D(A^{1/2}(t)) \neq \{\emptyset\}$. We construct the Hilbert space $W(t)$ on $D(A^{1/2}(t))$ for all $t \in I$, equipped with the norm

$$|u|_t = |A^{1/2}(t)u(t)|.$$

The operator generated by the problem (P) is denoted by $L_\mu = (\mathcal{L}, \ell_{0_\mu}, \ell_{1_\mu})$ with the domain

$$D(L_\mu) = \left\{ \begin{array}{c} u \in L_2(I, H), u(t) \in D(A(t)), \dfrac{d^ku(t)}{dt^k}, \\ A(t)u \in L_2(I, H), \sup_{t\in\bar{I}} |A(t)u| < \infty, \quad k = 1, 2 \end{array} \right\}.$$

We denote by E_μ the completion of $D(L_\mu)$ with respect to the norm

$$\|u\|_\mu^2 = (1 - e^{3aT}|\mu|^2) \sup_{t\in\bar{I}}(|du(t)/dt|^2 + |u(t)|_t^2).$$

We consider the Hilbert space $E = L_2(I, H) \times W(0) \times H$, whose elements $F = (f, \varphi, \psi)$ are such that $|||F|||^2 = \|f\|^2 + |\varphi|_0^2 + |\psi|^2$ is finite.

For the operator L_μ we present the following Lemma:

LEMMA 2. *If the conditions* (a)–(c) *hold then* $D(L_\mu)$ *is dense in* $L_2(I, H)$.

Proof. Let $v \in L_2(I,H)$ be such that $\int_I (u,v)\,dt = 0$ for all $u \in D(L_\mu)$. We set $u = A^{-1}(t)h$, $t \in I$, where h is any function in $L_2(I,H)$.

We can easily see that $\int_I (A^{-1}(t)h, v)\,dt = 0$. In particular if $v = h$, then, from the condition (a), we obtain $A^{-1}(t)v = 0$. Hence $v = 0$ on I.

THEOREM 3. *Under the conditions of Lemma* 2, *we get*

$$\|u\|_\mu^2 \le K \||L_\mu u\||^2 \qquad \text{for all } u \in D(L_\mu), \tag{3}$$

where $K > 0$ *is a constant independent on* μ *and on* u.

Proof. Since the operators $A(t)$ are not bounded, we approximate them by a bounded and strongly differentiable operators $A(t)A_\varepsilon^{-1}(t)$ where

$$A_\varepsilon^{-1} = (I + \varepsilon\, A)^{-1}; \qquad \varepsilon \ge 0.$$

The regularizing operators A_ε^{-1} have the following properties:

$$\frac{dA_\varepsilon^{-1}(t)}{dt} = \varepsilon A(t)A_\varepsilon^{-1}(t)\frac{dA^{-1}(t)}{dt}A(t)A_\varepsilon^{-1}(t), \tag{P1}$$

$$\varepsilon A(t)A_\varepsilon^{-1}(t) = I - A_\varepsilon^{-1}(t), \tag{P2}$$

$$\|\varepsilon A(t)A_\varepsilon^{-1}(t)u\| = \|u - A_\varepsilon^{-1}(t)u\| \to 0 \text{ when } \varepsilon \to 0, \forall u \in H, \tag{P3}$$

$$\frac{d(A(t)A_\varepsilon^{-1}(t))}{dt} = \frac{-1}{\varepsilon}\frac{dA_\varepsilon^{-1}(t)}{dt}, \tag{P4}$$

A_ε^{-1} is self-adjoint positive operator and commute with A.

$$(A_\varepsilon v, v) \ge (1 + \varepsilon c_1), \forall\, v \in D(A(t)) \text{ and } t \in I. \tag{P5}$$

We integrate by part the expression $2\,\mathrm{Re}\, e^{c(\tau-t)}(\mathcal{L}u, A_\varepsilon^{-1}\frac{du}{dt})$ over the interval $]0,\tau[$ to get:

$$\begin{aligned}
&\left[\left(\frac{du}{dt}, A_\varepsilon^{-1}(t)\frac{du}{dt}\right) + \left(u, A(t)A_\varepsilon^{-1}(t)u\right)\right]_{t=\tau} \\
&= e^{c\tau}\left[\left(\frac{du}{dt}, A_\varepsilon^{-1}(t)\frac{du}{dt}\right) + \left(u, A(t)A_\varepsilon^{-1}(t)u\right)\right]_{t=0} \\
&+ 2\,\mathrm{Re}\int_0^\tau e^{c(\tau-t)}(\mathcal{L}u, A_\varepsilon^{-1}(t)\frac{du}{dt})dt + \mathrm{Re}\int_0^\tau e^{c(\tau-t)}(u, \frac{d\left(A(t)A_\varepsilon^{-1}(t)\right)}{dt}u)dt \\
&+ \mathrm{Re}\int_0^\tau e^{c(\tau-t)}\left(\frac{dA_\varepsilon^{-1}(t)}{dt}\frac{du}{dt}, \frac{du}{dt}\right)dt \\
&- \mathrm{Re}\int_0^\tau e^{c(\tau-t)}\left[\left(\frac{du}{dt}, A_\varepsilon^{-1}(t)\frac{du}{dt}\right) + \left(u, A(t)A_\varepsilon^{-1}(t)u\right)\right]dt.
\end{aligned} \tag{4}$$

In order to bound the right side of (4), we use the properties of the regularizing operators. Using the Cauchy-Schwartz inequality, the δ-inequality with $\delta = 1$ for the second term and the condition (c) for the third one we obtain:

$$(5)\quad \left[\left(\frac{du}{dt}, A_\varepsilon^{-1}(t)\frac{du}{dt}\right) + \left(u, A(t)A_\varepsilon^{-1}(t)u\right)\right]_{t=\tau}$$
$$\leq e^{c\tau}\left[\left(\frac{du}{dt}, A_\varepsilon^{-1}(t)\frac{du}{dt}\right) + \left(u, A(t)A_\varepsilon^{-1}(t)u\right)\right]_{t=0}$$
$$+ 2\operatorname{Re}\int_0^\tau e^{c(\tau-t)}\left|\mathcal{L}u\right|^2 dt + 2\operatorname{Re}\int_0^\tau e^{c(\tau-t)}\left|A_\varepsilon^{-1}(t)\frac{du}{dt}\right|^2 dt +$$
$$+ c_2\operatorname{Re}\int_0^\tau e^{c(\tau-t)}\left|A_\varepsilon^{-1}(t)u\right|^2 dt + \operatorname{Re}\int_0^\tau e^{c(\tau-t)}\left(\frac{dA_\varepsilon^{-1}(t)}{dt}\frac{du}{dt}, \frac{du}{dt}\right) dt$$
$$- c\operatorname{Re}\int_0^\tau e^{c(\tau-t)}\left[\left(\frac{du}{dt}, A_\varepsilon^{-1}(t)\frac{du}{dt}\right) + \left(u, A(t)A_\varepsilon^{-1}(t)u\right)\right] dt.$$

Using P_3 in (5), allowing $\varepsilon \to 0$, we get

$$(6)\quad \left[\left|\frac{du}{dt}\right|^2 + |u|_t^2\right]_{t=\tau} \leq e^{c\tau}\left[\left|\frac{du}{dt}\right|^2 + |u|_t^2\right]_{t=0}$$
$$+ \int_0^\tau e^{c(\tau-t)}\left|\mathcal{L}u\right|^2 dt + (1-c)\int_0^\tau e^{c(\tau-t)}\left|\frac{du}{dt}\right|^2 dt + (c_2 - c)\int_0^\tau e^{c(\tau-t)}|u|_t^2\, dt.$$

We can see that for $c \geq \max(1, c_2) = a$ the last two terms are not positive. Then they can be omitted. For $c = a$, the inequality (6) implies

$$(7)\quad \left[\left|\frac{du}{dt}\right|^2 + |u|_t^2\right]_{t=\tau} \leq e^{aT}\left[\left|\frac{du}{dt}\right|^2 + |u|_t^2\right]_{t=0} + \int_0^\tau e^{a(\tau-t)}\left|\mathcal{L}u\right|^2 dt.$$

Now we consider the form $2\operatorname{Re}\int_\tau^T e^{c(\tau-t)}(\mathcal{L}u, A_\varepsilon^{-1}(t)\frac{du}{dt})dt$ and, by using the same reasoning as before, we prove that

$$(8)\quad \left[\left|\frac{du}{dt}\right|^2 + |u|_t^2\right]_{t=T} \leq e^{aT}\left[\left|\frac{du}{dt}\right|^2 + |u|_t^2\right]_{t=\tau} + e^{aT}\int_\tau^T e^{a(\tau-t)}\left|\mathcal{L}u\right|^2 dt.$$

From (7) and (8) we get

$$(9)\quad (e^{aT} - 1)\left[\left|\frac{du}{dt}\right|^2 + |u|_t^2\right]_{t=\tau} \leq e^{2aT}\left[\left|\frac{du}{dt}\right|^2 + |u|_t^2\right]_{t=0}$$
$$- e^{-aT}\left[\left|\frac{du}{dt}\right|^2 + |u|_t^2\right]_{t=T} + e^{2aT}\int_0^T \left|\mathcal{L}u\right|^2 dt.$$

We need the following

LEMMA 4. *Let $g : I \to H$ and define*

$$h = g|_{t=0} - \mu g|_{t=T}, \tag{10}$$

where μ is a complex parameter satisfying $|\mu| < e^{-2aT}$. Then we have

$$e^{3aT}\left|g|_{t=0}\right|^2 - \left|g|_{t=T}\right|^2 \leq \frac{e^{3aT}}{(1 - e^{3aT}|\mu|^2)}|h|^2. \tag{11}$$

Proof. From (10) and the ε – *inequality*, we deduce that

$$\left|g|_{t=0}\right|^2 \leq (1+\varepsilon)\,|\mu|^2 \left|g|_{t=T}\right|^2 + \left(1+\varepsilon^{-1}\right)\,|h|^2.$$

It is enough to choose

$$\varepsilon = \frac{(1 - e^{3aT}|\mu|^2)}{e^{3aT}\,|\mu|^2}$$

to get (11).

Now we come back to the proof of the Theorem (3). Applying the Lemma 4 to the inequality (9) we then obtain

$$(e^{aT} - 1)\left[\left|\frac{du}{dt}\right|^2 + |u|_t^2\right]_{t=\tau} \leq \frac{e^{2aT}}{(1 - e^{3aT}|\mu|^2)}[|||\mathcal{L}u||^2 + |\varphi|_{(0)}^2 + |\psi|^2]. \tag{12}$$

Multiplying (12) by $\dfrac{1 - e^{3aT}|\mu|^2}{e^{2aT}}$ (we remark that $0 < 1 - e^{3aT}|\mu|^2 < 1$) and taking the least upper bounds for both sides of the resultant inequality with respect to τ, we obtain the inequality (3) with

$$K = \frac{e^{2aT}}{e^{aT} - 1}.$$

LEMMA 5. *Assume that the conditions of Theorem 3 hold. Then L_μ is closable.*

We denote by $\overline{L}_\mu$ its closure with domain of definition

$$D(\overline{L}_\mu) = \overline{D(L_\mu)}.$$

DEFINITION 6. A function u satisfying

$$\overline{L}_\mu u = F,$$

is called a strong solution of the considered problem.

By passing to the limit we extend the inequality (3) to strong solutions $u \in D(\overline{L}_\mu)$:

$$|||u|||_\mu^2 \leq K|||\overline{L}_\mu u|||^2, \qquad \forall u \in D\left(\overline{L}_\mu\right). \tag{13}$$

We can easily prove that

$$R(\overline{L_\mu}) = \overline{R(L_\mu)} \qquad \text{and} \qquad (\overline{L_\mu})^{-1} = (\overline{L_\mu^{-1}}).$$

REMARK 1. From the inequality (13), we deduce the uniqueness of the strong solution and the closure of $R(\overline{L_\mu})$.

For the existence of the strong solution, it remains to prove that the range $R(L_\mu)$ is dense in E, which is equivalent to $R(L_\mu)^\perp = \{0\}$.

3. Existence of the strong solution

THEOREM 7. *Let the conditions* (a)–(c) *hold. Then for any* $F = (f, \varphi, \psi) \in E$, *there exists an unique strong solution* u *to the problem* (P) *satisfying* $\|u\|_\mu^2 \le K \||\overline{L}_\mu u\||^2$ *for all* $u \in D(\overline{L}_\mu)$.

Before proving this Theorem 7 we give an intermediate result:

PROPOSITION 8. *Let the conditions of Theorem 7 hold. Assume moreover, that* $u \in D_0(L_\mu) = \{u \in D(L_\mu)$ *such that* $\ell_{0_\mu} u = \ell_{1_\mu} u = 0\}$. *Then an equality*

$$\int_0^T (\mathcal{L}u, v)dt = 0, \forall v \in L_2(I, H), \tag{14}$$

holds only for $v = 0$.

Proof. Let h be any function in $H_2(I, H)$ such that $u(t) = A^{-1}(t)h(t)$. Then (14) is equivalent to

$$\int_0^T \left(\frac{d^2A^{-1}(t)}{dt^2}h + 2\frac{dA^{-1}(t)}{dt}\frac{dh}{dt} + A^{-1}(t)\frac{d^2h}{dt^2} + h, v\right)dt = 0. \tag{15}$$

Next, let w be a solution of the Cauchy problem:

$$(T-t)\frac{dw}{dt} = e^{c(t-T)}v, \qquad w(0) = 0.$$

Then we obtain

$$\begin{aligned} &\int_0^T \left(\frac{d^2h}{dt^2}, (T-t)e^{(T-t)}A^{-1}(t)\frac{dw}{dt}\right)dt \\ &= -\int_0^T \left(\frac{d^2A^{-1}(t)}{dt^2}h, (T-t)e^{(T-t)}\frac{dw}{dt}\right)dt \\ &\quad - 2\int_0^T \left(\frac{dA^{-1}(t)}{dt}\frac{dh}{dt}, (T-t)e^{(T-t)}\frac{dw}{dt}\right)dt \\ &\quad - \int_0^T \left(h, (T-t)e^{c(T-t)}\frac{dw}{dt}\right)dt. \end{aligned} \tag{16}$$

Now, we integrate the left hand side of (16) by part, set $h = w$, take the double real part and then the resultant equality is

$$(17)\ \int_0^T e^{(T-t)}\left|A^{-\frac{1}{2}}(t)\frac{dw}{dt}\right|^2 dt = -c\operatorname{Re}\int_0^T (T-t)e^{(T-t)}\left|A^{-\frac{1}{2}}(t)\frac{dw}{dt}\right|^2 dt$$
$$- 3\operatorname{Re}\int_0^T (T-t)e^{(T-t)}\left(\frac{dA^{-1}(t)}{dt}\frac{dw}{dt}, \frac{dw}{dt}\right)dt$$
$$- 2\operatorname{Re}\int_0^T (T-t)e^{(T-t)}\left(\frac{d^2A^{-1}(t)}{dt^2}w, \frac{dw}{dt}\right)dt$$
$$- 2\operatorname{Re}\int_0^T (T-t)e^{c(T-t)}\left(w, \frac{dw}{dt}\right)dt.$$

We need the following Lemma:

LEMMA 9. *If* $A^{\frac{1}{2}}(t)\frac{d^2A^{-\frac{1}{2}}(t)}{dt^2} \in L_\infty(I, \mathcal{L}(H))$, *then there exists a constant* c_4, *not depending on* u *and* t *such that:*

$$\left|A^{\frac{1}{2}}(t)\frac{d^2A^{-\frac{1}{2}}(t)}{dt^2}u\right|^2 \leq c_3\left|A^{-\frac{1}{2}}(t)w\right|^2.$$

Proof. It is based on the Heinz inequality (see [7]). We can choose $c_3 = \operatorname{ess\,sup}\left\|A^{\frac{1}{2}}(t)\frac{d^2A^{-\frac{1}{2}}(t)}{dt^2}\right\|^2$.

Using the condition (c), Lemma 9 and that $w(0) = 0$, we get

$$(18)\qquad \int_0^T e^{(T-t)}\left|A^{-\frac{1}{2}}(t)\frac{dw}{dt}\right|^2 dt$$
$$\leq (-c + 3c_2 + 1)\int_0^T (T-t)e^{(T-t)}\left|A^{-\frac{1}{2}}(t)\frac{dw}{dt}\right|^2 dt$$
$$- \int_0^T [(c - c_3)(T-t) + 1]\, e^{(T-t)}|w|^2 dt.$$

One can see that if $c \geq \max(1 + 3c_3, c_4)$, then we have

$$\int_0^T e^{(T-t)}\left|A^{-\frac{1}{2}}(t)\frac{dw}{dt}\right|^2 dt = 0.$$

But this implies that $\frac{dw}{dt} = 0$ and consequently $v = 0$.

Proposition 8 is proved.

Proof of Theorem 7. Let $V = (v, \varphi_1, \psi_1) \in R(L_\mu)^\perp$ and u any function in $D(L_\mu)$ such that $(L_\mu u, V) = 0$. We have to prove that $V = 0$.

Firstly and without lose of generality, we suppose that $u \in D_0(L_\mu)$. Then $(L_\mu u, V) = 0$ implies that $\int_0^T (\mathcal{L}u, v)dt = 0$. From Proposition 8, we conclude that $v = 0$. Next

(19) $(L_\mu u, V) = 0$ becomes $(\ell_{0_\mu} u, \varphi_1)_0 + (\ell_{1_\mu} u, \psi_1) = 0,\ \forall u \in D(L_\mu)$.

Let $u(t) = t(T-t)\chi(t)\ A^{-1}(t)h$, where $\chi(t) = 1$ if $t \in [0, \varepsilon[(\varepsilon > 0)$ and 0 eleswhere. Then (19) gives $(A^{-1}(0)h, \psi_1) = 0$. Taking into account that $D(A(0))$ is dense in H, we conclude that $\psi_1 = 0$.

Finally, if we choose $u(t) = \chi(t)\ A^{-1}(t)h$, we obtain that $(\ell_{0_\mu} u, \varphi_1)_0 = 0$. Then $(A^{-1}(0)h, \varphi_1)_0 = 0$. Since $D(A(0))$ is dense in $W(0)$ we deduce that $\varphi_1 = 0$.

Now we give some examples.

EXAMPLE 1. Let Ω be an open bounded domain in $\mathbb{R}^n$ with C^1-boundary Γ, Σ denotes the open cylinder $\Omega \times I$ and let $H = L_2(\Omega)$.

Consider operators $A(t)$ generated by the differential expression

$$\mathcal{A}(t)u(x,t) = -\Delta_x u(x,t), \quad \forall (x,t) \in \Sigma$$

and the boundary conditions

$$u(x,t)|_{x\in\Gamma} = 0, \text{ and } \frac{\partial u}{\partial \eta}|_{x\in\Gamma} + a(t)u(x,t)|_{x\in\Gamma} = 0,$$

where the function $a \in C^2(I)$ is positive. One can prove that $\{A(t)\}$ satisfies the conditions of Theorem 3.

EXAMPLE 2. We can also choose for $A(t)$ the operators generated by

$$\mathcal{A}(t)u(x,t) = -\sum_{i,j=i}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial u}{\partial x_j}) + b(x,t)u(x,t)$$

$$u(x,t)|_{x\in\Gamma} = 0, \text{ and } \frac{\partial u}{\partial \eta}|_{x\in\Gamma} = 0.$$

As in (2), we set

$$\ell_{0_\mu} u(x,t) = u_0(x), \quad \ell_{1_\mu} u(x,t) = u_1(x),$$

where the functions $u_0(x)$ and $u_1(x)$ are given, $a_{ij} = a_{ji}$ and $b(x,t)$ are positive functions in $C^2(\Sigma)$, satisfying $\sum_{i,j}(a_{ij}\xi_i, \xi_j) \geq a|\xi_i|^2$, where $a = \inf_{x\in\Omega} |a_{ij}(x,t)|$.

For those two operators we can prove that the strong solution exists and is unique.

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