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ON THE EXISTENCE OF GLOBAL SOLUTIONS OF EVOLUTION EQUATIONS

Abstract. In this paper a sufficient condition for the existence of global solutions of evolution equations is proved. In the proof a modification of the Bihari type integral inequality to the case of a weakly singular nonlinear integral inequality is used. An application to a reaction-diffusion problem is given.

1. Introduction

The inequalities of Gronwall–Bihari type play an important role in the study of asymptotic properties of integral and differential equations. The analysis of asymptotic behaviour of ordinary differential equations on infinite-dimensional state spaces, associated with continuous and analytic semigroups of operators, requires modified versions of that type inequalities. Their so-called mild solutions coincide with continuous solutions of integral equations with weakly singular kernels (see [12], [13]). D. Henry [13] proved a modification of the Gronwall lemma covering the case of linear integral inequality with weakly singular kernel (see also [18], [25], [28]. A new approach to an analysis of nonlinear integral inequalities with singular kernels is proposed in the paper [19] and using this method a modification of the the well known Bihari lemma and also a result concerning a modification of the Ou - Iang - Pachpatte inequality (see [26]) are proved. This method has been applied in the paper [21] also to nonlinear integral inequalities for functions in two and n independent variables with singular kernels. A discrete analogue of that type of inequalities suitable for discretizations of parabolic equations is derived in the paper [22]. Using this method some

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results concerning the existence of global solutions and asymptotic stability of solutions of semilinear parabolic equations are proved in the papers [14], [20] and an exponential decay for a semilinear problem with memory is obtained in the paper [29].

In the paper [8] by A. Constantin and S. Peszat a sufficient condition for the existence of global solutions of semilinear evolution equations is proved. The paper [23] also contains a sufficient condition for the existence of such solutions, however with completely different proof from that in [8]. It is proved by using a modification of a result on an integral inequality with singular kernel published in the paper [19]. This problem was solved for special classes of nonlinear equations, e. g. by H. Amann [1]–[3], A. Constantin [7], M. Mizoguchi and E. Yamaniga [24] and A. H. Martin [17]. Sufficient conditions for the boundedness of global solutions of semilinear parabolic equations are proved by M Fila and H. A. Levine in the papers[9], [10]. A necessary and sufficient condition for the global existence of solutions of a scalar integral equations with weakly singular kernel is proved in the paper [6]. This result is used also in the paper [8].

The results contained in the papers [19], [21], [22] are proved under the assumption that the nonlinear function of the state variable appearing in the integral inequality, or difference inequality, respectively, satisfies a condition referred as the condition (q). However this condition is very restrictive. In this paper we define a class of couples of functions fulfilling a condition referred as the condition (r, q), more convenient for applications. We shall prove a sufficient condition for fulfilling such condition. This condition enables us to prove a modified version of [19, Theorem 1], which is a nonlinear version of the Henry inequality from the book [13]. Using this inequality we shall prove a new sufficient condition for the existence of global solutions of evolution equations which is different from those proved in the papers [6] and [23].

2. Couples of functions satisfying a condition (r, q)

DEFINITION 1. Let $r, q > 0, 0 < T \leqq \infty$ and $R^+ = \langle 0, \infty)$. We say that an ordered couple (ω, η) of functions $\omega : R^+ \to R^+$, $\eta : R^+ \to R^+$ satisfies a condition (r, q), if

$$(r,q) \qquad e^{-rt}\omega(u)^q \leqq R(t)\eta(e^{-rt}u^q), \ t \in \langle 0, T), \ u \in R^+,$$

where $R : \langle 0, T) \to R^+$ is a continuous function.

We have defined the condition (q) in [19] for one function ω, which coincides with the inequality (r, q), if $\omega = \eta$ and $r = q$. If $\omega(u) = u^m$, $m \geqq 1$, $q > 0$, then the inequality (r, q) is satisfied with $\omega = \eta$, $R(t) = e^{(m-1)rt}$.

Now we shall prove a sufficient condition for fulfilling the condition (r, q).

PROPOSITION 1. *Let* $r, q > 0$, $0 < T \leqq \infty$, $R : \langle 0, T) \to R^+$ *be a continuously differentiable, positive function,* $\omega : R^+ \to R^+$ *be a continuous, nondecreasing function and* $\eta : R^+ \to R^+$ *be a continuously differentiable, nondecreasing function satisfying the conditions*

$$\omega(u)^q \leqq R(0)\eta(u^q),\ u \in R^+, \tag{1}$$

$$\frac{d\eta(u)}{du}u - \left(1 + \frac{1}{r}\frac{R'(t)}{R(t)}\right)\eta(u) \leqq 0,\ u \in R^+,\ t \in \langle 0, T), \tag{2}$$

where $R'(t) = \frac{dR(t)}{dt}$. *Then the couple* (ω, η) *satisfies the condition* (r, q).

Proof. Let

$$h(t) = e^{rt}R(t)\eta(e^{-rt}u^q) - \omega(u)^q.$$

The condition (1) yields $h(0) \geqq 0$ and

$$h'(t) = -re^{rt}R(t)\left[\frac{d\eta(v)}{dv}v - \left(1 + \frac{1}{r}\frac{R'(t)}{R(t)}\right)\eta(v)\right],$$

where $v = e^{-rt}u^q$. From the condition (2) we obtain that $h'(t) \geqq 0$ for $t \in R^+$ and thus $h(t) \geqq h(0) \geqq 0$ for $t \in R^+$, i. e. the condition (r, q) is satisfied.

PROPOSITION 2. *The couple* (ω, η), *where* $\omega(u) = \sqrt{\ln(\kappa + u)}$ *and* $\eta(u) = \ln(\kappa + \sqrt{u}), \kappa > 1$ *satisfies the condition* (r, q) *with* $r = q = 2$ *and* $R(t) \equiv 1$.

Proof. Since $\omega(u)^2 = \eta(u^2), u \geqq 0$ the condition (1) is satisfied with $r = q = 2$, $R(t) \equiv 1$ and

$$\eta(u) - \frac{d\eta(u)}{du}u = \ln(\kappa + \sqrt{u}) - \frac{u}{2(\kappa + \sqrt{u})\sqrt{u}} = \ln(\kappa + \sqrt{u}) - \frac{\sqrt{u}}{2(\kappa + \sqrt{u})}.$$

Let $H(w) = \ln(\kappa + w) - \frac{w}{2(\kappa + w)}$, $w \in R^+$. Obviously $H(0) > 0$ and

$$\frac{dH(w)}{dw} = \frac{1}{\kappa + w} - \frac{\kappa}{2(\kappa + w)^2} = \frac{\kappa + 2w}{2(\kappa + w)^2} > 0,\ w \in R^+.$$

This yields $H(w) \geqq H(0) > 0$ for all $w \in R^+$ and thus $\frac{d\eta(u)}{du}u - \eta(u) < 0$ for all $u \in R^+$. Therefore the assertion of the proposition follows from Proposition 1.

PROPOSITION 3. *Let* $q > 1$ *and* $\kappa = e^{q-1}$. *Then the couple* (ω, η), *where* $\omega(u) = \ln(\kappa + u^q)$ *and* $\eta(u) = [\ln(\kappa + u)]^q$, *satisfies the condition* (r, q) *with* $r = q$ *and* $R(t) \equiv 1$.

Proof. Since $\omega(u)^q = \eta(u^q)$, the condition (1) is satisfied with $r = q$, $R(t) \equiv 1$ and

$$\frac{d\eta(u)}{du}u-\eta(u)=\big[\ln(\kappa+u)\big]^{q-1}\big[\frac{qu}{\kappa+u}-\ln(\kappa+u)\big]=\big[\ln(\kappa+u)\big]^{q-1}\frac{\psi(u)}{\kappa+u},$$

where $\psi(u) = qu - (\kappa + u)\ln(\kappa + u)$. Since $\ln\kappa = q - 1$ we have $\psi(0) = -\kappa\ln\kappa < 0$ and $\frac{d\psi(u)}{du} = q - 1 - \ln(\kappa + u) < 0$ for all $u \geqq 0$. This yields $\psi(u) < 0$ for all $u \geqq 0$ and thus $\frac{d\eta(u)}{du}u - \eta(u) < 0$ for all $u \in R^+$. By Proposition 1 the couple (ω, η) satisfies the condition (r, q) with $r = q$ and $R(t) \equiv 1$.

PROPOSITION 4. *The couple* (ω, η), *where* $\omega(u) = \big[\ln(\kappa+u^q]^{\frac{1}{q}}$, $\kappa > 1$, $q > 1$, $\eta(u) = \ln(\kappa + u)$ *satisfies the condition* (r, q) *with* $r = q$ *and* $R(t) \equiv 1$.

Proof. Since $\omega(u)^q = \eta(u^q)$, the condition (1) is satisfied with $r = q$ and $R(t) \equiv 1$ and

$$\frac{d\eta(u)}{du}u-\eta(u)=\frac{a(u)}{\kappa+u},\ \text{where } a(u)=u-(\kappa+u)\ln(\kappa+u).$$

Obviously $a(0) = -\kappa\ln\kappa$, $\frac{da(u)}{du} = -\ln(\kappa + u) < 0$ for all $u \geqq 0$. This yields $a(u) < 0$ for all $u \geqq 0$ and thus we have proved that $\frac{d\eta(u)}{du}u - \eta(u) < 0$ for all $u \geqq 0$. The assertion of the proposition follows from Proposition 1.

3. Integral inequalities

We shall reformulate and improve the main results from the paper [19].

THEOREM 1. *Let* $0 < T \leqq \infty, a : \langle 0, T) \to R^+$ *be a nondecreasing* C^1-*function,* $F : \langle 0, T) \to R^+$ *be a continuous function,* $\omega, \eta : R^+ \to R^+$ *be continuous, nondecreasing functions,* $\eta(u) > 0$ *for* $u > 0$. *Let* $u : \langle 0, T) \to R^+$ *be a continuous function satisfying the inequality*

$$u(t)\leqq a(t)+\int_0^t(t-s)^{\beta-1}F(s)\omega(u(s))ds,\ t\in\langle 0,T), \tag{3}$$

where $0 < \beta < 1$ *and let* ϵ *be a positive number. Then the following assertions hold:*

(i) *If* $\beta > \frac{1}{2}$ *and the couple* (ω, η) *satisfies the condition* (r, q) *with* $r = 2\epsilon$ *and* $q = 2$, *then*

$$\Omega([u(t)e^{-\epsilon t}]^2)\leqq\Omega(2a(t)^2)+g_1(t,\epsilon),\ t\in\langle 0,T), \tag{4}$$

or

$$u(t)\leqq e^{\epsilon t}\{\Omega^{-1}[\Omega(2a(t)^2)+g_1(t,\epsilon))]\}^{\frac{1}{2}},\ t\in\langle 0,T_1\rangle,$$

where

$$g_1(t,\epsilon)=\frac{\Gamma(2\beta-1)}{4^{\beta-1}\epsilon^{2\beta-1}}\int_0^t R(s)F(s)^2ds,$$

Γ is the Eulerian Gamma function, $\Omega(v) = \int_{v_0}^{v} \frac{dy}{\eta(y)}$, $v \geqq v_0 > 0, \Omega^{-1}$ is the inverse of Ω and $T_1 > 0$ is such that $\Omega(2a(t)^2) + g_1(t) \in Dom(\Omega^{-1})$ for all $t \in \langle 0, T_1\rangle$.

(ii) *Let $\beta = \frac{1}{1+z}$, $z \geqq 1$ and the couple (ω, η) satisfies the condition (r, q) with $q = 1 + z + \delta$, $r = q\epsilon$, where δ is a positive number. Then*

$$\Omega([u(t)e^{-\epsilon t}]^q) \leqq \Omega(2^{q-1}a(t)^q) + g_2(t,\epsilon),\ t \in \langle 0, T) \tag{5}$$

or

$$u(t) \leqq e^{\epsilon t}\{\Omega^{-1}[\Omega(2^{q-1}a(t)^q) + g_2(t,\epsilon)]\}^{\frac{1}{q}},\ t \in \langle 0, T_2\rangle,$$

where

$$g_2(t,\epsilon) = \frac{2^{q-1}\Gamma(1-\alpha p)}{(p\epsilon)^{1-\alpha p}}\int_0^t R(s)F(s)^q ds,\ \alpha = 1-\beta,\ p = \frac{1+z+\delta}{z+\delta}$$

and $T_2 > 0$ is such that $\Omega(2^{q-1}a(t)^q) + g_2(t) \in Dom(\Omega^{-1})$ for all $t \in \langle 0, T_2\rangle$.

Proof. We shall prove the assertion (i) using the method of desingularization presented in [19]. Applying the Cauchy - Schwarz inequality we obtain from (3)

$$\begin{aligned} u(t) &\leqq a(t) + \int_0^t (t-s)^{\beta-1}e^{\epsilon s}F(s)e^{-\epsilon s}\omega(u(s))ds \leqq \\ &\leqq a(t) + \left[\int_0^t (t-s)^{2\beta-2}e^{2\epsilon s}ds\right]^{\frac{1}{2}}\left[\int_0^t F(s)^2e^{-2\epsilon s}\omega(u(s))^2ds\right]^{\frac{1}{2}}. \end{aligned} \tag{6}$$

For the first integral in (6) we have an estimate

$$\begin{aligned} \int_0^t (t-s)^{2\beta-2}e^{2\epsilon s}ds &= \int_0^t \tau^{2\beta-2}e^{2\epsilon(t-\tau)}d\tau = e^{2\epsilon t}\int_0^t \tau^{2\beta-2}e^{-2\epsilon\tau}d\tau = \\ &= \frac{1}{(2\epsilon)^{2\beta-1}}e^{2\epsilon t}\int_0^{2\epsilon t}\sigma^{2\beta-2}e^{-\sigma}d\sigma < e^{2\epsilon t}\frac{2}{4^\beta\epsilon^{2\beta-1}}\Gamma(2\beta-1). \end{aligned}$$

Using this estimate, the condition (r, q) with $r = 2\epsilon$, $q = 2$ and the inequality $(A+B)^2 \leqq 2(A^2+B^2)$, we obtain

$$v(t) \leqq \alpha(t) + K(\epsilon)\int_0^t F(s)^2R(s)\eta(v(s))ds,\ t \in \langle 0, T), \tag{7}$$

where

$$v(t) = (e^{-\epsilon t}u(t))^2,\ \alpha(t) = 2a(t)^2,\ K(\epsilon) = \frac{1}{4^{\beta-1}\epsilon^{2\beta-1}}\Gamma(2\beta-1).$$

Then applying the Bihari lemma (see e. g. [4, 5, 11, 15, 16]) we obtain the inequality (5).

Now let us prove the assertion (ii). If $p = \frac{1+z+\delta}{z+\delta}$ then $\frac{1}{p} + \frac{1}{q} = 1$ and using the Hölder inequality we obtain

$$(8)\qquad u(t) \leqq a(t) + \left[\int_0^t (t-s)^{-\alpha ps} e^{p\epsilon s} ds\right]^{\frac{1}{p}} \left[\int_0^t F(s)^q e^{-q\epsilon s} \omega(u(s))^q ds\right]^{\frac{1}{q}},$$

where $\alpha = 1 - \beta$. For the first integral in (8) we have the estimate

$$\int_0^t (t-s)^{-\alpha p} e^{p\epsilon s} ds = e^{p\epsilon t} \int_0^{p\epsilon t} \tau^{-\alpha p} e^{-p\epsilon\tau} d\tau < \frac{e^{p\epsilon t}}{(p\epsilon)^{1-\alpha p}} \Gamma(1-\alpha p).$$

Obviously, $1 - \alpha p = \frac{\delta}{(1+z)(z+\delta)} > 0$ and so $\Gamma(1-\alpha p) < \infty$. Using the condition (r, q) with $q = (1+z+\delta), r = q\epsilon$ and the inequality $(A+B)^q \leqq 2^{q-1}(A^q + B^q),\ A, B \geqq 0$ we obtain

$$v(t) \leqq \Phi(t) + L(\epsilon) \int_0^t F(s)^q R(s) \eta(v(s)) ds,$$

where

$$v(t) = (e^{-\epsilon t} u(t))^q,\ \Phi(t) = 2^{q-1} a(t)^q,\ L(\epsilon) = \frac{2^{q-1}\Gamma(1-\alpha p)}{(p\epsilon)^{1-\alpha p}}.$$

Then applying the Bihari lemma to this inequality we obtain the inequality (5).

As a consequence of Theorem 1 we have

THEOREM 2. *Let* $0 < T \leqq \infty$, $a(t)$, $F(t)$ *be as in Theorem* 1 *and* $u : \langle 0, T) \to R^+$ *be a continuous, nonnegative function satisfying the inequality*

$$u(t) \leqq a(t) + \int_0^t (t-s)^{\beta-1} F(s) u(s) ds,\ \ \langle 0, T),$$

where $0 < \beta < 1$. *Then the following assertions hold:*

(i) *If* $\beta > \frac{1}{2}$ *and* ϵ *is an arbitrary positive number, then*

$$u(t) \leqq \sqrt{2} a(t) \exp\left(\epsilon t + \frac{2\Gamma(2\beta-1)}{4^\beta} \int_0^t F(s)^2 ds\right),\ t \in \langle 0, T).$$

(ii) *If* $\beta = \frac{1}{1+z}, z \geqq 1$ *and* δ, ϵ *are arbitrary positive number, then*

$$u(t) \leqq 2^{\frac{q-1}{q}} a(t) \exp\left(\epsilon t + \frac{2^{q-1}\Gamma(1-\alpha p)}{q(p\epsilon)^{1-\alpha p}} \int_0^t F(s)^q ds\right),\ t \in \langle 0, T),$$

where $\alpha = 1 - \beta$, $p = \frac{1+z+\delta}{z+\delta}$, $q = (1+z+\delta)$.

REMARK. We need no other restrictions on ϵ, δ in Theorem 2, because the linear function $\omega(u) = u$ satisfies the condition (r, q) with any $r, q > 0$ and with $R(t) \equiv 1, \omega = \eta$. We also remark that all results presented in the papers [19], [21], [22] can be reformulated and improve in the style of Theorem 1.

4. Global solutions of semilinear evolution equations

In the papers [8] and [23] sufficient conditions for the existence of global solutions solutions of the evolution equation

$$\dot{x} + Ax = H(t, x), \; x(0) = x_0 \in E \tag{9}$$

are proved. It is assumed there that $-A$ is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t \geqq 0}$ on a Banach space V, $S(t) \in L(V, E), t > 0, E$ is a Banach space densely and continuously embedded into V with

$$||S(t)||_{L(V,E)} \leqq ct^{-\alpha}, \; t > 0, \tag{10}$$

where $c > 0, \alpha \in (0, 1)$ are constants, $H : R^+ \times E \to V$, is a continuous map (see [27]).

By the mild solution of the problem (9) on the interval $\langle 0, T)$ $(0 < T < \infty)$ we mean a map $x \in C(\langle 0, T), E)$ satisfying the integral equation

$$x(t) = S(t)x_0 + \int_0^t S(t-s)H(s, x(s))ds, \; 0 \leqq t < T. \tag{11}$$

We say that $x \in C(\langle 0, \infty), E)$ is a global solution of the problem (9) if it is the mild solution of (9) on any finite interval $\langle 0, T)$.

THEOREM 3. *Let $T > 0, H : R^+ \times E \to V$ be a continuous map satisfying the condition*

$$||H(t, v)||_V \leqq G(t)\omega(||v||_E), \; (t, v) \in R^+ \times E, \tag{12}$$

where $\omega : R^+ \to R_+$ is a continuous, nondecreasing function and $G : R^+ \to R^+$ is a continuous function. Let $\{S(t)\}_{t \geqq 0}$ be a C_0-semigroup satisfying the condition (10). Assume that $\eta : R^+ \to R^+$ is a continuous, nondecreasing function with $\int_0^\infty \frac{d\sigma}{\eta(\sigma)} = \infty$ and such that one of the following conditions is satisfied:

(a) $\beta > \frac{1}{2}$ *and the couple (ω, η) satisfies the condition (r, q) with $q = 2, r = 2\epsilon$, where $\epsilon > 0$.*

(b) $\beta = \frac{1}{1+z}$, *where $z \geqq 1$ and the couple (ω, η) satisfies the condition (r, q) with $q = 1 + z + \delta, r = q\epsilon$, where $\delta > 0, \epsilon > 0$.*

Then $\sup_{t \in \langle 0,T)} ||x(t)||_E < \infty$ for any mild solution $x(t)$ of the problem (9) defined on the interval $\langle 0, T)$.

Proof. Let $x(t)$ be a mild solution of the problem (9) defined on the interval $\langle 0, T)$ with $\lim_{t \to T^-} ||x(t)||_E = \infty$. From the conditions (10), (12) and the equation (11) we have

$$(13)\qquad ||x(t)||_E \leqq ||S(t)x_0||_E + \int_0^t c(t-s)^{\beta-1}G(s)\omega(||x(s)||_E)ds,$$

where $\beta = 1-\alpha$. Applying Theorem 1 with

$$u(t) = ||x(t)||_E,\ a(t) \equiv a := max_{t\in\langle 0,T\rangle}||S(t)x_0||_E,\ F(t) = cG(t)$$

we obtain that the assertion (i) with $\epsilon = \frac{r}{2}$, if $\beta > \frac{1}{2}$ and the assertion (ii) with $\epsilon = \frac{r}{q}$, if $0 < \beta \leqq \frac{1}{2}$, of this theorem. If $\beta > \frac{1}{2}$, then by (i)

$$(14)\qquad \Omega([e^{-\epsilon t}u(t)]^2) \leqq \Omega(2a^2) + g_1(t,\epsilon),$$

where

$$g_1(t,\epsilon) = \frac{\Gamma(2\beta-1)}{4^{\beta-1}\epsilon^{2\beta-1}}\int_0^t R(s)F(s)^2 ds.$$

Since $\lim_{t\to T^-}[\Omega(2a^2) + g_1(t,\epsilon)] < \infty$ we obtain from the inequality (14) that $\lim_{t\to T^-} \Omega([e^{-\epsilon t}u(t)]^2) < \infty$. However

$$\lim_{t\to T^-} \Omega([e^{-\epsilon t}u(t)]^2) = \lim_{t\to T^-} \int_{||x_0||}^{[e^{-\epsilon t}u(t)]^2} \frac{d\sigma}{\eta(\sigma)} = \int_{||x_0||}^{\infty} \frac{d\sigma}{\eta(\sigma)} = \infty.$$

This contradiction can also be obtained in the case $0 < \beta \leqq \frac{1}{2}$ by using the assertion (ii).

THEOREM 4. *Let $T > 0$ and $H : R^+ \times E \to V$ be a continuous map satisfying the condition (12) with $\omega(u) = \ln(\kappa + \sqrt{u})$, $\kappa > 1$, i.e.*

$$||H(t,v)||_V \leqq G(t)\ln(\kappa + \sqrt{||v||_E}),\ (t,v) \in R^+ \times E,$$

where $G : R^+ \to R^+$ is a continuous function and $\{S(t)\}_{t\geqq 0}$ is a C_0-semigroup satisfying the condition (10) defined on the interval $\langle 0,T)$. Then

$$\sup_{t\in\langle 0,T)} ||x(t)||_E < \infty$$

for any mild solution $x(t)$ of the problem (9) defined on the interval $\langle 0,T)$.

Proof. By Proposition 2 the couple (ω,η) satisfies the condition (r,q) with $r = q = 2$ and $R(t) \equiv 1$. Since $\kappa > 1$, obviously

$$\int_0^\infty \frac{d\sigma}{\eta(\sigma)} > \int_0^\infty \frac{d\sigma}{\kappa+\sigma} = \int_{\ln\kappa}^\infty \frac{e^\tau}{\tau}d\tau = \infty$$

and the assertion of the theorem follows from Theorem 3.

THEOREM 5. *Let $T > 0, q > 1$ and $H : R^+ \times E \to V$ be a continuous map satisfying the condition* (12) *with $\omega(u) = \ln(\kappa + u^q), \kappa = e^{q-1}$, i. e.*

$$||H(t,v)||_V \leqq G(t)\ln(\kappa + ||v||_E^q),\ (t,v) \in R^+ \times E,$$

where $G : R^+ \to R^+$ is a continuous function and $\{S(t)\}_{t\geqq 0}$ be a C_0-semigroup satisfying the condition (10). *Then*

$$\sup_{t\in\langle 0,T)} ||x(t)||_E < \infty$$

for any mild solution $x(t)$ of the problem (9) *defined on the interval $\langle 0,T)$.*

P r o o f. By Proposition 3 the couple (ω, η), where $\eta(u) = [\ln(\kappa+u)]^q$ satisfies the condition (r,q) with $r = q$ and $R(t) \equiv 1$. Obviously

$$I := \int_0^\infty \frac{d\sigma}{\eta(\sigma)} = \int_0^\infty \frac{d\sigma}{[\ln(\kappa+\sigma)]^q} = \int_{\ln\kappa}^\infty \frac{e^\tau}{\tau^q} d\tau.$$

If $m(t) = \frac{e^t}{t^q}$, then $\frac{dm(t)}{dt} = \frac{e^t t^{q-1}}{t^{2q}}[t-q] > 0$ for all $t > q$ and this yields $m(t) > \frac{e^q}{q^q}$ for all $t > q = \ln\kappa + 1 > \ln\kappa$. Thus we obtain that

$$I > \int_q^\infty \frac{e^\tau}{\tau^q} d\tau > \int_q^\infty \frac{e^q}{q^q} d\tau = \infty.$$

We have proved that all assumptions of Theorem 3 are satisfied and so the assertion of the theorem follows from this result.

THEOREM 6. *Let $T > 0$ and $H : R^+ \times E \to V$ be a continuous map satisfying the condition* (12) *with $\omega(u) = \left[\ln(\kappa + u^q)\right]^{\frac{1}{q}},\ \kappa > 1, q > 1.$, i. e.*

$$||H(t,v)||_V \leqq G(t)\left[\ln(\kappa + ||v||_E^q)\right]^{\frac{1}{q}},\ (t,v) \in R \times E,$$

where $G : R^+ \to R^+$ is a continuous function and $\{S(t)\}_{t\geqq 0}$ is a C_0-semigroup satisfying the condition (10). Then

$$\sup_{t\in\langle 0,T)} ||x(t)|| < \infty$$

for any mild solution $x(t)$ of the problem (9) *defined on the interval $\langle 0,T)$.*

P r o o f. By Proposition 4 the couple (ω,η), where $\eta(u) = \ln(\kappa + u)$, $\kappa > 1$, satisfies the condition (r,q), with $r = q$ and $R(t) \equiv 1$. Since the function $\eta(u)$ is the same as in Theorem 4 we have $\int_0^\infty \frac{d\sigma}{\eta(\sigma)} = \infty$ and thus the assertion of theorem follows from Theorem 3.

Obviously

$$\int_0^\infty \frac{d\sigma}{\eta(\sigma)} = \int_0^\infty \frac{d\sigma}{\ln(\kappa+\sigma)} = \int_{\ln\kappa}^\infty \frac{e^\tau}{\tau} d\tau > \int_{\ln\kappa}^\infty d\tau = \infty$$

and the assertion of theorem follows from Theorem 3.

5. Applications to reaction–diffusion problems

In this section we apply Theorems 3–6 to the reaction diffusion problem considered in the paper [8] as an example 1. The reaction - diffusion problems studied in [8] as examples 2 - 4 can also be solved analogously by using these theorems and we do not formulate the corresponding results.

Consider the perturbed heat equation

$$\partial_t u = \Delta u + f(t, Du), \ u|\partial\Omega = 0, \tag{15}$$

where $\Omega \subset R^d$ is a bounded domain with C^∞-boundary, $f : R^+ \times R^d \to R$ is continuous, $f(t,0) \equiv 0$ and D is the gradient operator. In [8] the same problem is studied, however the function f is independent of t. One can rewrite (15) in the form (9), where $E = \{u \in C^1(\bar\Omega) : u = 0, Du = 0 \,\text{on}\, \partial\Omega\}$, $V = \{u \in C(\bar\Omega) : u = 0, \,\text{on}\, \partial\Omega\}$ and $H : R^+ \times E \to V, (t,v) \mapsto f(t, Dv(.))$. The map H is obviously continuous and the Laplace operator Δ with the Dirichlet boundary condition is the generator of the compact, C_0-semigroup

$$S(t)\phi(x) = \int_\Omega G(t,x,y)\phi(y)dy,$$

where G is the corresponding Green function such that

$$\left|\frac{\partial}{\partial x_j} G(t,x,y)\right| \leqq K_1 t^{-\frac{1}{2}} \mathcal{H}(K_2, |x-y|), \ j = 1,2,\ldots,d, \ t > 0, \tag{16}$$

$\mathcal{H}(t,z) = (2\pi t)^{-\frac{d}{2}} \exp\{-\frac{|z|^2}{2t}\}$, $z \in R^d, K_1, K_2 > 0$. This yields $S(t) \in L(V,E)$ and it satisfies the condition (10) with $\alpha = \frac{1}{2}$ (see [8]).

THEOREM 7. *Let $q = 2+\delta, \delta > 0, r = q, f$ being as in* (15),

$$|f(t,s)| \leqq G(t) f_1(s), \ t \in R^+, s \in R^d,$$

where $G : R^+ \to R^+$, $f_1 : R^d \to R^+$ are continuous functions, $F : R^+ \to R^+, F(u) = 1 + \sup_{|s|\leqq u} f_1(s) \leqq \omega(u)$ for all $u \in R^+$. Assume that there are continuous functions $\omega, \eta : R^+ \to R^+$ such that the couple (ω,η) satisfies the condition (r,q) and $\int_0^\infty \frac{d\sigma}{\eta(\sigma)} = \infty$. Then

$$\sup_{t\in\langle 0,T)} ||x(t)||_E < \infty$$

for any $T > 0$ and any mild solution $x(t)$ of the problem (15).

Proof. Since the condition (10) is satisfied with $\alpha = \frac{1}{2}$, we have $\beta = 1-\alpha = \frac{1}{2}$, i.e. β in Theorem 3 is equal to $\frac{1}{1+z}$ with $z = 1$ and thus $q = 1+z+\delta = 2+\delta$. Obviously

$$||H(t,v)||_V \leqq \sup_{x\in\bar\Omega} |f(t, Dv(x))| \leqq G(t)F(||v||_E) \leqq G(t)\omega(||v||_E), \ v \in E,$$

i.e. the condition (12) is satisfied. Since also all other assumptions of Theorem 3 are satisfied, the proof is finished.

As a consequence of Theorems 4–7 we obtain

THEOREM 8. *Let the assumptions of Theorem 7 be satisfied with* $\omega(u) = \ln(\kappa + \sqrt{u})$, $\kappa > 1$, *or* $\omega(u) = \ln(\kappa + u^q)$, $\kappa = e^{q-1}, q > 1$ *or* $\omega(u) = \left[\ln(\kappa + u^q)\right]^{\frac{1}{q}}$, $\kappa > 1, q > 1$. *Then*

$$\sup_{t\in\langle 0,T)} ||x(t)||_E < \infty$$

for any $T > 0$ *and any mild solution* $x(t)$ *of the problem* (15).

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