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ON HEROD'S QUADRATIC DIFFERENTIAL SYSTEM WITH INFINITELY MANY UNKNOWNS

1. Purpose of this paper

With x_k continuous functions $\mathbb{R} \to \mathbb{R}$ and $\dot{x}_k = dx_k/dt$, the following system was introduced about fifteen years ago by James Herod [1]:

$$\dot{x}_n + x_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k x_{n-k}, \qquad n = 0, 1, 2, \ldots, \tag{1a}$$

$$x_0(0) = 1, \quad x_1(0) = 0, \quad x_k(0) = c_k, \quad (k \geq 2). \tag{1b}$$

The system (1) was formulated in the course of a comprehensive investigation of a class of differential systems related to the Helmholtz equation. It has a number of very interesting properties and seems well worth investigating for its own sake. Added to this is the fact that a question connected with it was mentioned in [1] as an open problem and, so far as I know, has been open ever since. Namely, if the sequence $\{c_k\}$ is in ℓ^2, is the same true of the sequence $\{x_n(t)\}$ for each $t \geq 0$?

One purpose of this paper is to settle this question in the negative. Another purpose is to study further properties of (1) and, in particular, to discover under what additional restrictions the ℓ^2 result is true. For example Theorem 7 below shows that it is true if

$$|c_k| \leq \frac{1}{(\log k)^a}, \qquad k \geq 2,$$

where $a > 1/2$ is constant. In a context involving $\{c_k\} \in \ell^2$ it seems surprising that such a weak condition could make any difference. Nevertheless the conclusion fails if we assume only

$$|c_k| \leq \frac{C}{(\log k)^a}, \qquad k \geq 2,$$

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for a moderately large constant C. Here a can be as large as we please and the conclusion still fails.

In the course of the work we also study an equation, implicitly contained in [1], that is less restrictive than (1) but follows from it.

Although the paper is sharply focused on these two equations, an additional objective is to illustrate a variety of methods that may apply to other differential systems of similar quadratic structure. With this thought in mind we have not hesitated to supply different proofs of related results.

2. Preliminary remarks

Equation (1) admits a reformulation that is more convenient for the purposes at hand. The initial conditions give in succession

$$x_0 = 1, \quad x_1 = 0, \quad x_2 = c_2 e^{-t/3}, \quad x_3 = c_3 e^{-t/2}.$$

Since $x_0 = 1$ and $x_1 = 0$, Equation (1) reduces to

$$\dot{x}_n + a_n x_n = \frac{1}{n+1} \sum_{k=2}^{n-2} x_k x_{n-k}, \qquad n \geq 4, \tag{2a}$$

where, as throughout this paper,

$$a_k = \frac{k-1}{k+1}, \qquad k \geq 1. \tag{2b}$$

All empty sums are interpreted as 0. With this convention Equation (2a) remains valid for $1 \leq n < 4$. Since the equation gives x_n in terms of x_k with $k \leq n-2$, existence of the solution for $0 \leq t < \infty$ follows by induction.

Our results are hardest to prove when x_n is as large as possible, and this condition is also desirable for the construction of counterexamples. Hence we assume all $c_k \geq 0$.

For any vector $v = (v_1, v_2, v_3, \cdots)$ we set

$$(\|v\|_n)^2 = \sum_{k=2}^{n} (v_k)^2, \quad \|v\|^2 = \sum_{k=2}^{\infty} (v_k)^2,$$

taking the nonnegative square root, both here and below. For vectors associated with sequences such as

$$\{c_i\}, \{m_i\}, \{x_i\}, \{x_i(t)\}, \{u_i\}, \{y_k\}, \{y_k(t)\},$$

we use corresponding letters c, m, x, $x(t)$, u, y, $y(t)$.

We now summarize a few elementary results that are needed later. With a, c constant and $p = p(t)$ continuous, the solution of

$$\dot{y} + ay = p(t), \quad y(0) = c \tag{3a}$$

is given by

$$\text{(3b)} \qquad y(t) = e^{-at}\int_0^t e^{as}p(s)\,ds + ce^{-at}.$$

This expression shows that if both equal signs in (3a) are replaced by $\leq$ or $\geq$, then (3b) remains valid with the same replacement of its equal sign. More general results of the same kind are well known from the theory of differential inequalities, but the proof in this case is so simple that it seemed best to include it.

If a, b are constant with $a \geq 0$ then

$$\text{(4)} \qquad y' + ay \leq ab, \quad y(0) = c \Rightarrow y \leq \max(b,\, c).$$

As another result of this kind, suppose

$$\text{(5a)} \qquad y' + ay \leq pe^{-(a+h)t}, \quad y(0) = c$$

where a, p, h are positive constants and $c \geq 0$. Then

$$\text{(5b)} \qquad y < e^{-at}\left(\frac{p}{h} + c\right).$$

Both (4) and (5) follow from (3) and the accompanying remarks.

We will use (4) to show that all solutions of (2) are ≥ 0 and bounded. This holds for x_2 and x_3 by inspection. Suppose it holds for x_k with $k \leq n-2$, thus

$$0 \leq x_k \leq m_k, \qquad 2 \leq k \leq n-2,$$

where $m_k = \sup x_k(t) < \infty$. The differential equation gives $x_n \geq 0$ and (4) gives

$$\text{(6)} \qquad m_n \leq \max\left(c_n,\, \frac{1}{n-1}\sum_{k=2}^{n-2} m_k m_{n-k}\right), \qquad n \geq 4.$$

The proof is completed by induction.

As noted above, global existence of the solution follows trivially without any need for a local Lipschitz condition, although such a condition is easily estsblished for finite segments

$$x_2,\, x_3,\, \cdots,\, x_n, \quad y_2,\, y_3, \cdots,\, y_n$$

of solutions on a finite interval. But the Referee stated a global Lipschitz condition which applies to the full solutions

$$x = (x_0,\, x_1,\, x_2, \cdots), \quad y = (y_0,\, y_1,\, y_2, \cdots)$$

whenever $\|x\|$ and $\|y\|$ are bounded. His result is not used here but is reproduced because of its collateral interest.

In this paragraph only, we return to the original equation (1) and we change the definition of $\|v\|$ to

$$\|v\|^2 = \sum_{k=0}^{\infty} v_k{}^2.$$

Let $f(x)$ be defined by $f(x) = (u_0,\ u_1,\ u_2, \cdots)$ where

$$u_n = -x_n + \frac{1}{n+1} \sum_{k=0}^{n} x_k x_{n-k}.$$

Then the Referee's result (included by permission) is

$$\|f(x) - f(y)\| \le \frac{\pi}{\sqrt{6}} \|x+y\|\,\|x-y\| + \|x-y\|.$$

Concluding these preliminary remarks, we determine the exact values of m_4 and m_5. These values are not needed for the subsequent analysis, but they shed light on the nature of the difficulties involved.

With constants $c \ge 0$, $d \ge 0$, $a > 0$, $h > 0$ suppose

$$\dot{y} + ay = hde^{-(a+h)t}, \qquad y(0) = c.$$

Then $\max y(t) = c$ if $dh \le ac$ and

$$\max y(t) = h\left(\frac{a}{d}\right)^{a/h} \left(\frac{c+d}{a+h}\right)^{(1+a/h)}$$

if $dh \ge ac$. In this case the location of the max is at t^* where

$$e^{ht^*} = \frac{d(a+h)}{a(c+d)}.$$

(The proof is not difficult and is omitted.) Since

$$\dot{x}_4 + \frac{3}{5} x_4 = \frac{1}{5} c_2{}^2 e^{-(2/3)t}, \qquad x_4(0) = c_4,$$

the above result gives $m_4 = c_4$ if $3c_4 \ge c_2{}^2$ and otherwise

$$m_4 = \left(\frac{3}{c_2{}^2}\right)^9 \left(\frac{3c_2{}^2 + c_4}{10}\right)^{10}, \quad e^{t^*} = \left(\frac{10c_2{}^2}{9c_2{}^2 + 3c_4}\right)^{15}.$$

Similarly from

$$\dot{x}_5 + \frac{2}{3} x_5 = \frac{1}{3} c_2 c_3 e^{-(5/6)t}, \qquad x_5(0) = c_5$$

follows $m_5 = c_5$ if $2c_5 \ge c_2 c_3$ and otherwise

$$m_5 = \left(\frac{2}{c_2 c_3}\right)^4 \left(\frac{c_5 + 2c_2 c_3}{5}\right)^5, \quad e^{t^*} = \left(\frac{5c_2 c_3}{2c_5 + 4c_2 c_3}\right)^6.$$

A similar attempt to evaluate m_6 leads to hopeless complication. However these results are already enough to show that m_n is an elaborate function of the c_i and that the location of the maximum of $x_n(t)$ can depend in the wildest way on n.

3. Bounds for the solution

Our first two theorems require the following lemma:

LEMMA 1. *For $n \geq 4$ and $2 \leq k \leq n-2$ we have $a_k + a_{n-k} \geq a_n + 1/15$, that is,*

$$\frac{k-1}{k+1} + \frac{n-k-1}{n-k+1} - \frac{n-1}{n+1} \geq \frac{1}{15}. \tag{7}$$

Proof. We show that the minimum of the left side is attained when $k = 2$ or $n-2$ and in that case the inequality holds. The derivative with respect to k is

$$\frac{2}{(k+1)^2} - \frac{2}{(n-k+1)^2}$$

which is positive for $k < n/2$ and negative for $k > n/2$. This gives the first assertion. When $k = 2$ the left side of (7) is

$$\frac{1}{3} - \frac{4}{n^2-1}.$$

This is least when $n = 4$ and then has the value 1/15.

THEOREM 1. *With x as in (2) there exist constants d_n such that*

$$c_n e^{-a_n t} \leq x_n(t) \leq d_n e^{-a_n t}, \qquad t \geq 0, \quad n \geq 2.$$

We can choose $d_n = 1/30$ independent of n if $\sup c_k \leq 1/60$.

The left-hand inequality follows from $\dot{x}_n + a_n x_n \geq 0$, so we consider only the right-hand inequality.

Proof. The desired result holds for x_2 and x_3 with $d_2 = c_2$ and $d_3 = c_3$. If it holds for x_k with $k \leq n-2$, then

$$\dot{x}_n + a_n x_n \leq \frac{1}{n+1} \sum_{k=2}^{n-2} d_k e^{-a_k t} d_{n-k} e^{-a_{n-k} t}, \qquad n \geq 4.$$

By Lemma 1 we can replace $a_k + a_{n-k}$ by $a_n + h$ where $h = 1/15$. Thus

$$\dot{x}_n + a_n x_n \leq p_n e^{-(a_n+h)t}$$

where

$$p_n = \frac{1}{n+1} \sum_{k=2}^{n-2} d_k d_{n-k}.$$

Equations (5) with $h = 1/15$ yield

$$x_n \le (15p_n + c_n)e^{-a_n t}.$$

This gives $x_n \le d_n e^{-a_n t}$ if $d_n = 15p_n + c_n$. Thus the d_n can be determined by recursion, giving the first statement in Theorem 1.

To get $d_n = d$, a constant, let $c_k \le \bar{c}$ for all k. Then it suffices to have

$$d \ge 15\frac{n-3}{n+1}d^2 + \bar{c} \quad \text{or also} \quad d \ge 15d^2 + \bar{c}.$$

This holds when $d = 1/30$ and $\bar{c} \le 1/60$, giving the second statement.

Theorem 1 shows that $\lim x_n(t) = 0$ as $t \to \infty$ and hence $m_n = \max x_n(t)$ is attained for each $n \ge 2$. If the maximum exceeds c_n, it is attained at some point $t^* > 0$. At this point $\dot{x}_n(t^*) = 0$ and the differential equation gives

$$\max_t x_n(t) \le \max\left(c_n,\ \max_t \frac{1}{n-1}\sum_{k=2}^{n-2} x_k(t)x_{n-k}(t)\right).$$

The same result follows from (4).

Theorem 1 and its proof suggest the following:

THEOREM 2. *With x as in (2), let $0 < \|c\| < \infty$. Then*

$$\|x(t)\| \le 2\|c\| \quad \text{for} \quad 0 \le t \le \frac{0.3978}{\|c\|}.$$

It is rather remarkable that the same conclusion is obtained in Theorem 6 below, though the proof (which makes no use of Lemma 1) is entirely different, and the result is based on (11) rather than (2).

Proof. If $y_2 \ge x_2$, $y_3 \ge x_3$, $y_k(0) \ge c_k$ for $k \ge 4$, and

$$\dot{y}_n + a_n y_n \ge \frac{1}{n+1}\sum_{k=2}^{n-2} y_k y_{n-k}, \qquad n \ge 4, \tag{8}$$

it follows by induction that $y_k \ge x_k$ for $k \ge 2$. Equation (8) can be solved by setting

$$y_k = (r_k + c_k)e^{-a_k t} - r_k e^{-(a_k+h)t}, \qquad k \ge 2,$$

where each $r_k \ge 0$ and $h = 1/15$. This gives

$$y_k = e^{-a_k t}(r_k + c_k - r_k e^{-ht}) \le e^{-a_k t}(c_k + \bar{r}_k)$$

where $\bar{r}_k = htr_k$. Hence (8) holds if

$$r_n h e^{-(a_n+h)t} \ge \frac{1}{n+1}\sum_{k=2}^{n-2}(\bar{r}_k + c_k)e^{-a_k t}(\bar{r}_{n-k} + c_{n-k})e^{-a_{n-k}t}.$$

By Lemma 1 the inequality holds if

$$hr_n \geq \frac{1}{n+1} \sum_{k=2}^{n-2} (\bar{r}_k + c_k)(\bar{r}_{n-k} + c_{n-k})$$

or also if

$$hr_n \geq \frac{1}{n+1} \sum_{k=2}^{n-2} (\bar{r}_k + c_k)^2.$$

Since $(\|\bar{r}+c\|_{n-2})^2 \leq (\|\bar{r}\|_{n-2}+\|c\|_{n-2})^2 \leq (\|\bar{r}\|+\|c\|)^2$ and since $h = 1/15$, it suffices to have

$$r_n \geq \frac{15}{n+1}(\|\bar{r}\| + \|c\|)^2. \tag{9}$$

We choose $r_k = \beta/(k+1)$ where β is a constant. Then

$$\|r\|^2 = \sum_{k=3}^{\infty} \frac{\beta^2}{k^2} = \beta^2 B^2 \quad \text{where} \quad B^2 = \frac{\pi^2}{6} - \frac{5}{4}. \tag{10}$$

Thus (9) holds if $\beta \geq 15(\bar{B}\beta + \|c\|)^2$, where $\bar{B} = htB$. We want to choose β in such a way as to have $\|c\|$ as large as possible. If A, $\bar{B}$, C are positive constants, it is easily checked that

$$\beta = A(\bar{B}\beta + C)^2$$

has a solution β if and only if $C \leq 1/(4A\bar{B})$, and then $\beta = 1/(4A\bar{B}^2)$. In the present case this requires $\|c\| \leq 1/(4tB)$, which agrees with the hypothesis on t in Theorem 2. We then have $x_k \leq y_k$, hence

$$\|x\| \leq \|y\| \leq \|\bar{r} + c\| \leq \|\bar{r}\| + \|c\| \leq \beta\bar{B} + \|c\| = 2\|c\|.$$

4. A counterexample

The following theorem settles the question raised by Herod:

THEOREM 3. *If c_2 is sufficiently large and $c_k = 0$ for $k \geq 3$, the solution of (2) satisfies $\|x(1)\| = \infty$.*

Proof. The functions y_k defined for $k \geq 2$ by $y_k = e^{a_k t}x_k$ satisfy $y_2 = c_2$, $y_3 = 0$ and

$$\dot{y}_n = \frac{e^{a_n t}}{n+1} \sum_{k=2}^{n-2} e^{-a_k t} y_k e^{-a_{n-k} t} y_{n-k}.$$

We assume $0 \leq t \leq 1$. Since $y_k/e \leq x_k \leq y_k$, the conditions $\|y(1)\| = \infty$ and $\|x(1)\| = \infty$ are equivalent.

The inequalities $a_k \le a_n < 1$ give

$$\dot{y}_n \ge \frac{(1/e)}{n+1}\sum_{k=2}^{n-2} y_k y_{n-k}, \qquad n \ge 4,$$

where, as above, $e = 2.718\cdots$ is the base of natural logarithms. Let $c_2 = p^2 e$ where p is a somewhat large number to be determined later. Then $\dot{y}_4 \ge p^4 e/5$, which gives

$$y_4 \ge \frac{p^4 et}{5} \ge p^2 et \quad \text{if} \quad p^2 \ge 5.$$

In the series

$$y_2, \quad y_4, \quad y_8, \quad y_{16}, \cdots$$

each $\dot{y}_k$ beyond the first involves the square of its predecessor. Hence we confine attention to y_n when n is a power of 2. This gives $\dot{y}_8 \ge p^4 et^2/9$ so

$$y_8 \ge \frac{p^4 et^3}{27} \ge p^3 et^3 \quad \text{if} \quad p \ge 27.$$

The same process gives

$$y_{16} \ge p^4 et^7 \quad \text{if} \quad p^2 \ge 7 \times 17.$$

The last two equations agree with the inequality

$$y_{2^n} \ge p^n et^{2^{n-1}-1}$$

for $n = 3$ and 4 respectively. If this holds for a given $n \ge 4$ then

$$\dot{y}_{2^{n+1}} \ge \frac{1}{2^{n+1}+1} p^{2n} et^{2^n-2}$$

and hence

$$y_{2^{n+1}} \ge \frac{1}{2^{n+1}+1}\frac{p^{2n}e}{2^n-1}t^{2^n-1}.$$

This gives the desired result

$$y_{2^{n+1}} \ge p^{n+1} et^{2^n-1}$$

provided $p^{n-1} \ge (2^n-1)(2^{n+1}+1)$. When $n = 3$ this agrees with the condition $p^2 \ge 7 \times 17$ obtained above for the passage from y_8 to y_{16}. A sufficient condition in general is

$$p \ge 2^{(2n+1)/(n-1)}.$$

Since the right side decreases in n, the inequality $p \ge 5$ suffices when $n \ge 16$. This completes the proof.

5. A weaker hypothesis

By the Schwarz inequality (2) implies

$$\text{(11a)} \qquad \dot{x}_n + a_n x_n \leq \frac{1}{n+1} \sum_{k=2}^{n-2} x_k{}^2, \qquad n \geq 4,$$

which is, in several respects, a much weaker condition. Although we use the same letter x, the results obtained now are based on (11) rather than (2), with the same initial conditions as those for (2); namely

$$\text{(11b)} \qquad x_2 = c_2 e^{-a_2 t}, \quad x_3 = c_3 e^{-a_3 t}, \quad x_k(0) = c_k, \qquad k \geq 4.$$

In view of our convention regarding empty sums, (11a) remains valid for $n = 2$ and 3.

The analog of Theorem 1 for (11) is

$$x_n(t) \leq d_n e^{-(1/3)t}, \qquad t \geq 0, \quad n \geq 2.$$

This is easily proved by induction. It shows that each x_n is bounded and tends to 0; hence

$$m_k = \max x_k(t)$$

is attained for each $k \geq 2$. If we replace x_k in (11) by m_k, then apply (4), and finally take the max of the left side, we get

$$\text{(12)} \qquad m_n \leq \max\left(c_n, \frac{1}{n-1} \sum_{k=2}^{n-2} (m_k)^2\right).$$

The equality corresponding to (11a) is

$$\dot{x}_n + a_n x_n = \frac{1}{n+1} \sum_{k=2}^{n-2} x_k{}^2, \qquad n \geq 4.$$

Under the boundary conditions

$$x_2 = c_2 e^{-a_2 t}, \quad c_k = 0 \quad \text{for} \quad k \geq 3,$$

Theorem 3 shows that the solution satisfies $\|x(1)\| = \infty$ provided c_2 is sufficiently large. Here, however, we consider x_n for all even n, not just for n of the form 2^j. The result is that $\|x(1)\|$ can have an extremely rapid rate of growth; indeed we get a lower bound of the form

$$x_{2n}(1) > A^{2^n}$$

no matter how large A may be. (The proof is similar to the proof of Theorem 3 and need not be repeated.) In spite of this spectacular failure of the condition $\|x(t)\| < \infty$, it will be seen that (11) admits much the same positive theorems as were valid for (2). This is one of several surprises provided by problems associated with [1].

For example if x is as in (11) and $c_k \le 1$, then an easy induction depending on (12) gives $m_n \le 1$. As another illustration, suppose $\sup c_k = \bar{c} < \infty$. Then $x_n(t) \le 2\bar{c}$ on an interval of length $1/(4\bar{c})$. To see why, suppose by some means we have found a constant a such that

$$x_k(t) \le a, \qquad 0 \le t \le \frac{a-\bar{c}}{a^2}, \quad 2 \le k \le n-2.$$

By (11) we get $\dot{x}_n \le a^2$ on this interval, hence $x_n \le \bar{c} + a^2 t \le a$ on the same interval. By induction this holds for all $n \ge 2$. To get started we must have $a \ge \max(c_2, c_3)$. The choice $a = 2\bar{c}$ satisfies this condition and maximizes the interval.

6. Sums of squares revisited

The theorems given in the rest of this paper pertain to x in (11), hence they apply also to x in (2).

THEOREM 4. *Let x be as in* (11). *Then* $\|c\| \le 0.7085 \Rightarrow \sup \|x(t)\| < \infty$.

Proof. Let $(s_n)^2 = (\|x(t)\|_n)^2$. Then

$$s_n \dot{s}_n = \sum_{k=2}^{n} x_k(t) \dot{x}_k(t).$$

For $k \ge 2$ Equation (11a) gives

$$\dot{x}_k \le \frac{1}{k+1} \sum_{j=2}^{k-2} {x_j}^2 - a_k x_k.$$

(In view of the initial conditions this holds as an equality when $k = 2$ or 3.) Hence for $n \ge 4$

$$s_n \dot{s}_n \le \sum_{k=4}^{n} \frac{x_k}{k+1} \sum_{j=2}^{k-2} {x_j}^2 - \sum_{k=2}^{n} a_k {x_k}^2. \tag{13}$$

We start at $k = 4$ because the middle sum is empty for $k < 4$. By the Schwarz inequality

$$\sum_{k=4}^{n} \frac{x_k}{k+1} \le a s_n \quad \text{where} \quad a^2 = \frac{\pi^2}{6} - \frac{5}{4} - \frac{1}{9} - \frac{1}{16}.$$

Since each $a_n \ge 1/3$ and since $s_{k-2} \le s_n$ Equation (13) gives

$$\dot{s}_n \le a {s_n}^2 - \frac{1}{3} s_n$$

after division by s_n. Thus $y = s_n$ satisfies a Riccati inequality $\dot{y} \le a y^2 - b y$ with $b = 1/3$ and $y(0) \le \|c\|$. Such an inequality is easily solved by setting $y = 1/u$. It turns out that y is bounded (and actually tends to 0 as $t \to \infty$)

if $ac < b$. The resulting condition $3a\|c\| < 1$ agrees with the hypothesis of Theorem 2. This completes the proof.

The following is related to Theorem 4, but the proof is entirely different:

THEOREM 5. *Let x satisfy* (11) *with $\pi\|c\| \le \sqrt{3/2}$. Then the vector m given by $m_i = \sup x_i(t)$ satisfies $\|m\| \le \sqrt{3}/\pi$.*

The condition $\|m\| < \infty$ is much stronger than $\sup \|x(t)\| < \infty$.

Proof. Suppose we have found a sequence $\{u_k\}$ such that $u_2 = m_2 = c_2$, $u_3 = m_3 = c_3$ and

$$u_n \ge \max\left(c_n, \frac{1}{n-1}\sum_{k=2}^{n-2}(u_k)^2\right), \qquad n \ge 4. \tag{14}$$

We then get in succession $u_4 \ge m_4$, $u_5 \ge m_5$ and so on. Thus $u_n \ge m_n$ for $n \ge 4$.

Let us set $A = \|c\|^2$, $B \ge \|u\|^2$ assuming for the moment that the latter is finite. For (14) it then suffices to have

$$u_n = \max\left(c_n, \frac{B}{n-1}\right), \qquad n \ge 2. \tag{15}$$

Whenever $c_n \ge 0$ and $d_n \ge 0$ we have

$$\big(\max(c_n, d_n)\big)^2 = \max(c_n^2, d_n^2) \le c_n^2 + d_n^2,$$

and hence $M_n = \max(c_n, d_n)$ satisfies $\|M\|^2 \le \|c\|^2 + \|d\|^2$. (Equality holds if $c_i d_i = 0$ for all i.) In (15) this gives

$$\|u\|^2 \le \|c\|^2 + \frac{\pi^2}{6}B^2.$$

We want this to be $\le B$ or equivalently

$$\|c\|^2 \le B - \frac{\pi^2}{6}B^2,$$

so $\{u_i\}$ satisfies the needed condition $\|u\|^2 \le B$. Choosing $B = 3/\pi^2$, we get Theorem 5.

THEOREM 6. *With x as in* (11), *let* $0 < \|c\| < \infty$. *Then*

$$\|x(t)\| \le 2\|c\| \quad \text{for} \quad 0 \le t \le \frac{0.3978}{\|c\|}.$$

Proof. The function $y_k = x_k - c_k e^{-a_k t}$ satisfies

$$\dot{y}_n + a_n y_n \le \frac{1}{n+1}\sum_{k=2}^{n-2}(y_k + c_k e^{-a_k t})^2$$

together with the initial conditions $y_k(0) = 0$ and $y_2 = y_3 = 0$. If $y_n \geq 0$ then

$$\dot{y}_n \leq \frac{1}{n+1} \sum_{k=2}^{n-2} (y_k + c_k)^2.$$

The latter sum is $(\|y + c\|_{n-2})^2$, so

$$\dot{y}_n \leq \frac{1}{n+1} (\|y\|_{n-2} + \|c\|_{n-2})^2, \quad n \geq 4 \tag{16}$$

provided $y_n \geq 0$. As shown in [2], the condition $y_n \geq 0$ has no effect on the use we shall make of (16) and can be ignored until later.

Let M be a constant such that $\|y(t)\|_{n-2} \leq M$ on a given interval $0 \leq t \leq t_0$. Then (16) gives an inequality for $\dot{y}_k$ which implies the right-hand inequality in

$$-c_k \leq y_k \leq \frac{t}{k+1} (M + C)^2, \qquad 4 \leq k \leq n-2,$$

where $C = \|c\|$. The left-hand inequality follows from $x_k \geq 0$. Hence

$$\|y\|_n \leq \max\left(C,\ tB(M+C)^2\right) \quad \text{where} \quad B^2 = \frac{\pi^2}{6} - \frac{5}{4}.$$

The inequality $\|y\|_n \leq M$ holds for $0 \leq t \leq t_0$ if

$$M \geq C \quad \text{and} \quad M \geq t_0 B (M+C)^2, \quad \text{or} \quad t_0 \leq \frac{1}{B} \frac{M}{(M+C)^2}.$$

Repetition gives the same for $n+1$, $n+2$ and so on. Taking $M = C$ yields the theorem, since $1/(4B) > 0.3978$ and $\|x\| \leq \|y\| + \|c\|$.

THEOREM 7. *With x as in (11), suppose $\|c\| < \infty$ and $c_n \leq 1/(\log n)^a$ for $n \geq 2$, where $a > 1/2$ is constant. Then $m_k = \max x_k(t)$ satisfies $\|m\| < \infty$.*

The definition of s_n in the following lemma is more convenient for present purposes than that in the proof of Theorem 4:

LEMMA 2. *With x as in (11) set $s_n = (\|m\|_n)^2$ and suppose $\|c\| < \infty$. Then*

$$\sum_{n=2}^{\infty} \frac{s_n}{n^2} < \infty \Rightarrow \sup s_n < \infty.$$

Proof. Equation (13) gives

$$m_n \leq \max\left(c_n, \frac{s_{n-2}}{n-1}\right) \leq \max\left(c_n, \frac{s_{n-1}}{n-1}\right).$$

Hence

$$s_n - s_{n-1} = {m_n}^2 \leq \max\left((c_n)^2, \left(\frac{s_{n-1}}{n-1}\right)^2\right). \tag{17}$$

Without loss of generality we take $c_2 > 0$, so $s_n \geq (c_2)^2 > 0$ for $n \geq 2$. By (17) for $n \geq 3$

$$s_n \leq s_{n-1}(1+\epsilon_n) \quad \text{where} \quad \epsilon_n = \left(\frac{c_n}{c_2}\right)^2 + \frac{s_{n-1}}{(n-1)^2}.$$

Since $\sum \epsilon_n < \infty$, the infinite product

$$\frac{s_n}{s_2} = \frac{s_n}{s_{n-1}}\frac{s_{n-1}}{s_{n-2}}\cdots\frac{s_3}{s_2} = \prod_{k=2}^{n}(1+\epsilon_k)$$

converges as $n \to \infty$, and this completes the proof.

Proof of Theorem 7. We take $2a > 1$ but close to 1 and assume for the moment that $c_k = 1/(\log k)^a$ for $k \geq 2$. It is a principal objective to show that m_n as given by (12) satisfies $m_n \leq 1/(\log n)^a$ for $n \geq 2$. This holds for $n = 2$ and 3, since $m_2 = c_2$ and $m_3 = c_3$. By a brief calculation it also holds for $3 \leq n \leq 8$. (A simple procedure is to set $a = 1/2$ and show that the desired inequalities are strict.) We assume that $m_n \leq 1/(\log n)^a$ for $2 \leq k \leq n$, where $n \geq 8$, and show that the same holds for $n+1$.

In view of (12) an adequate hypothesis and conclusion are respectively

$$\frac{1}{(\log n)^a} \geq \frac{1}{n-1}\sum_{k=2}^{n-2}(m_k)^2, \qquad \frac{1}{\big(\log(n+1)\big)^a} \geq \frac{1}{n}\sum_{k=2}^{n-1}(m_k)^2.$$

Since $(m_{n-1})^2 \leq 1/\big(\log(n-1)\big)^{2a}$, the second inequality follows from the first if

$$\frac{n-1}{(\log n)^a} + \frac{1}{\big(\log(n-1)\big)^{2a}} \leq \frac{n}{\big(\log(n+1)\big)^a}.$$

With $\phi(x) = x/\big(\log(x+1)\big)^a$, the desired inequality holds if

$$\frac{1}{(\log(n-1))^{2a}} \leq \phi(n) - \phi(n-1) = \phi'(\xi), \quad n-1 \leq \xi \leq n. \tag{18}$$

We have

$$\phi'(x) = \frac{1}{\big(\log(x+1)\big)^a}\left(1 - \frac{ax}{x+1}\frac{1}{\log(x+1)}\right).$$

Since $ax/(x+1) < a$ it is easily seen that

$$\phi'(\xi) \geq \frac{1}{\big(\log(n+1)\big)^a}\left(1 - \frac{a}{\log n}\right).$$

Hence (18) holds if

$$\frac{\big(\log(n+1)\big)^a}{\big(\log(n-1)\big)^{2a}} \leq 1 - \frac{a}{\log n}.$$

A short calculation shows that the left side is a decreasing function of n and the right side is clearly increasing. Hence if the inequality holds for any n it also holds for any larger n. Since it holds for $n = 9$, it holds for $n \geq 9$, completing the proof.

In all this we used the fact that $c_k = 1/(\log k)^a$, where $2a$ is close to 1. However the result also holds for $c_k \leq 1/(\log k)^a$, since diminishing any c_k usually diminishes both $x_n(t)$ and m_n and never increases either. The inequality is compatible with $\|c\| < \infty$ as the equality was not.

The result we have just obtained gives

$$s_n = \sum_{k=2}^{n} (m_k)^2 \leq (m_2)^2 + \int_2^n \frac{dx}{(\log x)^{2a}}.$$

Furthermore

$$\int_2^n \frac{dx}{(\log x)^{2a}} \leq \int_2^{\sqrt{n}} \frac{dx}{(\log 2)^{2a}} + \int_{\sqrt{n}}^n \frac{dx}{(\log \sqrt{n})^{2a}} < \frac{\sqrt{n}}{(\log 2)^{2a}} + \frac{2^{2a} n}{(\log n)^{2a}}.$$

This shows that the sum is $O(n/(\log n)^{2a})$ and hence that the hypothesis of Lemma 2 is fulfilled. The lemma gives $\sup s_n < \infty$, completing the proof.

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