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# LINEARLY INVARIANT CLASSES OF FUNCTIONS ANALYTIC IN A UNIT CIRCLE OF ZERO TYPE

## Introduction

The class of univalent functions was simultaneously narrowed up to some of its subclasses and vice versa, extended to wider classes (see [3], [4]).

One of such extensions was proposed by a German mathematician Ch. Pommerenke [1] and the author of the paper (see [3]).

Ch. Pommerenke gives a definition by a linearly univalent class of analytic functions in a unit circle $E$ which, in our notation, is formulated as follows.

Let $\tilde{A}_1(E)$ be a class of analytic in $E$ functions $f(z)$, normalized by conditions $f(0) = 0, f'(0) = 1$, for which $f'(z) \neq 0, \forall z \in E$ and let $\Lambda$ be a set of all linear-fractional functions

$$\omega = \omega(z) = \frac{e^{i\Theta}z + \zeta}{1 + \bar{\zeta}e^{i\Theta}z}, \quad \zeta \in E, \quad \Theta \in (-\infty, \infty),$$

bijectively mapping the unit circle $E$ onto itself. We introduce the operator

$$\Omega_1^{\omega}[f] = \frac{f(\omega(z)) - f(\zeta)}{e^{i\Theta}(1 - |\zeta|^2)f'(\zeta)}.$$

The operator transforms any function from class $\tilde{A}_1(E)$ to a function of the same class. We denote by $\tilde{\Im}_1(E)$ a class of analytic functions $f(z)$ from $\tilde{A}_1(E)$ bearing the following property: if the function $f(z) \in \tilde{\Im}_1(E)$, then the function $\Omega_1^{\omega}[f(z)] \in \tilde{\Im}_1(E)$, too, for any $\omega \in \Lambda$. Pommerenke called the class $\tilde{\Im}_1(E)$ a linearly invariant class. Class $\tilde{A}_1(E)$ as well as the class of univalent and normed in $E$ function may serve as examples of the linearly invariant class in the sense of Pommerenke. Pommerenke devoted a great part of his work [1] to a detailed study of the properties of functions from different linearly invariant classes $\tilde{\Im}_1(E)$.

Note, again, that the idea to maximally exploit analytic automorphism of the unit circle $E$ in combination with the Pommerenke operator for investigating the properties of univalent normalized in $E$ functions belongs to a French mathematician Marty [2] who succeeded in obtaining quite a number of important results (see also [3], [4]).

From the facts mentioned above we see that in order to construct a linearly invariant class of analytic functions in a unit circle one needs to introduce a special operator defined on this class and connect it with analytic automorphism of the unit circle aod normalization of the functions included in this class.

In the paper the author considers the class $\tilde{A}_0(E)$ of analytic functions $f(z)$ in the unit circle $E$ which is normalized by the condition $f(0) = 1$ and posseses the property $f(z) \neq 0$ in $E$. The operator

$$\Omega^{\omega}[f] = \frac{f(\omega(z))}{f(\omega(0))},$$

is introduced in the class $\tilde{A}_0(E)$ that transforms any function of the class $\tilde{A}_0(E)$ to a function belonging to this class. By $\tilde{\Im}(E)$ we denote a class of functions $f(z)$ from $\tilde{A}_0(E)$, that possesses the property: if a function $f(z) \in \tilde{\Im}(E)$, then the function $\Omega^{\omega}[f(z)] \in \tilde{\Im}(E)$ for any $\omega \in \Lambda$. Following Pommerenke's examples the author also calls this class a linearly invariant class of analytic functons in the unit circle.

## 1. Notation, definitions and the particular properties

1. Let $A_0(E)$ be a class of analytic in a unit circle $E$ functions $f(z)$ possessing the property $f(z) \neq 0$ in $E$.

Let us denote by $\tilde{A}_0(E)$ a class of analytic in a unit circle $E$ functions $f(z)$ from $A_0(E)$, normalized by the condition $f(0) = 1$.

Let $\Lambda$ be a set of all linear fractional functions of the type

$$\omega = \omega(z) = \frac{e^{i\Theta}z + \zeta}{1 + \bar{\zeta}e^{i\Theta}z}, \quad \zeta \in E, \quad \Theta \in (-\infty; \infty).$$

Let us call $\omega = \omega(z)$ an omega-transform (in the sequel 0.-t.). It can be easily seen that the set $\Lambda$ of omega-transforms $\omega = \omega(z)$ is a group, if we use the operation of multiplication $\bigotimes$ of two omega-transforms $\omega_1$ and $\omega_2$ according to the rule $\omega_1 \bigotimes \omega_2 = \omega_1(\omega_2)$.

We introduce the omega-operator

$$\Omega^{\omega}\Big[f(z)\Big] = \frac{f(\omega(z))}{f(\omega(0))}, \quad \omega \in \Lambda, \quad f(z) \in \tilde{A}_0(E).$$

This operator transforms any function $f(z)$ from the class $\tilde{A}_0(E)$ into a function also belonging to the class $\tilde{A}_0(E)$.

Let us describe some properties of the omega-operator introduced above.
If $f \in A_0(E)$ and $c = const \neq 0$, then $\Omega^{\omega}[cf] = \Omega^{\omega}[f]$,
if $f_1, f_2 \in A_0(E)$, then $f_1 f_2 \in A_0(E)$ and $\Omega^{\omega}[f_1 f_2] = \Omega^{\omega}[f_1]\Omega^{\omega}[f_2]$,
if $f_1, f_2 \in \tilde{A}_0(E)$, then $f_1/f_2 \in A_0(E)$ and $\Omega^{\omega}[f_1/f_2] = \Omega^{\omega}[f_1]/\Omega^{\omega}[f_2]$,
if $f_1, f_2 \in \tilde{A}_0(E)$, and $f_1 \neq f_2$, then $\Omega^{\omega}[f_1] \neq \Omega^{\omega}[f_2]$, $\forall \omega \in \Lambda$.

We may call a set $\tilde{\Im}(E)$ of functions $f(z) \in \tilde{A}_0(E)$ a linearly invariant class if from $f(z) \in \tilde{\Im}(E)$ it follows that $\Omega^{\omega}[f(z)] \in \tilde{\Im}(E)$ for any $\omega \in \Lambda$.

Let us denote the number

$$\delta = \delta(\tilde{\Im}(E)) = \sup_{f(z) \in \tilde{\Im}(E)} \left| f'(0) \right|$$

the bound of the class $\tilde{\Im}(E)$ and denote class $\tilde{\Im}(E)$ of the bound $\delta$ by $\tilde{\Im}(E;\delta)$.

We also denote by $\tilde{U}(E;\delta)$ the set of all linearly invariant classes $\tilde{\Im}(E)$ the bound of which do not exceed the number $\delta$. The validity of the following statements is quite evident.

The class $\tilde{A}_0(E)$ is a linearly invariant class.

One function $f(z) \equiv 1$ forms a linearly invariant class.

2. LEMMA 1.1. *Let $f_1(z) \in \tilde{A}_0(E)$ and $\omega_1, \omega_2 \in \Lambda$. If $f_2(z) = \Omega^{\omega_1}[f_1(z)]$, $f_3(z) = \Omega^{\omega_2}[f_2(z)]$, then $f_3(z) = \Omega^{\omega_1(\omega_2)}[f_1(z)]$.*

Proof. Let

$$\omega_1(z) = \frac{e^{i\Theta_1}z + \zeta_1}{1 + \bar{\zeta}_1 e^{i\Theta_1}z}, \quad \omega_2(z) = \frac{e^{i\Theta_2}z + \zeta_2}{1 + \bar{\zeta}_2 e^{i\Theta_2}z}.$$

Since $\omega_1, \omega_2 \in \Lambda$, we have $\omega_3 = \omega_1 \bigotimes \omega_2 \in \Lambda$. The function $\omega_3 = \omega_3(z)$ is written as follows:

$$\omega_3(z) = \frac{e^{i\Theta_3}z + \zeta_3}{1 + \bar{\zeta}_3 e^{i\Theta_3}z} \quad e^{i\Theta_3} = \frac{e^{i\Theta_2}(e^{i\Theta_1} + \zeta_1\bar{\zeta}_2)}{1 + \bar{\zeta}_1\zeta_2 e^{i\Theta_2}}, \; \zeta_3 = \frac{e^{i\Theta_1}\zeta_2 + \zeta_1}{1 + \bar{\zeta}_1\zeta_2 e^{i\Theta_1}}.$$

Basing ourselves on the properties of the operator we obtain the following sequence of equalities:

$$f_3(z) = \Omega^{\omega_2}[f_2(z)] = \Omega^{\omega_2}[\Omega^{\omega_1}[f_1(z)]] =$$
$$\Omega^{\omega_2}[f_1(\omega_1(z))] = \frac{f_1(\omega_1(\omega_2(z)))}{f_1(\omega_1(\omega_2(0)))} = \frac{f_1(\omega_3(z))}{f_1(\omega_3(0))} = \Omega^{\omega_3}[f_1(z)].$$

The lemma is proved.

LEMMA 1.2. *The equality*

$$\Omega^{\omega_1} \bigotimes \Omega^{\omega_2} = \Omega^{\omega_2 \bigotimes \omega_1}$$

*holds.*

Indeed, for any function $f(z) \in \tilde{A}_0(E)$ we have

$$\Omega^{\omega_1}[f(z)] \bigotimes \Omega^{\omega_2}[f(z)] = \Omega^{\omega_1}[\Omega^{\omega_2}[f(z)]] = \Omega^{\omega_1}\left[\frac{f(\omega_2(z))}{f(\omega_2(0))}\right] =$$

$$\Omega^{\omega_1}[f(\omega_2(z))] = \frac{f(\omega_2(\omega_1(z)))}{f(\omega_2(\omega_1(0)))} = \Omega^{\omega_2 \bigotimes \omega_1}[f(z)].$$

It follows from Lemma 1.2 that the product of two operators $\Omega^{\omega_1}$ and $\Omega^{\omega_2}$, taken from the set of operators $\Omega^{\omega}$, where $\omega$ runs over the whole set of transforms from $\Lambda$, is also an operator from $\Omega^{\omega}$. For the operator $\Omega^{\omega_0}$. where $\omega_0$ is a unit transformation of $\omega_0 = \omega_0(z) \equiv z$, we have $\Omega^{\omega_0}[f] = f$ for any function $f(z) \in \tilde{A}_0(E)$. If $\omega(z) \in \Lambda$, then an inverse transformation denoted by $\omega^*(z)$ belongs to $\Lambda$ and it can be written as

$$\omega^*(z) = \frac{e^{-i\Theta}z - \zeta e^{-i\Theta}}{1 - \overline{\zeta}z}.$$

Further, $\Omega^{\omega} \otimes \Omega^{\omega^*} = \Omega^{\omega^*} \otimes \Omega^{\omega} = \Omega^{\omega_0}$ and $\Omega^{\omega^*} \otimes \Omega^{\omega} = \Omega^{\omega \bigotimes \omega^*} = \Omega^{\omega_0}$. Note, that for any $\omega_1, \omega_2 \in \Lambda$ the inequalities

$$(\Omega^{\omega_1} \bigotimes \Omega^{\omega_2}) \bigotimes \Omega^{\Omega_3} = \Omega^{\omega_2 \bigotimes \omega_1} \bigotimes \Omega^{\omega_3} = \Omega^{\omega_1} \bigotimes (\Omega^{\omega_2} \bigotimes \Omega^{\omega_3})$$

hold. If $\omega_1 = \omega_2$, then, obviously, $\Omega^{\omega_1} = \Omega^{\omega_2}$.

Conversely, if $\Omega^{\omega_1}[f] = \Omega^{\omega_2}[f]$, $\forall f \in \tilde{A}_0(E)$, then $\omega_1 = \omega_2$. It follows from the latter equality that

$$\Omega^{\omega_1}[f] \bigotimes \Omega^{\omega_2^*}[f] = \Omega^{\omega_2}[f] \bigotimes \Omega^{\omega_2^*}[f], \quad \forall f \in \tilde{A}_0(E).$$

Now, according to Lemma 1.2, we obtain

$$\Omega^{\omega_2^* \bigotimes \omega_1}[f] = \Omega^{\omega_2^* \bigotimes \omega_2}[f] = \Omega^{\omega_0}[f], \quad \forall f \in \tilde{A}_0(E).$$

By applying last equalities to the functions $f_0(z) = 1 + z$ and $f_1(z) = 1 - z$, belonging to the class $\tilde{A}_0(E)$ we obtain two equalities

$$\Omega^{\omega_2^* \bigotimes \omega_1}[f_0] = f_0, \quad \Omega^{\omega_2^* \bigotimes \omega_1}[f_1] = f_1,$$

which lead us to the equalities:

$$\frac{1 + \omega_2^*(\omega_1(z))}{1 + \omega_2^*(\omega_1(0))} = 1 + z = \frac{1 + \omega_0(z)}{1 + \omega_0(0)}, \quad \forall z \in E,$$

$$\frac{1 - \omega_2^*(\omega_1(z))}{1 - \omega_2^*(\omega_1(0))} = 1 - z = \frac{1 - \omega_0(z)}{1 - \omega_0(0)}, \quad \forall z \in E.$$

Considering these equalities we conclude that $\omega_2^*(\omega_1) = \omega_0$ and, therefore, $\omega_1 = \omega_2$.

The above consideration allow us to formulate the following statement:

THEOREM 1.1. *A set of operators $\Omega^{\omega}, \omega \in \Lambda$, defined on the class $\tilde{A}_0(E)$, forms a group of transformations if the operation $\bigotimes$ of multiplication of two* *operators* $\Omega^{\omega_1}$ *and* $\Omega^{\omega_2}$, *where* $\omega_1, \omega_2 \in \Lambda$ *is performed following the rule* $\Omega^{\omega_1} \bigotimes \Omega^{\omega_2} = \Omega^{\omega_1}[\Omega^{\omega_2}]$.

3. Any *o. – t.* can be represented as $\omega(z) = \omega_*(z) \bigotimes \omega_{**}(z)$, where

$$\omega_*(z) = \frac{z+\zeta}{1+\bar{\zeta}z}, \quad \omega_{**}(z) = e^{i\Theta}z.$$

Then

$$\Omega^{\omega} = \Omega^{\omega_* \bigotimes \omega_{**}} = \Omega^{\omega_{**}} \bigotimes \Omega^{\omega_*}.$$

It follows from the definition of omega-operator $\Omega^{\omega_*}[f(z)]$ that it transforms the function

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \in \tilde{A}_0(E)$$

into the function

$$f(z,\zeta) = 1 + \sum_{k=1}^{\infty} a_k(\zeta) z^k \in \tilde{A}_0(E).$$

Besides, $\Omega^{\omega_{**}}[f(z)] = f(e^{i\Theta}z) \in \tilde{A}_0(E)$. It means, that we can write $f(e^{i\Theta}z;\zeta) = \Omega^{\omega}[f(z)]$.

LEMMA 1.3. *The expansion of the function* $f(e^{i\Theta}z;\zeta)$ *in power series of* $z$ *has the form*

$$f(e^{i\Theta}z;\zeta) = 1 + \sum_{k=1}^{\infty} a_k(\zeta;\Theta) z^k, \tag{1.1}$$

*where*

$$a_k(\zeta;\Theta) = \sum_{m=0}^{k-1} \frac{(k-1)!(-1)^m}{m!(k-1-m)!}\left(1-|\zeta|^2\right)^{k-m}\bar{\zeta}^m e^{ik\Theta} \frac{f^{(k-m)}(\zeta)}{(k-m)!f(\zeta)}. \tag{1.2}$$

In particular, it follows from (1.2) that

$$a_1(\zeta;\Theta) = e^{i\Theta}\left(1-|\zeta|^2\right)\frac{f'(\zeta)}{f(\zeta)}. \tag{1.3}$$

If we add all the functions of the form $f\left(e^{i\Theta}z;\zeta\right), \zeta \in E, \Theta \in (-\infty,\infty)$ to the function $f(z) \in A_0(E)$, we will obtain a class of functions that is called a simple class and is denoted by $\tilde{\Pi}(E;f)$.

LEMMA 1.4. *A simple class is a linearly invariant class.*

Our statement follows from Lemma 1.1.

Let us denote a simple class $\tilde{\Pi}(E;f)$, whose bound is $\delta$ by $\tilde{\Pi}(E;f;\delta)$. It can easily be seen that the simple class $\tilde{\Pi}(E;f;\delta)$ is also a linearly invariant class.

LEMMA 1.5. *If* $f_1 \in \tilde{\Pi}(E; f_2)$ *then* $f_2 \in \tilde{\Pi}(E; f_1)$.

To show the validity of the given Lemma, it suffices to apply Lemma 1.1.

4. Let us introduce an operator $\Delta_1[f]$ into class $\tilde{A}_0(E)$ by the formula

$$\Delta_1[f(z)] = \left(1 - |z|^2\right)\frac{f'(z)}{f(z)}.$$

For the fixed $z \in E$ we deal with a functional defined on class $\tilde{A}_0(E)$. If we fix the function $f(z) \in \tilde{A}_0(E)$, then $\Delta_1[f(z)]$ will be a function of $z$ which is not analytic in $E$.

Let us denote

$$\delta_f = \sup_{z\in E} |\Delta_1[f(z)]|.$$

The following lemma holds true:

LEMMA 1.6. *The bound of a linearly invariant class* $\tilde{\Im}(E)$ *may be calculated by the formula*

$$\delta = \sup_{f(z)\in\tilde{\Im}(E)} \delta_f.$$

REMARK 1.1. If it is clear from the context which function forms a simple class, then we frequently write $\tilde{\Pi}(E; f)$ and $\tilde{\Pi}(E; \delta)$ instead of $\tilde{\Pi}(E)$ and $\tilde{\Pi}(E; \delta)$ without indicating the function itself. Basing ourselves on Lemmas 1.1, 1.5., and 1.6. we arrive at the following statement.

THEOREM 1.2. *The functional* $\delta_f$ *assumes a constant value in the functions that belong to one and the same simple class, i.e.,* $\delta_f = const$, $\forall f \in \tilde{\Pi}(E)$.

COROLLARY 1.1. *One can find the bound of the simple class* $\tilde{\Pi}(E; f)$ *by the formula*

$$\delta_f = \delta(\tilde{\Pi}(E; f)) = \sup_{z\in E} |\Delta_1[f(z)]|.$$

REMARK 1.2. Obviously, two simple classes either do not have a common function, or they are coincide. Each function of a simple class serves as a generator of this class. The association of simple classes is a linearly invariant class. Conversely, any linearly invariant class $\tilde{\Im}(E)$ is a simple class or a association of simple classes. Hence, there follows

LEMMA 1.7. *The universal class* $\tilde{U}(E; \delta)$ *is a linearly invariant class.*

## 2. Major criteria of referring the functions to a particular class

1. The following theorem gives the conditions for the function to belong to a universal class.

THEOREM 2.1. *In order that the function $f(z)$ belong to $\tilde{U}(E;\delta)$ it is necessary and sufficient that the inequality*

$$\delta_f \leq \delta \tag{2.1}$$

*be satisfied.*

Proof. Let the function $f(z) \in \tilde{U}(E;\delta)$. By means of the function $f(z)$ we form a simple class $\tilde{\Pi}(E)$. According to Lemma 1.4 it is a linearly invariant class consisting of all the functions of the form (1.1). In addition, by Lemma 1.7, $\tilde{U}(E;\delta)$ is a universal linearly invariant class and, therefore, $\tilde{\Pi}(E) \subset \tilde{U}(E,\delta)$. Then, for the coefficients (1.3) of the functions (1.1) the estimate

$$|a_1(\zeta,\Theta)| \leq \delta, \quad \forall\zeta \in, \quad \forall\Theta \in (-\infty,\infty) \tag{2.2}$$

holds. Hence, the inequality (2.1) follows.

Now, let $f(z) \in \tilde{A}_0(E)$ and let the condition (2.1) be fulfilled. By means of the function $f(z)$ we form a simple linearly invariant class $\tilde{\Pi}(E)$ consisting of the functions (1.1). According to (2.1), (2.2) holds for the coefficients (1.3) of these functions (2.2) holds, i.e., the bound of a simple class $\tilde{\Pi}(E)$ does not exceed $\delta$. Then, however, it is contained in $\tilde{U}(E;\delta)$, along with the function $f(z)$.

2. We will need in the sequel the functions analytic in $E$ of the form

$$\Phi_{t,a,b}(z) = \left(\frac{1-\bar{a}z}{1-\bar{b}z}\right)^{\frac{t}{\bar{b}-\bar{a}}}, \quad \Phi_{t,a,a}(z) = e^{\frac{tz}{1-\bar{a}z}},$$

where $|a| \leq 1, |b| \leq 1$ and $t$ is a complex number. Moreover, it is assumed that $\Phi_{t,a,b}(0) = 1$, if $a \neq b$ and $\Phi_{t,a,a}(0) = 1$. Let us call these functions the basic functions of linearly invariant classes. Some of the particular basic functions are

$$\Phi_{t,-1,0}(z) = (1+z)^t, \quad \Phi_{t,0,1} = \frac{1}{(1-z)^t}, \quad \Phi_{t,0,0}(z) = e^{tz}.$$

The product and quotient of two basic functions are also as basic functions. For the derivative at the basic functions we have the formulas

$$\Phi'_{t,a,b}(z) = t\left(\frac{1-\bar{a}z}{1-\bar{b}z}\right)^{\frac{t}{\bar{b}-\bar{a}}} \frac{t}{(1-\bar{a}z)(1-\bar{b}z)}, \quad a \neq b,$$

$$\Phi'_{t,a,a}(z) = \frac{1}{(1-\bar{a}z)^2} e^{\frac{tz}{1-\bar{a}z}}.$$

Also note that

$$\frac{\Phi'_{t,a,b}(z)}{\Phi_{t,a,b}(z)} = \frac{t}{(1-\bar{a}z)(1-\bar{b}z)}, \quad a \neq b, \quad \frac{\Phi'_{t,a,a}(z)}{\Phi_{t,a,a}(z)} = \frac{t}{(1-\bar{a}z)^2}.$$

Using the latter equalities in calculation one can see that, for $a \neq b$, the formula

$$(2.3)\quad \Delta_1[\Phi_{t,a,b}(z)] = \frac{(1-|z|^2)t}{(1-\overline{a}z)(1-\overline{b}z)} = \frac{t}{(\overline{b}-\overline{a})}\left(\frac{-\overline{a}+\overline{z}}{1-\overline{a}z} - \frac{-\overline{b}+\overline{z}}{1-\overline{b}z}\right),$$

holds, while, for $a = b$, the formula

$$(2.4)\qquad \Delta_1[\Phi_{t,a,a}(z)] = \frac{(1-|z|^2)t}{(1-\overline{a}z)^2}$$

is valid.

LEMMA 2.1. *For $\Delta_1[\Phi_{t,a,b}(z)]$ where $a \neq b$, $|a| = |b| = 1$, the equalities*

$$\sup_{z\in E}\left|\Delta_1[\Phi_{t,a,b}(z)]\right| = \frac{2|t|}{|\overline{b}-\overline{a}|}, \quad \inf_{z\in E}\left|\Delta_1[\Phi_{t,a,b}(z)]\right| = 0$$

*hold.*

Proof. From the formula (2.3) we get

$$\left|\Delta_1[\Phi_{t,a,b}(z)]\right|^2 = \frac{|t|^2}{|\overline{b}-\overline{a}|^2}\left(\frac{-\overline{a}+\overline{z}}{1-\overline{a}z} - \frac{-\overline{b}+\overline{z}}{1-\overline{b}z}\right)\left(\frac{-a+z}{1-a\overline{z}} - \frac{-b+z}{1-b\overline{z}}\right) =$$

$$= \frac{|t|^2}{|\overline{b}-\overline{a}|^2}(2+2\mathrm{Re}\{W(z)\}),$$

where

$$W(z) = \frac{-\overline{a}+\overline{z}}{1-\overline{a}z}\,\frac{b-z}{1-b\overline{z}}.$$

It is not difficult to observe that the function $w = W(z)$ maps the diameter $\Theta$ of the unit circle $E$, with the ends at the points $\pm\sqrt{ab}$, onto the unit circumference $|w| = 1$, where both end points are mapped into the point $w = -1$.

Let us expand the function $\Phi_{t,a,b}(z)$ into a power series

$$\Phi_{t,a,b}(z) = 1 + \sum_{k=1}^{\infty} g_{k,a,b}(t)z^k.$$

Then, for the $k$-th coefficient, the recurrent formula

$$(2.5)\qquad g_{k,a,b}(t) = \frac{1}{k}\Big(tg_{k,a,b}(t) + (k-1)(\overline{a}+\overline{b})g_{k-1,a,b}(t) \\ -(k-2)\overline{a}\overline{b}g_{k-2,a,b}(t),\Big)$$

holds, where it is assumed that $g_{-1,a,b}(t) \equiv 0$ and $g_{0,a,b}(t) = 1$. In particular, $g_{1,a,b}(t) \equiv t$. Note that $\Phi_{0,a,b}(z) \equiv 1$ and hence, $g_{k,a,b}(0) = 0, \quad \forall k \geq 1$. If $a = -1$ and $b = 1$, then we assume, for the sake of brevity, that $\Phi_{t,-1,1}(z) \equiv$

$\Phi_t(z)$ and $g_{k,-1,1}(t) \equiv g_k(t)$. In this case,

$$\Phi_t(z) = 1 + \sum_{k=1}^{\infty} g_k(t) z^k,$$

where

$$g_k(t) = \frac{1}{k}\Big(t g_{k-1}(t) + (k-2) g_{k-2}(t)\Big).$$

Note that $\Phi_0(z) \equiv 1$, hence, $g_k(0) = 0, \forall k \geq 1$. It can be easily seen that the basic function is the unique solution of the linear homogeneous differential equation

$$(1-\bar{a}z)(1-\bar{b}z)Z'(z) - tZ(z) = 0$$

of the first order with the initial condition $Z(0) = 1$.

The following two lemmas provide the conditions that the basic functions belong to the class $\tilde{U}(E;\delta)$.

LEMMA 2.2. *Let $|a| = |b| = 1$ and $a \neq b$. In order that the basic function $\Phi_{t,a,b}(z)$ belong to the class $\tilde{U}(E;\delta)$ it is necessary and sufficient that the inequality*

$$\frac{2|t|}{|\bar{b}-\bar{a}|} \leq \delta \tag{2.6}$$

*be satisfied.*

Proof. Let (2.6) hold. Then

$$\left|(1-|z|^2)\frac{\Phi'_{t,a,b}(z)}{\Phi_{t,a,b}(z)}\right| = \frac{|t|}{|\bar{b}-\bar{a}|}\left|\frac{-\bar{a}+\bar{z}}{1-\bar{a}z} - \frac{-\bar{b}+\bar{z}}{1-\bar{b}z}\right| \leq \frac{2|t|}{|\bar{b}-\bar{a}|} \leq \delta$$

and by Theorem 2.1 the function $\Phi_{t,a,b}(z)$ belongs to $\tilde{U}(E;\delta)$.

Now, assume (2.6) does not hold. Then in case $z = ar$, where $0 < r < 1$, we obtain

$$\lim_{r\to 1}\left|(1-r^2)\frac{\Phi'_{t,a,b}(ar)}{\Phi_{t,a,b}(ar)}\right| = \lim_{r\to 1}\frac{(1+r)|t|}{|\bar{b}r-\bar{a}|} = \frac{2|t|}{|\bar{b}-\bar{a}|} > \delta.$$

Then there is such a number $z_0 = ar_0$, where $0 < r_0 < 1$, for which we obtain

$$\left|\Big(1-|z_0|^2\Big)\frac{\Phi'_{t,a,b}(z_0)}{\Phi_{t,a,b}(z_0)}\right| > \delta.$$

Hence, it follows that $\Phi_{t,a,b}(z) \notin \tilde{U}(E;\delta)$.

LEMMA 2.3. *In order that the function $\Phi_{t,a,a}(z), |a| < 1$ belong to the class $\tilde{U}(E;\delta)$ it is necessary and sufficient that*

$$|t| \leq \delta(1-|a|^2). \tag{2.7}$$

Proof. Let the inequality (2.7) holdd for some $t$. Then, we have

$$\sup_{|z|<1}\left(1-|z|^2\right)\frac{|\Phi'_{t,a,b}(z)|}{|\Phi_{t,a,b}(z)|}=\sup_{|z|<1}\frac{|t|(1-|z|^2)}{|1-\overline{a}z|^2}\le\frac{|t|}{1-|a|^2}\le\delta$$

and, according to Theorem 2.1, the function $\Phi_{t,a,b}(z)\in\tilde{U}(E;\delta)$.

Now, let the function $\Phi_{t,a,b}(z), |a|<1$ belong to the class $\tilde{U}(E;\delta)$ for some $t$. By Theorem 2.1. we obtain

$$\delta\ge\sup_{|z|<1}\left(1-|z|^2\right)\frac{|\Phi'_{t,a,b}(z)|}{\Phi_{t,a,b}(z)}=\sup_{|z|<1}\frac{|t|(1-|z|^2)}{|1-\overline{a}z|^2}=\frac{|t|}{1-|a|^2},$$

which leads us to the inequality (2.7)

LEMMA 2.4. *A simple class formed by the basic function $\Phi_{t,a,a}(z)$, where $a=e^{i\alpha}$ and $t\neq 0$, has the bound $\delta=\infty$. A simple class formed by the basic function $\Phi_{t,0,0}(z)$ has the bound $\delta=|t|$.*

As a matter of fact, for the first basic function, by the formula (2.4), we have

$$\sup_{z\in E}\left|\Delta_1\left[\Phi_{t,a,a}(z)\right]\right|=\sup_{z\in E}\frac{(1-|z|^2)|t|}{|1-e^{i\alpha}z|^2}=\infty.$$

For the second basic function we obtain

$$\sup_{z\in E}\left|\Delta_1\left[\Psi_{t,0,0}(z)\right]\right|=\sup_{z\in E}(1-|z|^2)|t|=|t|.$$

With the aid of Lemma 2.4 it is easy to prove the following

THEOREM 2.2. *The bound $\delta$ of any linearly invariant class satisfies a double inequality $0\le\delta\le\infty$. Any number from the interval $[0,\infty]$ can be the bound of a particular linearly invariant class.*

REMARK 2.1. A linearly invariant class has the bound equal to zero if and only if it consists of the only one function $f(z)\equiv 1$. Really, let some function $f(z)$ belong to the linearly invariant class $\Im(E;0)$. Then $\delta_f\le 0$, i.e.

$$\left(1-|z|^2\right)\frac{|f'(z)|}{|f(z)|}=0,\quad\forall z\in E.$$

Since $f(z)\neq 0$ for any $z\in E$ and $f(0)=1$, it follows from the latter equality that $f(z)\equiv 1$.

3. We will present more conditions in order that the function belong to the class $\tilde{U}(E;\delta)$.

THEOREM 2.3. *Let $f(z)\in\tilde{A}_0(E)$. If*

$$\sup_{|z|<1}\left|(1-z^2)\frac{f'(z)}{f(z)}\right|\le\delta,\tag{2.8}$$

*then $f(z) \in \tilde{U}(E;\delta)$. Not every function of the class $\tilde{U}(E;\delta)$ satisfies the condition (2.8).*

Proof. Basing ourselves on (2.8) we get the inequalities

$$\delta_f = \sup_{|z|<1} \left|\left(1-|z|^2\right)\frac{f'(z)}{f(z)}\right| \le \sup_{|z|<1} \left|(1-z^2)\frac{f'(z)}{f(z)}\right| \le \delta$$

and, by Theorem 2.1, we get that $f(z) \in \tilde{U}(E;\delta)$. To prove the second part of the theorem, let us take the basic function $\Phi_{t,a,b}(z), t = \delta i, a = i, b = -i$, which belongs to the class $\tilde{U}(E;\delta)$. This function does not satisfy the inequality (2.8). Indeed, for $z = i/2$, we obtain a number, on the lefthand side of (2.8), that is larger than $\delta$.

THEOREM 2.4. *If $f_m(z) \in \tilde{U}(E;\delta), m = 1,\dots,k$ and $\lambda_1 + \dots + \lambda_k = 1$, where $\lambda_1,\dots,\lambda_k$ are positive numbers, then the function*

$$f(z) = \prod_{m=1}^{k} f_m^{\lambda_m}(z)$$

*belongs to $\tilde{U}(E;\delta)$.*

Proof. Evidently, $f(z) \in \tilde{A}_0(E)$. Next, we have

$$\frac{f'(z)}{f(z)} = \sum_{m=1}^{k} \lambda_m \frac{f'_m(z)}{f_m(z)}.$$

Consequently,

$$(1-|z|^2)\frac{|f'(z)|}{|f(z)|} \le \sum_{m=1}^{k} \lambda_m (1-|z|^2)\frac{|f'_m(z)|}{|f_m(z)|} \le \sum_{m=1}^{k} \lambda_m \delta = \delta.$$

By Theorem 2.1 it follows that $f(z) \in \tilde{U}(E;\delta)$.

4. Let us introduce the operator

$$\Delta_1^0\left[f(z)\right] = (1-z^2)\frac{f'(z)}{f(z)}, \quad f(z) \in \tilde{A}_0(E)$$

which differs slightly from the operator $\Delta_1[f]$. Note that

$$(1-|z|^2)\Delta_1^0[f(z)] = (1-z^2)\Delta_1[f(z)], \quad \forall z \in E,$$
$$\Delta_1[f(z)] = \Delta_1^0[f(x)], \ \forall x \in (-1,1).$$

For the basic functions we have

$$\Delta_1^0\left[\Phi_{t,a,b}(z)\right] = \frac{(1-z^2)t}{(1-\bar{a}z)(1-\bar{b}z)} = \frac{t}{(\bar{b}-\bar{a})}\left(\frac{-\bar{a}+z}{1-\bar{a}z} - \frac{-\bar{b}+z}{1-\bar{b}z}\right),$$
$$\Delta_1^0\left[\Phi_{t,a,a}(z)\right] = \frac{(1-z^2)t}{(1-\bar{a}z)^2}.$$

For the fixed $z \in E$ we deal with a functional derived on the class $\tilde{A}_0(E)$. If we fix the function $f(z) \in \tilde{A}_0(E)$, then $\Delta_1^0[f(z)]$ will be an analytic in $E$ function

$$h(z) = (1 - z^2)\frac{f'(z)}{f(z)}.$$

Let us denote

$$\delta_f^0 = \sup_{z \in E} \left|\Delta_1^0[f(z)]\right|.$$

The following theorem holds.

THEOREM 2.5. *If* $\delta_f^0 \leq \delta$, *then* $f(z) \in \tilde{U}(E;\delta)$.

Indeed, it can be easily seen that $\delta_f \leq \delta_f^0 \leq \delta$, and by Theorem 2.1. the function $f(z)$ belongs to $\tilde{U}(E;\delta)$.

THEOREM 2.6. *Let* $h(z)$ *be a function analytic in* $E$ *satisfying the inequality* $|h(z)| \leq \delta, \forall z \in E$. *Then, the function*

$$f(z) = \exp\left\{\int_0^z \frac{h(z)}{1 - z^2} dz\right\}, \quad f(0) = 1,$$

*belongs to the class* $\tilde{U}(E;\delta)$.

Indeed, since

$$\delta_f^0 = \sup_{z \in E} |h(z)| \leq \delta,$$

by Theorem 2.5 we conclude that $f(z) \in \tilde{U}(E;\delta)$.

## 3. A set of values of some functionals

Let us find the set of values of the following two functionals.

THEOREM 3.1. *Let* $z_0 \in E$ *and be fixed. Then, all the values of the functional*

$$\left(1 - |z_0^2|\right)\frac{f'(z_0)}{f(z_0)},$$

*defined on the class* $\tilde{U}(E;\delta)$, *are in the disk* $|w| \leq \delta$, *completely filling it.*

Proof. Since $f(z) \in \tilde{U}(E;\delta)$ then, by Theorem 2.1 the inequality

$$\left(1 - |z_0|^2\right)\left|\frac{f'(z_0)}{f(z_0)}\right| \leq \delta \tag{3.1}$$

holds. By virtue of an arbitrary choice of the function $f(z)$ in the class $\tilde{U}(E;\delta)$ we obtain that all the values of the functional (3.1) are situated in the disk $|w| \leq \delta$. Let now $c$ be an arbitrary complex number subject to the condition $|c| \leq \delta$. Let us take the function

$$\Phi_{t,a,b}(z), \quad t = c, \quad a = -e^{i\Theta_0}, \quad b = e^{i\Theta_0}, \ \Theta_0 = arg\, z_0.$$

By Lemma 2.2, this function belongs to the class $\tilde{U}(E;\delta)$.

Calculations show that

$$\left(1-|z_0|^2\right)\frac{\Phi'_{c,a,b}(z_0)}{\Phi_{c,a,b}(z)}=c.$$

This means that the functional (3.1) assumes any value from the circle $|w|\leq\delta$.

COROLLARY 3.1. *Let $z_0\in E$ be fixed. Then all the values of the functional*

$$\frac{f'(z_0)}{f(z_0)},\quad f(z)\in\tilde{U}(E;\delta)$$

*defined on the class $\tilde{U}(E;\delta)$ are situated in the disk,*

$$|w|\leq\frac{\delta}{1-|z_0|^2}$$

*completely filling it.*

## 4. Some estimates

1. Let us estimate $|\ln f(z)|, |f(z)|, |f'(z)|, |arg\, f(z)|$, where $f(z)\in\tilde{U}(E;\delta)$. We need the following lemma:

LEMMA 4.1. *Let $u(x)$ be a complex-valued function of a real variable $x$, continuous in the interval $[a,b]$. In order to fulfil the equality*

$$\left|\int_a^b u(x)dx\right|=\int_a^b|u(x)|dx, \tag{4.1}$$

*it is necessary and sufficient that all the values of the function $u(x)$ be situated on a segment of the ray $l(\beta)$ going out of the origin of coordinates and inclined to the real axis at a certain angle $\beta$, i.e., that $u(x)=|u(x)|e^{i\beta}$ for any $x\in[a,b]$.*

Proof. Let the equality (4.1) be fulfilled for the function $u(x)$ indicated in the Lemma. Assume that

$$\int_a^b u(x)dx=\left|\int_a^b u(x)dx\right|e^{i\beta}.$$

Then the equality (4.1) can be rewritten in the form

$$\int_a^b \operatorname{Re}\{e^{-i\beta}u(x)\}dx=\int_a^b|e^{i\beta}u(x)|dx. \tag{4.2}$$

In addition,

$$\operatorname{Re}\left\{e^{-i\beta}u(x)\right\}dx\leq|e^{i\beta}u(x)|,\quad\forall x\in[a,b]. \tag{4.3}$$

The functions $\operatorname{Re}\{e^{-i\beta}u(x)\}$ and $|e^{i\beta}u(x)|$ are continuous in the interval $[a,b]$. Therefore, from (4.2) and (4.3) it readily follows that

$$\operatorname{Re}\{e^{-i\beta}u(x)\} = |e^{i\beta}u(x)|, \quad \forall x \in [a,b].$$

In such a case, it can be easily seen that all the values of the function $u(x)$ are situated on the ray $l(\beta)$.

Let all the values of the function $u(x)$ be situated on a certain radius $l(\beta)$. Then, it is clear that

$$\left|\int_a^b u(x)dx\right| = \left|\int_a^b |u(x)|e^{i\beta}dx\right| = \int_a^b |u(x)|dx.$$

THEOREM 4.1. *For any function $f(z) \in \tilde{A}_0(E)$, the estimate*

$$|\ln f(z)| \le \ln \Phi_{\delta_f}(|z|) = \frac{1}{2}\delta_f \ln \frac{1+|z|}{1-|z|}, \quad \forall z \in E \tag{4.4}$$

*holds. The sign of equality in (4.4) for $z = z_0 = r_0 e^{i\gamma_0}$, where $0 < r_0 < 1$ and $0 \le \gamma_0 < 2\pi$, is realized only by the basic functions of the form $\Phi_{t,a,b}(z)$, $a = -e^{i\gamma_0}$, $b = e^{i\gamma_0}$, where $t$ is any complex number that belongs to the class $\tilde{A}_0(E)$.*

Proof. Let $f(z) \in \tilde{A}_0(E)$. For $z = re^{i\gamma}, 0 \le r < 1$, $0 \le \gamma < 2\pi$, we have the equality

$$\frac{\partial}{\partial r}\ln f(z) = \frac{f'(z)}{f(z)}e^{i\gamma}.$$

Therefore,

$$|\ln f(z)| = \left|\int_0^r \Big(\frac{\partial}{\partial}\ln f(re^{i\gamma})\Big)dr\right| = \left|\int_0^r \frac{1}{1-r^2}\Delta_1[f(re^{i\gamma})]dr\right|$$

$$\le \int_0^r \frac{1}{1-r^2}\Big|\Delta_1[f(re^{i\gamma})]\Big|dr.$$

Basing ourselves on the definition of the function $\delta_f$ we get

$$\int_0^r \frac{1}{1-r^2}\Big|\Delta_1[f(re^{i\gamma})]\Big|dr \le \int_0^r \frac{\delta_f}{1-r^2}dr = \frac{1}{2}\delta_f \ln\frac{1+r}{1-r} \tag{4.5}$$

and the inequality (4.4) is established.

Let us return to the problem of the sign of equality in (4.4). Let there be the sign of equality for the function $\varphi(z) \in \tilde{A}_0(E)$ in (4.4) as $z = z_0 = r_0 e^{i\gamma_0} \ne 0$, i.e.,

$$\Big|\ln\varphi(z_0)\Big| = \frac{1}{2}\delta_\varphi \ln\frac{1+|z_0|}{1-|z_0|}. \tag{4.6}$$

Then, we obtain two equalities

$$(4.7)\qquad \left|\int\limits_0^{r_0} \frac{1}{1-r^2}\Delta_1[\varphi(re^{i\gamma_0})]dr\right| = \int\limits_0^{r_0} \frac{1}{1-r^2}\Big|\Delta_1[\varphi(re^{i\gamma_0})]\Big|dr,$$

$$(4.8)\qquad \left|\int\limits_0^{r_0} \frac{1}{1-r^2}\Big|\Delta_1[\varphi(re^{i\gamma})]\Big|dr\right| = \int\limits_0^{r_0} \frac{\delta_\varphi}{1-r^2}dr.$$

By considering (4.5) and (4.8) we get the equality

$$\Big|\Delta_1[\varphi(re^{i\gamma_0})]\Big| = \delta_\varphi, \quad \forall r \in [0, r_0].$$

Taking into account the above equality and applying Lemma 4.1 to (4.7) we have

$$(4.9)\qquad \Big|\Delta_1[\varphi(re^{i\gamma_0})]\Big| = \delta_\varphi e^{i\beta}, \quad \forall r \in [0, r_0],$$

where $\beta$ is a real number. Let us write (4.9) as an equality

$$(4.10)\qquad \frac{\varphi'(z)}{\varphi(z)} = \frac{\delta_\varphi e^{i\beta}}{1 - e^{-2i\gamma_0}z^2}$$

valid for any $z = re^{i\gamma_0}, 0 \le r \le r_0$. By virtue of the analytical character of the functions on the left-hand and right-hand sides of the equality (4.10) we obtain that this equality holds for all $z \in E$. In solving (4.10) relative to $\varphi(z)$ we get

$$\varphi(z) = \left(\frac{1 + e^{-i\gamma_0}z}{1 - e^{-i\gamma_0}z}\right)^{\frac{1}{2}\delta_\varphi e^{i\beta} e^{i\gamma_0}}$$

or

$$(4.11)\qquad \varphi(z) = \Phi_{t,a,b}(z), \quad \text{where } a = -e^{i\gamma_0},\ b = e^{i\gamma_0},\ t = \delta_\varphi e^{i\beta}.$$

Thus, if the function $\varphi(z) \in \tilde{A}_0(E)$ realizes the sign of equality in (4.4), then, it is of the form (4.11). We will show that actually any function of the form (4.11) satisfies the condition (4.6). First, we will find $\delta_\varphi$ of the function $\varphi(z) = \Phi_{t,a,b}(z)$. Using Lemma 4.1, we can easily see that

$$\delta_\varphi = \sup_{z\in E}\Big|\Delta_1\Big[\Phi_{t,a,b}(z)\Big]\Big| = |t|.$$

Substituting now the function of the form (4.11) into (4.6) and taking into consideration (4.9), we obtain

$$|\ln \Phi_{t,a,b}(z_0)| = \left|\ln\left(\frac{1+r_0}{1-r_0}\right)^{\frac{1}{2}\delta_\varphi e^{i\beta} e^{i\gamma_0}}\right| = \frac{1}{2}|\delta_\varphi| \ln\frac{1+r_0}{1-r_0} = \frac{1}{2}|t| \ln\frac{1+|z_0|}{1-|z_0|}.$$

Hence, it follows that any function of the form (4.11) satisfies the condition (4.6). Thus the functions $\Phi_{t,a,b}(z)$ indicated in the theorem, where $a = -e^{i\gamma_0}, b = e^{i\gamma_0}$, and $t$ – is any complex number, are the only functions which realize the sign of equality in (4.4). These functions belong to the class $\tilde{A}_0(E)$. Note also that, if the sign of equality in (4.4) is realized at some point $z_0 \neq 0$, then it is realized at all the points of the radius of the circle $E$ passing through the point $z_0$.

COROLLARY 4.1. *For any function $f(z)$ belonging to the simple class $\tilde{\Pi}(E;\delta)$ the inequality $|\ln f(z)| \leq \ln \Phi_\delta(|z|)$, $\forall z \in E$, holds.*

COROLLARY 4.2. *For any function $f(z) \in \tilde{\Im}(E;\delta)$, the inequality $|\ln f(z)| \leq \ln \Phi_\delta(|z|)$, $\forall z \in E$, holds.*

COROLLARY 4.3. *For any function $f(z) \in \tilde{U}(E;\delta)$, the inequality $|\ln f(z)| \leq \ln \Phi_\delta(|z|)$, $\forall z \in E$, holds. As $z = z_0 = r_0 e^{i\gamma_0}$, where $0 < r_0 < 1$ and $0 \leq \gamma_0 < 2\pi$, the sign of equality is realized only by the basic functions $\Phi_{t,a,b}(z)$, where $a = -e^{i\gamma_0}, b = e^{i\gamma_0}, t = \delta e^{i\Theta}, \Theta \in [0, 2\pi]$, that belong to the class $\tilde{U}(E;\delta)$.*

THEOREM 4.2. *For the module of any function $f(z) \in \tilde{A}_0(E)$, the inequalities*

$$\Phi_{-\delta_f}(|z|) \leq |f(z)| \leq \Phi_{\delta_f}(|z|), \quad \forall z \in E, \tag{4.12}$$

*hold, i.e.,*

$$\left(\frac{1-|z|}{1+|z|}\right)^{\frac{1}{2}\delta_f} \leq |f(z)| \leq \left(\frac{1+|z|}{1-|z|}\right)^{\frac{1}{2}\delta_f}.$$

*The signs of equality in (4.12), for $z = z_0 = r_0 e^{i\gamma_0}$, where $0 < r_0 < 1$ and $0 \leq \gamma_0 < 2\pi$, are realized only by the basic functions*

$$\Phi_{t,a,b}(z), \quad t = \pm|t|e^{-i\gamma_0}, \quad a = -e^{i\gamma_0}, \quad b = e^{i\gamma_0},$$

*that belong to the class $\tilde{A}_0(E)$.*

Proof. Obviously, $|\ln|f(z)|| \leq |\ln f(z)|$, $\forall z \in E$. Hence, according to Theorem 4.1, we have the inequality $|\ln|f(z)|| \leq \ln \Phi_{\delta_f}(|z|)$, $\forall z \in E$ or

$$-\ln \Phi_{\delta_f}(|z|) \leq \ln|f(z)| \leq \ln \Phi_{\delta_f}(|z|), \quad \forall z \in E.$$

However, it can be easily seen that $-\ln \Phi_{\delta_f}(|z|) = \ln \Phi_{-\delta_f}(|z|)$. It means, that the latter double inequality is equivalent to double inequality (4.12) indicated in given Theorem.

Let us now go back to the problem of the sign of equality in (4.12). Let, on the right-hand side of (4.12), for $z = z_0 = r_0 e^{i\gamma_0}$, with $0 < r_0 < 1$ and $0 \leq \gamma_0 < 2\pi$, there hold the sign of equality for some function $\varphi(z) \in \tilde{A}_0(E)$, i.e.

$$\ln|\varphi(z_0)| = \ln \Phi_{\delta_\varphi}(|z_0|). \tag{4.13}$$

Taking into consideration (4.4) and (4.13), we obtain

$$\ln|\varphi(z_0)| = \ln \Phi_{\delta_\varphi}(|z_0|). \tag{4.14}$$

Since the function $\varphi(z)$belong to $\tilde{A}_0(E)$ and satisfies (4.14), according to Theorem 4.1 it is of the form

$$\varphi(z) = \Phi_{t,a,b}(z), \quad \text{where } a = -e^{i\gamma_0},\ b = e^{i\gamma_0}. \tag{4.15}$$

Consequently, $\ln\left|\Phi_{t,a,b}(z_0)\right| = \left|\ln \Phi_{t,a,b}(z_0)\right|$. Hence,

$$\arg \Phi_{t,a,b}(z_0) = \arg\left(\frac{1+r_0}{1-r_0}\right)^{\frac{1}{2}te^{i\gamma_0}} = 0$$

and therefore $te^{i\gamma_0} = \pm|t|$, while $te^{i\gamma_0} = \pm|t|e^{-i\gamma_0}$. It means, that in (4.15) we take only those functions for which $te^{i\gamma_0} = \pm|t|e^{-i\gamma_0}$. These are the functions considered in Theorem 4.2.

Based on (4.13) we can conclude that only the functions $\Phi_{t,a,b}(z)$, where $t = |t|e^{-i\gamma_0}, a = -e^{i\gamma_0}, b = e^{i\gamma_0}$, realize the sign of equality on the right-hand side of (4.12), when as $z = z_0$ and belong to the class $\tilde{A}_0(E)$.

In a similar way, we get convinced that only the functions $\Phi_{t,a,b}(z)$, with $t = |t|e^{-i\gamma_0}, a = -e^{i\gamma_0}, b = e^{i\gamma_0}$, realize the sign of equality on the left-hand side of (4.12) when $z = z_0$ and belong to class $\tilde{A}_0(E)$.

Note also that, if the sign of equality is realized at the point $z_0 \neq 0$, then it is realized at all the points of the radius of the unit circle $E$ crossing this point.

COROLLARY 4.4. *For the modulus of any function $f(z)$ belonging to the simple class $\tilde{\Pi}(E;\delta)$, the estimates $\Phi_{-\delta}(|z|) \leq |f(z)| \leq \Phi_\delta(|z|)$, $\forall z \in E$ hold.*

COROLLARY 4.5. *For any function $f(z) \in \tilde{\Im}(E;\delta)$, the inequality*

$$\Phi_{-\delta}(|z|) \leq |f(z)| \leq \Phi_\delta(|z|), \quad \forall z \in E \tag{4.16}$$

*holds.*

COROLLARY 4.6. *For any function $f(z) \in \tilde{U}(E;\delta)$, the inequality*

$$\Phi_{-\delta}(|z|) \leq |f(z)| \leq \Phi_\delta(|z|), \quad \forall z \in E \tag{4.17}$$

*holds. The sign of equality for $z = z_0 = r_0 e^{i\gamma_0}$, where $0 < r_0 < 1$ and $0 \leq \gamma_0 < 2\pi$ is realized only by the basic functions*

$$\Phi_{t,a,b}(z), \quad a = -e^{i\gamma_0}, \quad b = e^{i\gamma_0}, \quad t = \pm\delta e^{-i\gamma_0},$$

*that belong to the class $\tilde{U}(E;\delta)$.*

THEOREM 4.3. *For the modulus of the argument of any function $f(z) \in \tilde{A}_0(E)$ the following inequality holds:*

$$\left|\arg f(z)\right| \leq \Phi_{\delta_f}(|z|) = \frac{\delta_f}{2}\ln\frac{1+|z|}{1-|z|}, \quad \forall z \in E. \tag{4.18}$$

*The sign of equality for* $z = z_0 = r_0 e^{i\gamma_0}$, *where* $0 < r_0 < 1$ *and* $0 \le \gamma_0 < 2\pi$ *is realized only by the basic functions*

$$\Phi_{t,a,b}(z), \quad \text{where } t = \pm i|t|e^{-i\gamma_0},\ a = -e^{i\gamma_0},\ b = e^{i\gamma_0}, \tag{4.19}$$

*belonging to class* $\tilde{A}_0(E)$.

Proof. According to Theorem 4.1, we have a double inequality

$$-\frac{1}{2}\delta_f \ln\frac{1+|z|}{1-|z|} \le \arg f(z) \le \frac{1}{2}\delta_f \ln\frac{1+|z|}{1-|z|}, \quad \forall z \in E, \tag{4.20}$$

equivalent to the inequality (4.18) considered in the given Theorem.

Let us return to the problem of the sign of equality in (4.18). Let, on the right-hand side of (4.20), for $z = z_0 = r_0 e^{i\gamma_0}$, with $0 < r_0 < 1$ and $0 \le \gamma_0 < 2\pi$, there hold the sign of equality for some function $\Psi(z) \in \tilde{A}_0(E)$, i.e.,

$$\arg\Psi(z_0) = \frac{1}{2}\delta_\Psi \frac{1+|z_0|}{1-|z_0|}. \tag{4.21}$$

Considering (4.4) and (4.21) we obtain $\ln\Big|\Psi(z_0)\Big| = 0$ and therefore,

$$\ln|\Psi(z_0)| = \frac{1}{2}\delta_\psi \ln\frac{1+|z_0|}{1-|z_0|}. \tag{4.22}$$

By Theorem 4.1, the function $\Psi(z)$ of the class $\tilde{A}_0(E)$ satisfying (4.22) is of the shape $\Psi(z) = \Phi_{t,a,b}(z)$, where $a = -e^{i\gamma_0}, b = e^{i\gamma_0}$. It has been established above that

$$\ln\Big|\Psi(z_0)\Big| = \ln\Psi_{t,a,b}(z_0) = \ln\left(\frac{1+r_0}{1-r_0}\right)^{\frac{1}{2}te^{i\gamma_0}} = 0.$$

Hence, it follows that $te^{i\gamma_0} = \pm|t|i$ or $t = \pm i|t|e^{-i\gamma_0}$. Let us now consider the functions (4.19). It is easy to verify that these functions belong to the class $\tilde{A}_0(E)$ and realize the sign of equality in (4.18), when $z = z_0$. Thus, it has been established that only the functions indicated in Theorem 4.3 realize the sign of equality in (4.18) and belong to the class $\tilde{A}_0(E)$. Moreover, the functions $\Phi_{t,a,b}(z)$, with $t = i|t|e^{-i\gamma_0}, a = -e^{i\gamma_0}, b = e^{i\gamma_0}$, are the only functions that realize the sign of equality on the right-hand of (4.20), when $z = z_0$, while the functions $\Phi_{t,a,b}(z)$, with $t = -i|t|e^{-i\gamma_0}, a = -e^{i\gamma_0}, b = e^{i\gamma_0}$, are the only functions realizing the sign of equality on the left-hand of (4.20), when $z = z_0$.

Note also that, if the sign of equality is realized at the point $z_0 \ne 0$ then it is realized at all the points of the radius of the unit circle $E$ passing through this point.

COROLLARY 4.7. *For the modulus of the argument of any function $f(z)$ belonging to the simple class $\tilde{\Pi}(E;\delta)$, the estimate*

$$\left|\arg f(z)\right| \le \Phi_\delta(|z|) = \frac{\delta}{2}\ln\frac{1+|z|}{1-|z|}, \quad \forall z \in E$$

*holds.*

COROLLARY 4.8. *For the modulus of the argument of any function $f(z) \in \tilde{\mathfrak{I}}(E;\delta)$ the estimates*

$$\left|\arg f(z)\right| \le \Phi_\delta(|z|) = \frac{\delta}{2}\ln\frac{1+|z|}{1-|z|}, \quad \forall z \in E$$

*are valid.*

COROLLARY 4.9. *For the modulus of the argument of any function $f(z) \in \tilde{U}(E;\delta)$ the estimates*

$$\left|\arg f(z)\right| \le \Phi_\delta(|z|) = \frac{\delta}{2}\ln\frac{1+|z|}{1-|z|}, \quad \forall z \in E$$

*hold. When $z = z_0 = r_0 e^{i\gamma_0}$, with $0 < r_0 < 1$ and $0 \le \gamma_0 < 2\pi$, the sign of equality is realized only by the basic functions $\Phi_{t,a,b}(z)$, with $t = \pm i\delta e^{i\gamma_0}, a = -e^{i\gamma_0}, b = e^{i\gamma_0}$, that belong to the class $\tilde{U}(E;\delta)$.*

THEOREM 4.4. *For the modulus of the derivative of any function $f(z) \in \tilde{A}_0(E)$ the inequalities*

$$0 \le |f'(z)| \le \frac{\delta_f}{1-|z|^2}\Phi_{\delta_f}(|z|), \quad \forall z \in E, \tag{4.23}$$

*are valid.*

The sign of equality on the left-hand side of (4.23) is realized by the function $f(z) \equiv 1$ belonging to the class $\tilde{A}_0(E)$. On the right-hand side of (4.23), for $z = z_0 = r_0 e^{i\gamma_0}$, with $0 < r_0 < 1$ and $0 \le \gamma_0 < 2\pi$, the sign of equality is realized only by the basic functions

$$\Phi_{t,a,b}(z), \quad \text{where } t = |t|e^{-i\gamma_0}, a = -e^{i\gamma_0}, b = e^{i\gamma_0}, \tag{4.24}$$

that belong to the class $\tilde{A}_0(E)$.

Proof. Since $f(z) \in \tilde{A}_0(E)$, by Theorem 4.1

$$\left|f'(z)\right| \le \frac{\delta_f}{1-|z|^2}|f(z)|, \quad \forall z \in E, \tag{4.25}$$

and by Theorem 4.2

$$\left|f(z)\right| \le \left(\frac{1+|z|}{1-|z|}\right)^{\frac{1}{2}\delta_f}, \quad \forall z \in E. \tag{4.26}$$

Hence, (4.23) follows. Obviously, the function $f(z \equiv 1)$ belongs to the class $\tilde{A}_0(E)$ and realizes the sign of equality on the left-hand side of (4.23), with any value of $z \in E$. Let now the sign of equality hold on the right-hand side of (4.23), when $z = z_0 = r_0 e^{i\gamma_0}$, with $0 < r_0 < 1$ and $0 \le \gamma_0 < 2\pi$, i.e.

$$\left|f'(z_0)\right| = \frac{\delta_f}{1-|z_0|^2}\left(\frac{1+|z_0|}{1-|z_0|}\right)^{\frac{1}{2}\delta_f}. \tag{4.27}$$

From (4.25) and (4.27) we get that

$$\left|f(z_0)\right| \ge \frac{1-|z_0|^2}{\delta_f}\left|f'(z_0)\right|\left(\frac{1+|z_0|}{1-|z_0|}\right)^{\frac{1}{2}\delta_f}.$$

In view of the inequality (4.26), we obtain

$$\left|f(z_0)\right| = \left(\frac{1+|z_0|}{1-|z_0|}\right)^{\frac{1}{2}\delta_f}.$$

Now by Theorem 4.2 we arrive at the basic functions of (4.24). These functions belong to the class $\tilde{A}_0(E)$ and only they realize the sign of equality in (4.23) when $z = z_0$.

Note, also, that if the sign of equality in (4.23) is realized at the point $z_0 \neq 0$, then it is realized at all the points of the radius of the unit circle $E$ as above this point.

COROLLARY 4.10. *For the modulus of the derivative of any function $f(z)$ belonging to the simple class $\tilde{\Pi}(E;\delta)$ the estimates*

$$0 \le |f'(z)| \le \frac{\delta}{1-|z|^2}\Phi_\delta(|z|), \quad \forall z \in E$$

*are valid.*

COROLLARY 4.11. *For the modulus of the derivative of any function $f(z) \in \tilde{\mathfrak{S}}(E;\delta)$ the estimates*

$$0 \le |f'(z)| \le \frac{\delta}{1-|z|^2}\Phi_\delta(|z|), \quad \forall z \in E$$

*are valid*

COROLLARY 4.12. *For the modulus of the derivative of any function $f(z) \in \tilde{U}(E;\delta)$ the estimates*

$$0 \le |f'(z)| \le \frac{\delta}{1-|z|^2}\Phi_\delta(|z|), \quad \forall z \in E,$$

*are valid. The sign of equality for $z = z_0 = r_0 e^{i\gamma_0}$, where $0 < r_0 < 1$ and $0 \le \gamma_0 < 2\pi$ is realized only by the basic functions*

$$\Phi_{t,a,b}(z), \quad t = \pm i\delta e^{i\gamma_0}, \ a = -e^{i\gamma_0}, \ b = e^{i\gamma_0},$$

*that belong to class $\tilde{U}(E;\delta)$.*

THEOREM 4.5. *For any function $f(z) \in \tilde{A}_0(E)$ the inequalities*

$$\ln \frac{\left|f(z_1)\right|}{\left|f(z_2)\right|} \leq \frac{\delta_f}{2} \ln \frac{1+\left|\frac{z_2-z_1}{1-\overline{z}_1 z_2}\right|}{1-\left|\frac{z_2-z_1}{1-\overline{z}_1 z_2}\right|}, \quad \forall z_1, z_2 \in E, \tag{4.28}$$

$$\left|\arg \frac{f(z_2)}{f(z_1)}\right| \leq \frac{\delta_f}{2} \ln \frac{1+\left|\frac{z_2-z_1}{1-\overline{z}_1 z_2}\right|}{1-\left|\frac{z_2-z_1}{1-\overline{z}_1 z_2}\right|}, \quad \forall z_1, z_2 \in E, \tag{4.29}$$

*hold.*

Proof. Let

$$z_3 = \frac{z_2 - z_1}{1 - \overline{z}_1 z_2},$$

then

$$f(z_3; z_1) = \frac{f(z_2)}{f(z_1)}.$$

Applying the inequality (4.4) to the function $f(z; z_1)$ belonging to the class $\tilde{A}_0(E)$ as $z = z_3$, we obtain

$$\left|\ln f(z_3; z_1)\right| \leq \frac{\delta_f}{2} \ln \frac{1+|z_3|}{1-|z_3|}.$$

From this inequalities (4.28), (4.29) follows $\ln\left|f(z_3)\right| \leq s_1\eta(s_1) + s_2\eta(s_2)$.

2. Let $s$ be a positive number. We define a quantity $\eta(s)$ as follows:

$$\eta(s) = \sup_{f(z)\in\tilde{U}(E;\delta)} \max_{|z|=\tanh s} \frac{1}{s} \ln\left|f(z)\right|,$$

where

$$\tanh s = \frac{e^s - e^{-s}}{e^s + e^{-s}}.$$

THEOREM 4.6. *The following propositions hold:*

*1) For any $s_1 > 0$ and $s_2 > 0$ the inequality*

$$(s_1 + s_2)\eta(s_1 + s_2) \leq s_1\eta(s_1) + s_2\eta(s_2) \tag{4.30}$$

*are valid.*

*2) For any $s > 0$ the inequality*

$$\eta(s) \leq \delta \tag{4.31}$$

*holds.*

*3) If $\Phi_\delta(z) \in \tilde{\Im}(E;\delta)$ then $\eta(s) = \delta, \forall s > 0$.*

*4) If the function $f(z) \in \tilde{\Im}(E;\delta)$ and is such that $\eta(s_1) = \delta$, for which $s_1 > 0$, then $\eta(s) = \delta$ for any $s > 0$.*

Proof. Since $f(z) \in \tilde{\Im}(E;\delta)$, we have $f(z;\zeta) \in \tilde{\Im}(E;\delta)$ for any $\zeta \in E$. Then, it can be easily seen that

$$\ln\left|f(z_3)\right| = \ln\left|f(z_1;z_2)\right| + \ln\left|f(z_1)\right|, \quad \forall z_1, z_2, z_3 \in E,$$

where $z_3 = \dfrac{z_1+z_2}{1+\overline{z}_2 z_1}$.

Let $z_1 = r_1 e^{i\Theta}, z_2 = r_2 e^{i\Theta}, z_3 = r_3 e^{i\Theta}$, where $r_1 = \tanh s_1$, $r_2 = \tanh s_2$, $r_3 = \tanh s_3$, $0 \le \Theta < 2\pi$.

Then $\ln\left|f(z_3)\right| \le s_1\eta(s_1) + s_2\eta(s_2)$. The right-hand side of the latter inequality does not depend on the choice of the function in the class $\tilde{\Im}(E;\delta)$ or on $\Theta \in [0, 2\pi)$. Consequently,

$$\sup_{f(z)\in\tilde{\Im}(E;\delta)} \max_{|z|=\tanh(s_1+s_2)} \ln|f(z)| = (s_1+s_2)\eta(s_1+s_2) \le s_1\eta(s_1) + s_2\eta(s_2)$$

and the inequality (4.30) is proved. Let us prove the validity of the inequality (4.31). Indeed, if the function $f(z)$ belong $\tilde{\Im}(E;\delta)$, then $f(z) \in \tilde{U}(E;\delta)$. Therefore, for such a function $f(z)$, Theorem 4.2 holds, by which we obtain the inequality

$$\ln\left|f(z)\right| \le \frac{\delta}{2}\ln\frac{1+|z|}{1-|z|}, \quad \forall z \in E.$$

Assuming $|z| = \tanh s$ in the above inequality and dividing its both sides by $s$, we get $\ln|f(z)| = \delta s$. Hence, (4.31) follows. Now, suppose that $\Phi_\delta(z) \in \tilde{\Im}(E;\delta)$. In this case, calculations show that

$$\max_{|z|=\tanh s} \frac{1}{s}\ln\left|\Phi_\delta(z)\right| = \delta, \quad \forall s > 0.$$

Relying on (4.31), we obtain $\eta(s) = \delta$ for any $s > 0$. Now, let $f(z) \in \tilde{\Im}(E;\delta)$ and $\eta(s_0) = \delta$ for some $s_0 > 0$. It means that there exists $z_0 = r_0 e^{i\gamma_0}$ corresponding to the number $s_0$, at which $s_0\eta(s_0) = \ln\left|f(z_0)\right|$. Moreover, $2s_0 = \ln\frac{1+|z_0|}{1-|z_0|}$. Consequently,

$$\left|f(z)\right| = \left(\frac{1+|z_0|}{1-|z_0|}\right)^{\frac{\delta}{2}}.$$

According to Theorem 4.2 we have $f(z) \equiv \Phi_{t,a,b}(z)$ where $a = -e^{i\gamma_0}, b = e^{i\gamma_0}, t = \delta e^{-i\gamma_0}$. Besides, the function $f(z) = \Phi_{t,a,b}(z) \in \tilde{U}(E;\delta)$ and

$$\max_{|z|=\tanh s} \frac{1}{s}\ln\left|\Phi_{t,a,b}(z)\right| = \delta, \quad \forall s > 0.$$

Basing ourselves on (4.31) we obtain $\eta(s) = \delta$ for any $s > 0$. Theorem 2.6 thereby is completely proved.

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