DEMONSTRATIO MATHEMATICA
Vol. XXXVII No 4 2004

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PUSHOUTS OF PARTIAL HOMOMORPHISMS OF PARTIAL ALGEBRAS III: CLOSED-DOMAIN CLOSED QUOMORPHISMS

1. Introduction

This paper is the third in a series initiated with [4], [5] and concluded with [2] devoted to the study of the existence of pushouts in several categories of partial and total algebras over an arbitrary signature, with different types of (total and partial) homomorphisms as morphisms.

Our work belongs to the line of research that aims at an encyclopedic study of several categories of partial algebras, initiated by P. Burmeister and B. Wojdyło in [10–12]; other papers in this area are [3, 14–16]. So far, all results obtained in this study had consisted of characterizations of the signatures for which a certain category-theoretical construction relative to a certain type of homomorphisms always exists. In our work we address a further aspect of this kind of problems: when a construction, say a pushout, is known not to exist for all, say, pairs of morphisms of a given type, for which such pairs does it exist? Our choice of pushouts as the object of study comes from their applications in the field of algebraic transformation in programming theory.

The algebraic transformation of structures is a computation paradigm that was invented in the early seventies as a common generalization of string grammars and term rewriting systems to the transformation of multidimensional structures. One of the most popular algebraic approaches to transformation is the so-called *single-pushout* (SPO) approach, introduced by M. Löwe in [13]. Its name comes from the fact that the basic rewriting step is modelled by means of a pushout in a category of partial morphisms.

During the last years, there has been an increasing interest in the algebraic transformation of partial algebras over an arbitrary signature, because of the possibility of using it to specify software systems with complex states

2000 *Mathematics Subject Classification*: 08A55, 08A70, 68N30

and dynamic behavior. Unfortunately, as far as the SPO approach goes, the categories of algebras with some type of partial homomorphisms as morphisms that appear to be of interest in this field have never all pushouts or, in the best cases, they have all pushouts only when the signature contains only nullary and unary operation symbols. Thus, in order to use an SPO approach to the transformation of algebras in this context, one must find first an as general as possible application condition that answers the question: when can a given production rule be applied through a given morphism? In other words, it is necessary to characterize those pairs of morphisms having a pushout in the category where one plans to work.

Such an application condition was established by A. Wagner and M. Gogolla in their seminal paper on the SPO transformation of arbitrary partial algebras [17] for *quomorphisms* (homomorphisms defined on relative subalgebras of the source algebra), but only for partial algebras without nullary operations. In [4] we generalized their work to quomorphisms of partial algebras over an arbitrary signature, and we also established the application condition for *closed-domain quomorphisms*, while in [5] we studied another special type of quomorphisms, the *closed quomorphisms* (*closed* homomorphisms defined on relative subalgebras of the source algebra) and we also characterized the pairs of closed homomorphisms of partial algebras that have a pushout in the corresponding category.

In this paper we characterize the pairs of *closed-domain closed quomorphisms* of partial algebras that have a pushout in the category of partial Σ-algebras with this kind of morphisms, for an arbitrary signature Σ. Although we only consider in our work pushouts of *pairs* of morphisms, our results generalize in a straightforward way to pushouts of arbitrary non-empty families of morphisms with the same source algebra.

This paper, as well as [2, 4, 5], is based on the first-named author's PhD Thesis [1], although the proofs provided here are different from those given there and somehow more concise.

We assume the reader familiar with the basic language and methods of the theory of (finitary, many-sorted) partial algebras, as introduced for instance in [6, Chap. I] or [7, Parts 1,2]; Appendix A in [9] or Sect. 2 in [5] contain also all the basic notions and results needed to understand this paper. To close this introduction, we want to point out some conventions and notations that will be used throughout this paper without any further notice.

- A *signature* is a triple $\Sigma = (S, \Omega, \eta)$, where S is a non-empty set of *sorts*, Ω is a set of *operation symbols* and $\eta : \Omega \to S^* \times S$ is the *arity mapping*, that sends every $\varphi \in \Omega$ to its *arity* $(\omega(\varphi), \sigma(\varphi)) \in S^* \times S$.

- An operation symbol φ is *n-ary* when the length of the word $\omega(\varphi)$ is n. We shall denote by $\Omega^{(n)}$ the set of n-ary operation symbols in Σ, and by $\Omega^{(+)}$ the set $\Omega - \Omega^{(0)}$ of non-nullary operation symbols.
- Unless otherwise stated, whenever we denote an algebra by a capital letter in boldface type $(\mathbf{A}, \mathbf{B}, \ldots)$, we shall denote its carrier set by the same letter, but in slanted type $(A, B, \ldots)$, the realization of an operation in it by adding the algebra's name as a superscript to the operation symbol $(\varphi^{\mathbf{A}}, \psi^{\mathbf{B}}, \ldots)$, and, if necessary, its carrier of a given sort by the same capital letter in slanted type, but with the sort as a subscript $(A_s, B_t \ldots)$. Nevertheless, and in order to lighten the notations, whenever there is no danger of confusion we shall skip all subscripts corresponding to sorts in the names of the carriers of the algebras, the components of the homomorphisms or the congruences, etc.
- Given an S-set $A = (A_s)_{s\in S}$, let $A^{\lambda} = \{\emptyset\}$ (where the symbol λ stands for the empty word) and $A^{s_1\cdots s_p} = A_{s_1} \times \cdots \times A_{s_p}$ for every $s_1 \ldots s_p \in S^+$.
- We shall denote by $\Sigma^{(+)} = (\Omega^{(+)}, \eta|_{\Omega^{(+)}})$ the signature obtained from Σ by removing its nullary operation symbols. If $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ is a partial Σ-algebra, then $\mathbf{A}^{(+)} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega^{(+)}})$ will always stand for its $\Sigma^{(+)}$*-reduct*: the partial $\Sigma^{(+)}$-algebra obtained by simply omitting the nullary operations in $\mathbf{A}$.
- An operation $\varphi^{\mathbf{A}}$ in an algebra $\mathbf{A}$ is *total* when it is a total mapping, and *discrete* when it is the empty mapping. When a nullary operation in an algebra $\mathbf{A}$ is total, we say that it is *defined* in $\mathbf{A}$, and we identify it with its image. A partial Σ-algebra is *discrete* when all operations in it are so.
- Given an equivalence relation θ on an S-set A, we shall always denote by $nat_\theta : A \to A/\theta$ the corresponding quotient mapping of S-sets.

2. Preliminaries

To ease the task of the reader, in this section we gather some results that will be used later. The notations introduced in this section will also be used in the next section, usually without any further notice.

2.1. Pushouts of homomorphisms

Let $\Sigma = (S, \Omega, \eta)$ be a signature. It is well known that the category Alg_Σ of all partial Σ-algebras with homomorphisms as morphisms has all pushouts: see [6, §4.3] or [12]. Let us recall a construction of the pushout of two homomorphisms in Alg_Σ.

Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two homomorphisms of partial Σ-algebras. The *disjoint sum* of the $\Sigma^{(+)}$-reducts $\mathbf{A}^{(+)}$ and $\mathbf{B}^{(+)}$ of $\mathbf{A}$ and

$\mathbf{B}$ is the partial $\Sigma^{(+)}$-algebra

$$\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)} = (A \sqcup B, (\varphi^{\mathbf{A}^{(+)}\oplus\mathbf{B}^{(+)}})_{\varphi\in\Omega^{(+)}})$$

with carrier set the disjoint union[1] $A \sqcup B$ of their carrier sets A and B, and with operations defined in the following way: for every $\varphi \in \Omega^{(+)}$,

$$\operatorname{dom}\varphi^{\mathbf{A}^{(+)}\oplus\mathbf{B}^{(+)}} = \operatorname{dom}\varphi^{\mathbf{A}} \sqcup \operatorname{dom}\varphi^{\mathbf{B}}$$

and if $\underline{a} \in \operatorname{dom}\varphi^{\mathbf{A}}$ (resp., $\underline{b} \in \operatorname{dom}\varphi^{\mathbf{B}}$), then $\varphi^{\mathbf{A}^{(+)}\oplus\mathbf{B}^{(+)}}(\underline{a}) = \varphi^{\mathbf{A}}(\underline{a})$ (resp., $\varphi^{\mathbf{A}^{(+)}\oplus\mathbf{B}^{(+)}}(\underline{b}) = \varphi^{\mathbf{B}}(\underline{b})$).

Let now $\theta(f,g)$ be the congruence on $\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}$ generated by the relation $E_{f,g} = E(f,g) \cup X_0$, where

$$E(f,g) = \{(f(x), g(x)) \mid x \in K\},$$
$$X_0 = \{(\varphi_0^{\mathbf{A}}, \varphi_0^{\mathbf{B}}) \mid \varphi_0 \in \Omega^{(0)},\ \varphi_0^{\mathbf{A}}, \varphi_0^{\mathbf{B}} \text{ defined}\}.$$

Let $\mathbf{P}$ be the partial Σ-algebra whose $\Sigma^{(+)}$-reduct is the quotient

$$(\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)})/\theta(f,g)$$

and whose nullary operations are defined as follows: for every $\varphi_0 \in \Omega^{(0)}$, $\varphi_0^{\mathbf{P}}$ is defined if and only if $\varphi_0^{\mathbf{A}}$ or $\varphi_0^{\mathbf{B}}$ are defined, and if $\varphi_0^{\mathbf{A}}$ or $\varphi_0^{\mathbf{B}}$ are defined, then $\varphi_0^{\mathbf{P}}$ is the corresponding equivalence class modulo $\theta(f,g)$. Let finally $\tilde{g} : \mathbf{A} \to \mathbf{P}$ and $\tilde{f} : \mathbf{B} \to \mathbf{P}$ be the homomorphisms given by the restrictions to $\mathbf{A}$ and $\mathbf{B}$ of the quotient mapping $nat_{\theta(f,g)} : A \sqcup B \to (A \sqcup B)/\theta(f,g)$.

THEOREM 1. *With the previous notations, the cocone*

$$(\mathbf{P}, \tilde{g} : \mathbf{A} \to \mathbf{P}, \tilde{f} : \mathbf{B} \to \mathbf{P})$$

is a pushout of f and g in Alg_Σ. ∎

On the other hand, Burmeister and Wojdyło proved in [12] that the category C-Alg_Σ of all partial Σ-algebras with closed homomorphisms as morphisms has all pushouts only when $\Omega = \Omega^{(0)} \cup \Omega^{(1)}$ (actually, they proved it for one-sorted signatures, but their proof can be easily generalized to arbitrary, many-sorted signatures). In [5] we characterized the pairs of closed homomorphisms that have a pushout in C-Alg_Σ, for an arbitrary signature Σ. This characterization, which we recall below, uses the following notion.

DEFINITION 1. Let $\mathbf{A}$ and $\mathbf{B}$ be two partial Σ-algebras and let $\mathbf{C}$ be a relative subalgebra of $\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}$, with carrier set $C \subseteq A \sqcup B$.

A congruence θ on $\mathbf{C}$ is *separately closed* when it satisfies the following condition: for every $\varphi \in \Omega^{(+)}$, say with $\omega(\varphi) = s_1 \dots s_p$, if

$$\underline{x} = (x_1, \dots, x_p),\ \underline{y} = (y_1, \dots, y_p) \in (C \cap A)^{\omega(\varphi)} \cup (C \cap B)^{\omega(\varphi)}$$

[1] Formally, we define $A \sqcup B$ as $A \times \{1\} \cup B \times \{2\}$, but in order to simplify the notations we shall identify A and B with their images in $A \sqcup B$.

are such that $(x_i, y_i) \in \theta$, for every $i = 1, \ldots, p$, and $\underline{x} \in \operatorname{dom} \varphi^{\mathbf{C}}$, then $\underline{y} \in \operatorname{dom} \varphi^{\mathbf{C}}$.

THEOREM 2. *Two closed homomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ of partial Σ-algebras have a pushout in* C-Alg$_\Sigma$ *if and only if they satisfy the following two conditions:*

CHPO1) *The congruence $\theta(f, g)$ is separately closed on $\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}$.*
CHPO2) *For every $\varphi \in \Omega^{(+)}$,*

$$((A \sqcup B)/\theta(f,g))^{\omega(\varphi)} = nat_{\theta(f,g)}(A)^{\omega(\varphi)} \cup nat_{\theta(f,g)}(B)^{\omega(\varphi)}.$$

And if f and g satisfy these properties, their pushout in C-Alg$_\Sigma$ *is given by their pushout in* Alg$_\Sigma$. ∎

Notice that if f and g are closed, then $X_0 \subseteq E(f, g)$, and thus $\theta(f, g)$ is generated by $E(f, g)$. On the other hand, by [5, Lem. 3.1], the separate closedness of $\theta(f, g)$ is equivalent to closedness of the homomorphisms $\tilde{f}$ and $\tilde{g}$ in the pushout of f and g in Alg$_\Sigma$ described above.

2.2. Pushouts of different types of quomorphisms

Let $\Sigma = (S, \Omega, \eta)$ be a signature and $\mathbf{A}$ and $\mathbf{B}$ two partial Σ-algebras. A partial mapping $f : A \to B$ between their carrier sets is:

- a *quomorphism* from $\mathbf{A}$ to $\mathbf{B}$ when it is a homomorphism from the relative subalgebra of $\mathbf{A}$ supported on its *domain* $\operatorname{Dom} f$ to $\mathbf{B}$.
- a *closed-domain quomorphism* from $\mathbf{A}$ to $\mathbf{B}$, a *cd-quomorphism* for short, when it is a quomorphism and $\operatorname{Dom} f$ is a closed subset of $\mathbf{A}$.
- a *closed quomorphism* from $\mathbf{A}$ to $\mathbf{B}$, a *c-quomorphism* for short, when it is a closed homomorphism from the relative subalgebra of $\mathbf{A}$ supported on $\operatorname{Dom} f$ to $\mathbf{B}$.
- a *closed-domain closed quomorphism* from $\mathbf{A}$ to $\mathbf{B}$, a *cdc-quomorphism* for short, when it is simultaneously a c-quomorphism and a cd-quomorphism.

The categories of all partial Σ-algebras with quomorphisms, c-quomorphisms, cd-quomorphisms, and cdc-quomorphisms as morphisms will be denoted, respectively, by Q-Alg$_\Sigma$, CQ-Alg$_\Sigma$, CDQ-Alg$_\Sigma$, and CDCQ-Alg$_\Sigma$. The first two categories have been studied in [10, 11, 12], while the other two have been studied in [3]. We refer the interested readers to these papers for the basic information about these partial homomorphisms.

The pairs of quomorphisms, cd-quomorphisms and c-quomorphisms that have a pushout in Q-Alg$_\Sigma$, CDQ-Alg$_\Sigma$ and CQ-Alg$_\Sigma$, respectively, for an arbitrary signature Σ, have already been characterized in our previous papers [4, 5]. We recall in this subsection these characterizations for the last two categories, which are the ones used in the next section. But first, we need to introduce some notations.

Let S be a non-empty set (of sorts). Given two partial mappings $f : K \to A$ and $g : K \to B$ of S-sets:

- let $E(f,g)$ be the relation on the disjoint union $A \sqcup B$ defined by
$$E(f,g) = \{(f(x), g(x)) \mid x \in \operatorname{Dom} f \cap \operatorname{Dom} g\};$$
- let $\theta_e(f,g)'$ be the equivalence on $A \sqcup B$ generated by $E(f,g)$;
- let $D(f,g)$ be the S-subset of $A \sqcup B$ consisting of those $x \in A \sqcup B$ such that, for every $k \in K$, if $k \in \operatorname{Dom} f$ and $(x, f(k)) \in \theta_e(f,g)'$ or $k \in \operatorname{Dom} g$ and $(x, g(k)) \in \theta_e(f,g)'$, then $k \in \operatorname{Dom} f \cap \operatorname{Dom} g$;
- let $\theta_e(f,g)$ be the restriction of $\theta_e(f,g)'$ to $D(f,g)$.

It is straightforward to check that $f^{-1}(D(f,g) \cap A) = g^{-1}(D(f,g) \cap B)$; see [12, Lem. 8]. Notice also that if f and g are total, then $D(f,g) = A \sqcup B$ and the relation $E(f,g)$ is equal to the homonymous relation defined in §2.1.

As far as cd-quomorphisms goes, we shall not recall here the characterization established in [4] as it stands, but an equivalent one with a statement that is more suitable to our purposes.

THEOREM 3. *Two cd-quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ of partial Σ-algebras have a pushout in* CDQ-Alg$_\Sigma$ *if and only if they satisfy the following conditions:*

CDPO1) *There exist the greatest closed subsets A' and B' of $\mathbf{A}$ and $\mathbf{B}$, respectively, such that $f^{-1}(A') = g^{-1}(B')$.*

CDPO2) *Set $K' = f^{-1}(A') = g^{-1}(B')$, let $\mathbf{K}'$, $\mathbf{A}'$ and $\mathbf{B}'$ be the closed subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported on K', A' and B', respectively, and let $f' : \mathbf{K}' \to \mathbf{A}'$ and $g' : \mathbf{K}' \to \mathbf{B}'$ be the homomorphisms between these algebras induced by the cd-quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$. Let $\theta(f', g')$ be the congruence on $\mathbf{A}'^{(+)} \oplus \mathbf{B}'^{(+)}$ (the disjoint sum of the closed subalgebras of $\mathbf{A}^{(+)}$ and $\mathbf{B}^{(+)}$ supported on A' and B', respectively) generated by the relation $E_{f',g'} = E(f', g') \cup X_0$.*
For every closed subsets A_0 and B_0 of $\mathbf{A}'$ and $\mathbf{B}'$, respectively, such that $f^{-1}(A_0) = g^{-1}(B_0)$, if θ_0 is the congruence generated by $E_{f',g'} \cap (A_0 \sqcup B_0)^2$ on the closed subalgebra $\mathbf{A}_0^{(+)} \oplus \mathbf{B}_0^{(+)}$ of $\mathbf{A}'^{(+)} \oplus \mathbf{B}'^{(+)}$, then
$$\theta(f', g') \cap ((A_0 \sqcup B_0) \times (A' \sqcup B')) \subseteq \theta_0.$$

And if f and g satisfy conditions (CDPO1) and (CDPO2), then a pushout in CDQ-Alg$_\Sigma$ *of them can be obtained as follows. Let*
$$(\mathbf{P}, \bar{g} : \mathbf{A}' \to \mathbf{P}, \bar{f} : \mathbf{B}' \to \mathbf{P})$$
be a pushout of $f' : \mathbf{K}' \to \mathbf{A}'$ and $g' : \mathbf{K}' \to \mathbf{B}'$ in Alg$_\Sigma$*, and let $\tilde{g} : \mathbf{A} \to \mathbf{P}$*

and $\tilde{f} : \mathbf{B} \to \mathbf{P}$ *be the cd-quomorphisms from* $\mathbf{A}$ *and* $\mathbf{B}$ *to* $\mathbf{P}$ *defined by the homomorphisms* $\bar{g}$ *and* $\bar{f}$. *Then*

$$(\mathbf{P}, \tilde{g} : \mathbf{A} \to \mathbf{P}, \tilde{f} : \mathbf{B} \to \mathbf{P})$$

is a pushout of f *and* g *in* CDQ-Alg$_\Sigma$.

Let us prove that the condition (CDPO2) given in the last theorem is equivalent to the one given in [4, Thm. 9]. In the next statement, $E^t_{f',g'}$ stands for the inverse relation of $E_{f',g'}$.

PROPOSITION 1. *Let* $f : \mathbf{K} \to \mathbf{A}$ *and* $g : \mathbf{K} \to \mathbf{B}$ *be two cd-quomorphisms of partial* Σ*-algebras satisfying condition (CDPO1) in the previous theorem. Then, with the notations introduced in that theorem,* f *and* g *satisfy condition (CDPO2) if and only if they satisfy the following condition:*

> *For every closed subset* D_0 *of* $\mathbf{A}'^{(+)} \oplus \mathbf{B}'^{(+)}$ *containing all nullary operations defined in* $\mathbf{A}$ *or* $\mathbf{B}$ *and for every congruence* θ_0 *on the closed subalgebra* $\mathbf{D}_0$ *of* $\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}$ *supported on* D_0, *if*
>
> $$(E_{f',g'} \cup E^t_{f',g'}) \cap (D_0 \times (A' \sqcup B')) \subseteq \theta_0,$$
>
> *then*
>
> $$\theta(f', g') \cap (D_0 \times (A' \sqcup B')) \subseteq \theta_0.$$

Proof. To see that the condition given in the statement implies (CDPO2), it is enough to check that if two closed subsets A_0 and B_0 of $\mathbf{A}'$ and $\mathbf{B}'$ are such that $f^{-1}(A_0) = g^{-1}(B_0)$ and if θ_0 is the congruence generated by $E_{f',g'} \cap (A_0 \sqcup B_0)^2$ on the closed subalgebra $\mathbf{A}_0^{(+)} \oplus \mathbf{B}_0^{(+)}$ of $\mathbf{A}'^{(+)} \oplus \mathbf{B}'^{(+)}$ supported on $A_0 \sqcup B_0$, then

$$(E_{f',g'} \cup E^t_{f',g'}) \cap ((A_0 \sqcup B_0) \times (A' \sqcup B')) \subseteq \theta_0,$$

which is straightforward.

Let us prove now that (CDPO2) implies the condition in the statement. Let D_0 be a closed subset of $\mathbf{A}'^{(+)} \oplus \mathbf{B}'^{(+)}$ containing all nullary operations defined in $\mathbf{A}$ or in $\mathbf{B}$, and let θ_0 be a congruence on the closed subalgebra $\mathbf{D}_0$ supported on it such that

$$(E_{f',g'} \cup E^t_{f',g'}) \cap (D_0 \times (A' \sqcup B')) \subseteq \theta_0.$$

Set $A_0 = D_0 \cap A$ and $B_0 = D_0 \cap B$: these are closed subsets of $\mathbf{A}'$ and $\mathbf{B}'$, respectively, and, up to an isomorphism, $\mathbf{D}_0 = \mathbf{A}_0{}^{(+)} \oplus \mathbf{B}_0{}^{(+)}$. Then $f^{-1}(A_0) = g^{-1}(B_0)$. Indeed, if $x \in K$ is such that $f(x) \in A_0$, then $x \in K'$ and

$$(f(x), g(x)) \in E_{f',g'} \cap (D_0 \times (A' \sqcup B')) \subseteq \theta_0 \subseteq D_0 \times D_0,$$

which implies that $g(x) \in B \cap D_0 = B_0$. This proves that $f^{-1}(A_0) \subseteq g^{-1}(B_0)$: the other inclusion holds by symmetry.

Let θ_0' be the congruence on $\mathbf{D}_0$ generated by $E_{f',g'} \cap (D_0 \times D_0)$. Since

$$E_{f',g'} \cap (D_0 \times D_0) \subseteq E_{f',g'} \cap (D_0 \times (A' \sqcup B')) \subseteq \theta_0,$$

we have that $\theta_0' \subseteq \theta_0$. Now, condition (CDPO2) guarantees that

$$\theta(f',g') \cap (D_0 \times (A' \sqcup B')) \subseteq \theta_0',$$

and hence $\theta(f',g') \cap (D_0 \times (A' \sqcup B')) \subseteq \theta_0$, as we wanted to prove. ■

Finally, as far as c-quomorphisms go, in [5] we obtained the following characterization.

THEOREM 4. *Two c-quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ of partial Σ-algebras have a pushout in* CQ-Alg$_\Sigma$ *if and only if they satisfy the following two conditions:*

CQPO1) *The equivalence relation $\theta_e(f,g)$ is a separately closed congruence on the relative subalgebra $\mathbf{D}(f,g)$ of $\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}$ supported on $D(f,g)$.*

CQPO2) *For every $\varphi \in \Omega^{(+)}$,*

$$(D(f,g)/\theta_e(f,g))^{\omega(\varphi)} = nat_{\theta_e(f,g)}(D(f,g) \cap A)^{\omega(\varphi)} \cup nat_{\theta_e(f,g)}(D(f,g) \cap B)^{\omega(\varphi)}.$$

And if $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ are two c-quomorphisms satisfying conditions (CQPO1) and (CQPO2), then a pushout in CQ-Alg$_\Sigma$ *of them can be obtained as follows. Let K_D be the set $f^{-1}(D(f,g) \cap A) = g^{-1}(D(f,g) \cap B)$, let $\mathbf{K}_D$, $\mathbf{A}_D$ and $\mathbf{B}_D$ be the relative subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported on K_D, $D(f,g) \cap A$ and $D(f,g) \cap B$, respectively, and let $f_D : \mathbf{K}_D \to \mathbf{A}_D$ and $g_D : \mathbf{K}_D \to \mathbf{B}_D$ be the closed homomorphisms between these algebras induced by the c-quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$. Let*

$$(\mathbf{P}, \bar{g} : \mathbf{A}_D \to \mathbf{P}, \bar{f} : \mathbf{B}_D \to \mathbf{P})$$

be a pushout of f_D and g_D in Alg$_\Sigma$, *which turns out to be also their pushout in* C-Alg$_\Sigma$ *by Theorem 2, and let $\tilde{g} : \mathbf{A} \to \mathbf{P}$ and $\tilde{f} : \mathbf{B} \to \mathbf{P}$ be the c-quomorphisms from $\mathbf{A}$ and $\mathbf{B}$ to $\mathbf{P}$ defined by the closed homomorphisms $\bar{g}$ and $\bar{f}$. Then*

$$(\mathbf{P}, \tilde{g} : \mathbf{A} \to \mathbf{P}, \tilde{f} : \mathbf{B} \to \mathbf{P})$$

is a pushout of f and g in CQ-Alg$_\Sigma$.

3. Pushouts of cdc-quomorphisms

Let CDCQ-Alg$_\Sigma$ be the category of partial Σ-algebras with cdc-quomorphisms as morphisms. This category was studied in detail in [3] for one-sorted signatures Σ. In particular, it was proved therein that CDCQ-Alg$_\Sigma$ has all pushouts if and only if all operation symbols in Σ are unary: although it was only proved for one-sorted signatures, the proof can be easily generalized to arbitrary signatures. This was used in [8] to develop an SPO approach to the algebraic transformation of unary partial algebras using cdc-quomorphisms.

The goal of this section is to characterize the pairs of cdc-quomorphisms of partial Σ-algebras that have a pushout in CDCQ-Alg$_\Sigma$, for an arbitrary signature Σ. To begin with, we establish a result concerning pushouts in CDCQ-Alg$_\Sigma$ of closed homomorphisms that will be used in the proof of the main theorem.

PROPOSITION 2. *Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two closed homomorphisms of partial Σ-algebras having a pushout in* C-Alg$_\Sigma$. *Then, f and g have a pushout in* CDCQ-Alg$_\Sigma$ *if and only if they have a pushout in* CDQ-Alg$_\Sigma$. *And when f and g have a pushout in all three categories, they are all given by their pushout in* Alg$_\Sigma$.

Proof. Recall on the one hand that, by Theorem 2, the pushout of $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ in C-Alg$_\Sigma$ is given by their pushout in Alg$_\Sigma$

$$(\mathbf{P}, \tilde{g} : \mathbf{A} \to \mathbf{P}, \tilde{f} : \mathbf{B} \to \mathbf{P})$$

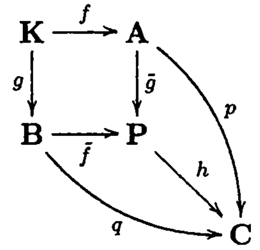
described in Theorem 1. On the other hand, since f and g are totally defined, they always satisfy condition (CDPO1) in Theorem 3, with $A' = A$ and $B' = B$, and then $K' = K$. Moreover, f and g being closed, for every $\varphi_0 \in \Omega^{(0)}$, $\varphi_0^{\mathbf{A}}$ is defined if and only $\varphi_0^{\mathbf{B}}$ is defined (actually, both assertions are equivalent to the fact that $\varphi_0^{\mathbf{K}}$ is defined), and if they are both defined, then

$$(\varphi_0^{\mathbf{A}}, \varphi_0^{\mathbf{B}}) = (f(\varphi_0^{\mathbf{K}}), g(\varphi_0^{\mathbf{K}})) \in E(f,g).$$

This shows that, in this case, the relation $E_{f',g'}$ used in condition (CDPO2) in Theorem 3 is equal to $E(f,g)$, and thus the congruence $\theta(f',g')$ defined in *loc. cit.* is equal to the congruence $\theta(f,g)$ used to construct the pushout of f and g in Alg$_\Sigma$.

$\Longleftarrow$) Assume first that $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ have a pushout in CDQ-Alg$_\Sigma$. Then, by Theorem 3 and the equalities $A' = A$, $B' = B$ and $\theta(f',g') = \theta(f,g)$ pointed out above, this pushout is also given by their pushout $(\mathbf{P}, \tilde{g} : \mathbf{A} \to \mathbf{P}, \tilde{f} : \mathbf{B} \to \mathbf{P})$ in Alg$_\Sigma$ and C-Alg$_\Sigma$. Let us prove that this cocone is also a pushout of them in CDCQ-Alg$_\Sigma$.

Since $\tilde{g}$ and $\tilde{f}$ are closed homomorphisms (because it is a pushout in C-Alg$_\Sigma$), they are in particular cdc-quomorphisms. Thus, it remains to prove that this cocone satisfies the universal property of pushouts in CDCQ-Alg$_\Sigma$. So, let $p : \mathbf{A} \to \mathbf{C}$ and $q : \mathbf{B} \to \mathbf{C}$ be two cdc-quomorphisms such that $p \circ f = q \circ g$. By the universal property of pushouts in CDQ-Alg$_\Sigma$, there exists one, and only one, cd-quomorphism $h : \mathbf{P} \to \mathbf{C}$ such that $h \circ \tilde{g} = p$ and $h \circ \tilde{f} = q$. If we prove that h is closed, we will be done.



Let φ_0 be a nullary operation symbol. If $\varphi_0^{\mathbf{C}}$ is defined, then, since, say, p is a c-quomorphism, $\varphi_0^{\mathbf{A}}$ is also defined and belongs to $\operatorname{Dom} p$, which implies, by the construction of $\mathbf{P}$, that $\varphi_0^{\mathbf{P}}$ is defined and equal to $\tilde{g}(\varphi_0^{\mathbf{A}})$, and then, since $h \circ \tilde{g} = p$, it belongs to $\operatorname{Dom} h$.

Let now $\varphi \in \Omega^{(+)}$ and $\underline{x} \in (\operatorname{Dom} h)^{\omega(\varphi)}$ be such that $h(\underline{x}) \in \operatorname{dom} \varphi^{\mathbf{C}}$. Since f and g satisfy condition (CHPO2) in Theorem 2, we have that $\underline{x} \in \tilde{g}(A)^{\omega(\varphi)}$ or $\underline{x} \in \tilde{f}(B)^{\omega(\varphi)}$. Assume that $\underline{x} = \tilde{g}(\underline{a})$ for some $\underline{a} \in A^{\omega(\varphi)}$: the other case is analogous. Since $\underline{x} \in (\operatorname{Dom} h)^{\omega(\varphi)}$ and $h \circ \tilde{g} = p$, we deduce that $\underline{a} \in (\operatorname{Dom} p)^{\omega(\varphi)}$ and $p(\underline{a}) = h(\underline{x}) \in \operatorname{dom} \varphi^{\mathbf{C}}$, which implies, p being a cdc-quomorphism, that $\underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}}$ and hence $\underline{x} = \tilde{g}(\underline{a}) \in \operatorname{dom} \varphi^{\mathbf{P}}$.

This proves that h is closed, as we wanted.

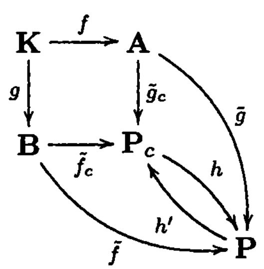
$\Longrightarrow$) Assume that f and g have a pushout

$$(\mathbf{P}_c, \tilde{g}_c : \mathbf{A} \rightharpoonup \mathbf{P}_c, \tilde{g}_c : \mathbf{B} \rightharpoonup \mathbf{P}_c)$$

in CDCQ-Alg$_\Sigma$; we want to prove that they have a pushout in CDQ-Alg$_\Sigma$. To begin with, let us prove that this cocone is, up to isomorphism, the pushout

$$(\mathbf{P}, \tilde{g} : \mathbf{A} \to \mathbf{P}, \tilde{f} : \mathbf{B} \to \mathbf{P})$$

of f and g in Alg$_\Sigma$ and C-Alg$_\Sigma$ given above. Indeed, since $\tilde{g}$ and $\tilde{f}$ are closed homomorphisms, and in particular cdc-quomorphisms, such that $\tilde{g} \circ f = \tilde{f} \circ g$, by the universal property of pushouts in CDCQ-Alg$_\Sigma$ we have that there exists a cdc-quomorphism $h : \mathbf{P}_c \rightharpoonup \mathbf{P}$ such that $\tilde{f} = h \circ \tilde{f}_c$ and $\tilde{g} = h \circ \tilde{g}_c$. Since $\tilde{g}$ and $\tilde{f}$ are homomorphisms, this implies that $\tilde{g}_c$ and $\tilde{f}_c$ are also totally defined, thus closed homomorphisms. We can apply then the universal property of pushouts again, this time in C-Alg$_\Sigma$, to deduce that there exists a closed homomorphism $h' : \mathbf{P} \to \mathbf{P}_c$ such that $h' \circ \tilde{f} = \tilde{f}_c$ and $h' \circ \tilde{g} = \tilde{g}_c$.



Then, $h' \circ h$ is a cdc-quomorphism such that $h' \circ h \circ \tilde{g}_c = \tilde{g}_c$ and $h' \circ h \circ \tilde{f}_c = \tilde{f}_c$. This implies, by the universal property of pushouts in CDCQ-Alg$_\Sigma$, that $h' \circ h = \mathrm{Id}_{\mathbf{P}_c}$. In particular, h is totally defined, and hence a closed homomorphism. Therefore, $h \circ h'$ is a closed homomorphism such that $h \circ h' \circ \tilde{g} = \tilde{g}$ and $h \circ h' \circ \tilde{f} = \tilde{f}$, and then, by the universal property of pushouts in C-Alg$_\Sigma$, $h \circ h' = \mathrm{Id}_{\mathbf{P}}$. This shows that h' and h are isomorphisms inverse to each other.

Thus, in the sequel we take the cocone

$$(\mathbf{P}, \tilde{g} : \mathbf{A} \to \mathbf{P}, \tilde{f} : \mathbf{B} \to \mathbf{P})$$

as the pushout of f and g in CDCQ-Alg$_\Sigma$. Let us prove now that f and g satisfy conditions (CDPO1) and (CDPO2) in Theorem 3, and thus that they have a pushout in CDQ-Alg$_\Sigma$.

As we mentioned at the beginning of this proof, f and g satisfy condition (CDPO1) taking $A' = A$ and $B' = B$. As far as condition (CDPO2) goes, we also saw there that $E_{f',g'} = E(f,g)$ and $\theta(f',g') = \theta(f,g)$, and thus, we have to prove that, for every closed subsets A_0 and B_0 of $\mathbf{A}$ and $\mathbf{B}$, respectively, such that $f^{-1}(A_0) = g^{-1}(B_0)$, if θ_0 is the congruence on $\mathbf{A}_0^{(+)} \oplus \mathbf{B}_0^{(+)}$ generated by $E(f,g) \cap (A_0 \sqcup B_0)^2$, then

$$\theta(f,g) \cap ((A_0 \sqcup B_0) \times (A \sqcup B)) \subseteq \theta_0.$$

Let $\mathbf{C}_0$ be the partial Σ-algebra whose $\Sigma^{(+)}$-reduct is $(\mathbf{A}_0^{(+)} \oplus \mathbf{B}_0^{(+)})/\theta_0$ and where, for each nullary operation symbol φ_0, $\varphi_0^{\mathbf{C}_0}$ is defined if and only if $\varphi_0^{\mathbf{A}}$ and $\varphi_0^{\mathbf{B}}$ are defined, and when they are defined $\varphi_0^{\mathbf{C}_0} = [\varphi_0^{\mathbf{A}}]_{\theta_0} = [\varphi_0^{\mathbf{B}}]_{\theta_0}$: it is well defined because, as we have seen at the beginning of this proof, $(\varphi_0^{\mathbf{A}}, \varphi_0^{\mathbf{B}}) \in E(f,g)$. Consider the cd-quomorphisms

$$p : \mathbf{A}^{(+)} \to \mathbf{C}_0^{(+)}, \quad q : \mathbf{B}^{(+)} \to \mathbf{C}_0^{(+)}$$

of partial $\Sigma^{(+)}$-algebras defined by the restrictions of the natural mapping nat_{θ_0} to the closed subsets A_0 and B_0, respectively.

Since, by construction, the restrictions to A and B of the quotient mapping $A \sqcup B \to (A \sqcup B)/\theta(f,g)$ are $\tilde{g} : \mathbf{A} \to \mathbf{P}$ and $\tilde{f} : \mathbf{B} \to \mathbf{P}$, and since they are closed homomorphisms, by [5, Lem. 3.1] the congruence $\theta(f,g)$ is separately closed on $\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}$. Then, since $\mathbf{A}_0^{(+)} \oplus \mathbf{B}_0^{(+)}$ is a closed subalgebra of $\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}$ and since $\theta_0 \subseteq \theta(f,g)$, by [5, Lem. 3.2] θ_0 is also separately closed on $\mathbf{A}_0^{(+)} \oplus \mathbf{B}_0^{(+)}$ and hence, again by [5, Lem. 3.1], p and q are cdc-quomorphisms of partial $\Sigma^{(+)}$-algebras. Moreover, the fact that A_0 and B_0 are closed subsets of $\mathbf{A}$ and $\mathbf{B}$ and the way the nullary operations in $\mathbf{C}_0$ have been defined, imply that p and q are actually cdc-quomorphisms of partial Σ-algebras $p : \mathbf{A} \to \mathbf{C}_0$, $q : \mathbf{B} \to \mathbf{C}_0$. And since $f^{-1}(A_0) = g^{-1}(B_0)$ and $E(f,g) \cap (A_0 \sqcup B_0)^2 \subseteq \theta_0$, it is clear that $p \circ f = q \circ g$.

Then, the universal property of pushouts in CDCQ-Alg$_\Sigma$ implies that there exists one and only one cdc-quomorphism $h : \mathbf{P} \to \mathbf{C}_0$ such that $h \circ \tilde{g} = p$ and $h \circ \tilde{f} = q$. Such a cdc-quomorphism makes the following diagram of partial mappings commutative:

$$\begin{array}{ccc} A \sqcup B & \xrightarrow{nat_{\theta(f,g)}} & (A \sqcup B)/\theta(f,g) \\ {\scriptstyle nat_{\theta_0}} \downarrow & \swarrow {\scriptstyle h} & \\ (A_0 \sqcup B_0)/\theta_0 & & \end{array}$$

(where $nat_{\theta_0} : A \sqcup B \to (A_0 \oplus B_0)/\theta_0$ stands for the partial mapping of domain $A_0 \sqcup B_0$ and defined on it by the quotient mapping nat_{θ_0}). By the Factorization Lemma for partial mappings (see, for instance, [10, Lem. 6], applied to a signature with an empty set of operation symbols, or [5, Lem. 5.1]), this implies that $\theta(f,g) \cap ((A_0 \sqcup B_0) \times (A \sqcup B)) \subseteq \theta_0$, as we wanted to prove. ∎

We now proceed to establish a first characterization of the pairs of cdc-quomorphisms of partial Σ-algebras that have a pushout in CDCQ-Alg$_\Sigma$.

THEOREM 5. *Two cdc-quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ of partial Σ-algebras have a pushout in* CDCQ-Alg$_\Sigma$ *if and only if the following two conditions are satisfied:*

a) *There exist the greatest closed subsets A_c and B_c of $\mathbf{A}$ and $\mathbf{B}$, respectively, such that:*

i) $f^{-1}(A_c) = g^{-1}(B_c)$; let us call K_c this subset of $\mathbf{K}$;

ii) if $\mathbf{K}_c$, $\mathbf{A}_c$, and $\mathbf{B}_c$ are the closed subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported on K_c, A_c and B_c, respectively, and if $f_c : \mathbf{K}_c \to \mathbf{A}_c$ and $g_c : \mathbf{K}_c \to \mathbf{B}_c$ are the closed homomorphisms between these algebras induced by f and g, then there exist two closed homomorphisms $p : \mathbf{A}_c \to \mathbf{C}$ and $q : \mathbf{B}_c \to \mathbf{C}$ to some partial Σ-algebra $\mathbf{C}$ such that $p \circ f_c = q \circ g_c$.

b) *The closed homomorphisms $f_c : \mathbf{K}_c \to \mathbf{A}_c$ and $g_c : \mathbf{K}_c \to \mathbf{B}_c$ have a pushout in* C-Alg$_\Sigma$ *and in* CDQ-Alg$_\Sigma$.

And if f and g satisfy conditions (a) and (b) and

$$(\mathbf{P}_c, \bar{g}_c : \mathbf{A}_c \to \mathbf{P}_c, \bar{f}_c : \mathbf{B}_c \to \mathbf{P}_c)$$

is a pushout of $f_c : \mathbf{K}_c \to \mathbf{A}_c$ and $g_c : \mathbf{K}_c \to \mathbf{B}_c$ in C-Alg$_\Sigma$, *and if we let $\tilde{g}_c : \mathbf{A} \to \mathbf{P}_c$ and $\tilde{f}_c : \mathbf{B} \to \mathbf{P}_c$ stand for the cdc-quomorphisms defined by $\bar{g}_c$ and $\bar{f}_c$, respectively, then*

$$(\mathbf{P}_c, \tilde{g}_c : \mathbf{A} \to \mathbf{P}_c, \tilde{f}_c : \mathbf{B} \to \mathbf{P}_c)$$

is a pushout of $f : \mathbf{K} \to \mathbf{A}$ *and* $g : \mathbf{K} \to \mathbf{B}$ *in* CDCQ-Alg$_\Sigma$.

Proof. $\Longleftarrow$) Assume first that $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ satisfy conditions (a) and (b). Then, by Proposition 2, the pushout

$$(\mathbf{P}_c, \bar{g}_c : \mathbf{A}_c \to \mathbf{P}_c, \bar{f}_c : \mathbf{B}_c \to \mathbf{P}_c)$$

of $f_c : \mathbf{K}_c \to \mathbf{A}_c$ and $g_c : \mathbf{K}_c \to \mathbf{B}_c$ in C-Alg$_\Sigma$ is also their pushout in CDCQ-Alg$_\Sigma$. We shall use it to prove that the cocone in CDCQ-Alg$_\Sigma$

$$(\mathbf{P}_c, \tilde{g}_c : \mathbf{A} \to \mathbf{P}_c, \tilde{f}_c : \mathbf{B} \to \mathbf{P}_c)$$

described in the statement is a pushout of $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ in CDCQ-Alg$_\Sigma$. It is clear that $\tilde{f}_c \circ g = \tilde{g}_c \circ f$.

Let now $p : \mathbf{A} \to \mathbf{C}$ and $q : \mathbf{B} \to \mathbf{C}$ be two cdc-quomorphisms such that $p \circ f = q \circ g$. Let $A_0 = \mathrm{Dom}\, p$, $B_0 = \mathrm{Dom}\, q$ and $K_0 = g^{-1}(B_0) = f^{-1}(A_0)$, let $\mathbf{K}_0$, $\mathbf{A}_0$ and $\mathbf{B}_0$ be the closed subalgebras of $\mathbf{K}$, $\mathbf{A}$, and $\mathbf{B}$ supported on these sets, respectively, and let $f_0 : \mathbf{K}_0 \to \mathbf{A}_0$ and $g_0 : \mathbf{K}_0 \to \mathbf{B}_0$ be the closed homomorphisms between these algebras induced by the cdc-quomorphisms f and g. Then, the cdc-quomorphisms $p : \mathbf{A} \to \mathbf{C}$ and $q : \mathbf{B} \to \mathbf{C}$ induce two closed homomorphisms $p_0 : \mathbf{A}_0 \to \mathbf{C}$ and $q_0 : \mathbf{B}_0 \to \mathbf{C}$ such that $p_0 \circ f_0 = q_0 \circ g_0$. Condition (a) implies then that $A_0 \subseteq A_c$ and $B_0 \subseteq B_c$. This entails that we can understand p and q as cdc-quomorphisms $p_c : \mathbf{A}_c \to \mathbf{C}$ and $q_c : \mathbf{B}_c \to \mathbf{C}$, still with domains A_0 and B_0, respectively, such that $p_c \circ f_c = q_c \circ g_c$.

Since $(\mathbf{P}_c, \bar{g}_c : \mathbf{A}_c \to \mathbf{P}_c, \bar{f}_c : \mathbf{B}_c \to \mathbf{P}_c)$ is a pushout of $f_c : \mathbf{K}_c \to \mathbf{A}_c$ and $g_c : \mathbf{K}_c \to \mathbf{B}_c$ in CDCQ-Alg$_\Sigma$, there exists one, and only one, cdc-quomorphism $h_c : \mathbf{P}_c \to \mathbf{C}$ such that $h_c \circ \bar{g}_c = p_c : \mathbf{A}_c \to \mathbf{C}$ and $h_c \circ \bar{f}_c = q_c : \mathbf{B}_c \to \mathbf{C}$. Since $A_0 \subseteq A_c = \mathrm{Dom}\, \tilde{g}_c$ and $B_0 \subseteq B_c = \mathrm{Dom}\, \tilde{f}_c$, the cdc-quomorphism $h_c : \mathbf{P}_c \to \mathbf{C}$ also satisfies that $h \circ \tilde{g}_c = p : \mathbf{A} \to \mathbf{C}$ and $h \circ \tilde{f}_c = q : \mathbf{B} \to \mathbf{C}$. Finally, the uniqueness of the cdc-quomorphism $h_c : \mathbf{P}_c \to \mathbf{C}$ such that $h_c \circ \bar{g}_c = p_c$ and $h_c \circ \bar{f}_c = q_c$ implies the uniqueness of the cdc-quomorphism $h_c : \mathbf{P}_c \to \mathbf{C}$ such that $h_c \circ \tilde{g}_c = p$ and $h_c \circ \tilde{f}_c = q$.

This proves that $(\mathbf{P}_c, \tilde{g}_c : \mathbf{A} \to \mathbf{P}_c, \tilde{f}_c : \mathbf{B} \to \mathbf{P}_c)$ is indeed a pushout of f and g in CDCQ-Alg$_\Sigma$.

$\Longrightarrow$) Assume that the cdc-quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ have a pushout

$$(\mathbf{Q}, \hat{g} : \mathbf{A} \to \mathbf{Q}, \hat{f} : \mathbf{B} \to \mathbf{Q})$$

in CDCQ-Alg$_\Sigma$. Set $A_c = \mathrm{Dom}\, \hat{g}$ and $B_c = \mathrm{Dom}\, \hat{f}$, and let us see first that these closed subsets of $\mathbf{A}$ and $\mathbf{B}$ are those asked for in condition (a).

- $f^{-1}(A_c) = \mathrm{Dom}\, \hat{g} \circ f = \mathrm{Dom}\, \hat{f} \circ g = g^{-1}(B_c)$; let us call K_c this closed subset of $\mathbf{K}$.

Let $\mathbf{A}_c$, $\mathbf{B}_c$ and $\mathbf{K}_c$ be the closed subalgebras of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{K}$ supported on A_c, B_c and K_c, respectively, and let $f_c : \mathbf{K}_c \to \mathbf{A}_c$ and $g_c : \mathbf{K}_c \to \mathbf{B}_c$ be the closed homomorphisms between these algebras induced by the cdc-quomorphisms f and g. Then, the cdc-quomorphisms $\hat{g} : \mathbf{A} \to \mathbf{Q}$ and $\hat{f} : \mathbf{B} \to \mathbf{Q}$ induce two closed homomorphisms $\hat{g}_c : \mathbf{A}_c \to \mathbf{Q}$ and $\hat{f}_c : \mathbf{B}_c \to \mathbf{Q}$ such that $\hat{g}_c \circ f_c = \hat{f}_c \circ g_c : \mathbf{K}_c \to \mathbf{Q}$. This shows that the subsets A_c and B_c satisfy the properties (i) and (ii).

- Let now A_0 and B_0 be two closed subsets of $\mathbf{A}$ and $\mathbf{B}$, respectively, satisfying (i) and (ii). Let K_0 be $f^{-1}(A_0) = g^{-1}(B_0)$, let $\mathbf{K}_0$, $\mathbf{A}_0$ and $\mathbf{B}_0$ be the closed subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported on K_0, A_0 and B_0 respectively, let $f_0 : \mathbf{K}_0 \to \mathbf{A}_0$ and $g_0 : \mathbf{K}_0 \to \mathbf{B}_0$ be the closed homomorphisms between these algebras induced by f and g, and let $p_0 : \mathbf{A}_0 \to \mathbf{Q}_0$ and $q_0 : \mathbf{B}_0 \to \mathbf{Q}_0$ be two closed homomorphisms such that $p_0 \circ f_0 = q_0 \circ g_0$. We want to see that $A_0 \subseteq A_c$ and $B_0 \subseteq B_c$. Consider the cdc-quomorphisms $p : \mathbf{A} \to \mathbf{Q}_0$ and $q : \mathbf{B} \to \mathbf{Q}_0$ defined by the closed homomorphisms $p_0 : \mathbf{A}_0 \to \mathbf{Q}_0$ and $q_0 : \mathbf{B}_0 \to \mathbf{Q}_0$. Since $f^{-1}(A_0) = g^{-1}(B_0)$ and $p_0 \circ f_0 = q_0 \circ g_0$, we have that $p \circ f = q \circ g$ as cdc-quomorphisms $\mathbf{K} \to \mathbf{Q}_0$. By the universal property of pushouts, there exists a cdc-quomorphism $h : \mathbf{Q} \to \mathbf{Q}_0$ such that $h \circ \hat{g} = p$ and $h \circ \hat{f} = q$, from where we deduce that $A_0 = \operatorname{Dom} p \subseteq \operatorname{Dom} \hat{g} = A_c$ and $B_0 = \operatorname{Dom} q \subseteq \operatorname{Dom} \hat{f} = B_c$, as we wanted.

Let still $\hat{g}_c : \mathbf{A}_c \to \mathbf{Q}$ and $\hat{f}_c : \mathbf{B}_c \to \mathbf{Q}$ be the closed homomorphisms induced by the cdc-quomorphisms $\hat{g} : \mathbf{A} \to \mathbf{Q}$ and $\hat{f} : \mathbf{B} \to \mathbf{Q}$, respectively. Using that

$$(\mathbf{Q}, \hat{g} : \mathbf{A} \to \mathbf{Q}, \hat{f} : \mathbf{B} \to \mathbf{Q})$$

is a pushout of $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ in CDCQ-Alg$_\Sigma$, it is straightforward to check that the cocone

$$(\mathbf{Q}, \hat{g}_c : \mathbf{A}_c \to \mathbf{Q}, \hat{f}_c : \mathbf{B}_c \to \mathbf{Q})$$

is a pushout in CDCQ-Alg$_\Sigma$ of $f_c : \mathbf{K}_c \to \mathbf{A}_c$ and $g_c : \mathbf{K}_c \to \mathbf{B}_c$. If we prove that it is also a pushout of them in C-Alg$_\Sigma$ then, by Proposition 2, it will be also a pushout of them in CDQ-Alg$_\Sigma$ and condition (b) will follow.

Let then $p_c : \mathbf{A}_c \to \mathbf{D}$ and $q_c : \mathbf{B}_c \to \mathbf{D}$ be two closed homomorphisms such that $p_c \circ f_c = q_c \circ g_c$. The cdc-quomorphisms $p : \mathbf{A} \to \mathbf{D}$ and $q : \mathbf{B} \to \mathbf{D}$ defined by these closed homomorphisms commute with $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$, and hence there exists one, and only one, cdc-quomorphism $h : \mathbf{Q} \to \mathbf{D}$ such that $h \circ \hat{g} = p : \mathbf{A} \to \mathbf{D}$ and $h \circ \hat{f} = q : \mathbf{B} \to \mathbf{D}$, and therefore one, and only one, cdc-quomorphism $h : \mathbf{Q} \to \mathbf{D}$ such that $h \circ \hat{g}_c = p_c$ and $h \circ \hat{f}_c = q_c$. If we can prove that h is totally defined, then it will be a closed homomorphism and we will be done.

Now, $\mathrm{Dom}\, h$ is a closed subset of $\mathbf{Q}$ containing $\hat{g}(A_c) \cup \hat{f}(B_c)$, and this set generates the carrier set Q of $\mathbf{Q}$, because Id_Q and $\mathrm{Id}_{C_{\mathbf{Q}}(\hat{g}(A_c)\cup\hat{f}(B_c))}$ (understood as a cdc-quomorphism from $\mathbf{Q}$ into itself) are cdc-quomorphisms $\mathbf{Q} \to \mathbf{Q}$ such that, when composed with $\hat{g}$ and $\hat{f}$, yield again $\hat{g}$ and $\hat{f}$, and therefore, by the universal property of pushouts, they must be equal. Thus, $C_{\mathbf{Q}}(\hat{g}(A_c) \cup \hat{f}(B_c)) = Q$ and hence $\mathrm{Dom}\, h = Q$, as we wanted. ∎

We can use now Theorems 2 and 3 to rephrase this characterization in a more intrinsic and operational way.

COROLLARY 1. *Two cdc-quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ of partial Σ-algebras have a pushout in* CDCQ-Alg$_\Sigma$ *if and only if they satisfy the following three conditions:*

CDCPO1) *There exist the greatest closed subsets A_c and B_c of $\mathbf{A}$ and $\mathbf{B}$, respectively, such that:*

i) $f^{-1}(A_c) = g^{-1}(B_c)$; let us call K_c this subset of $\mathbf{K}$;

ii) if $\mathbf{K}_c$, $\mathbf{A}_c$, and $\mathbf{B}_c$ are the closed subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported on K_c, A_c and B_c, respectively, and if $f_c : \mathbf{K}_c \to \mathbf{A}_c$ and $g_c : \mathbf{K}_c \to \mathbf{B}_c$ are the closed homomorphisms defined by the restrictions of f and g to these subalgebras, then the congruence $\theta(f_c, g_c)$ on $\mathbf{A}_c^{(+)} \oplus \mathbf{B}_c^{(+)}$ generated by $E(f_c, g_c)$ is separately closed.

CDCPO2) *For every $\varphi \in \Omega^{(+)}$,*

$$((A_c \sqcup B_c)/\theta(f_c, g_c))^{\omega(\varphi)} = nat_{\theta(f_c,g_c)}(A_c)^{\omega(\varphi)} \cup nat_{\theta(f_c,g_c)}(B_c)^{\omega(\varphi)}.$$

CDCPO3) *For all closed subsets A_0 and B_0 of $\mathbf{A}_c$ and $\mathbf{B}_c$, respectively, such that $f^{-1}(A_0) = g^{-1}(B_0)$, if θ_0 is the congruence on $\mathbf{A}_0^{(+)} \oplus \mathbf{B}_0^{(+)}$ generated by $E(f_c, g_c) \cap (A_0 \sqcup B_0)^2$, then*

$$\theta(f_c, g_c) \cap ((A_0 \sqcup B_0) \times (A_c \sqcup B_c)) \subseteq \theta_0.$$

Proof. Using [5, Lem. 3.1, 3.2] and [6, Prop. 2.4.5], it is easy to prove the equivalence between properties (ii) in (CDCPO1) and in condition (a) in Theorem 5, which proves the equivalence between these two conditions. Then condition (b) in *loc. cit.* is easily seen to be equivalent to the conjunction of (CDCPO2) and (CDCPO3) using Theorems 2 and 3 and the proof of Proposition 2. We leave the details to the reader. ∎

COROLLARY 2. *Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two cdc-quomorphisms of partial Σ-algebras that have a pushout in* CQ-Alg$_\Sigma$. *Then, f and g have a pushout in* CDCQ-Alg$_\Sigma$ *if and only if they have a pushout in* CDQ-Alg$_\Sigma$. *And when f and g have a pushout in all three categories, they are all equal.*

Proof. Let

$$(\mathbf{P}, \tilde{g} : \mathbf{A} \to \mathbf{P}, \tilde{f} : \mathbf{B} \to \mathbf{P})$$

be the pushout of $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ in CQ-Alg$_\Sigma$ described in Theorem 4, which, by [5, Thm. 5.2] is also their pushout in Q-Alg$_\Sigma$.

Let A_0 and B_0 be any two closed subsets of $\mathbf{A}$ and $\mathbf{B}$ such that $f^{-1}(A_0) = g^{-1}(B_0)$, let K_0 be this closed subset of $\mathbf{K}$, let $\mathbf{K}_0$, $\mathbf{A}_0$ and $\mathbf{B}_0$ be the closed subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$, respectively, supported on K_0, A_0 and B_0, and let $f_0 : \mathbf{K}_0 \to \mathbf{A}_0$ and $g_0 : \mathbf{K}_0 \to \mathbf{B}_0$ be the closed homomorphisms between these algebras induced by the cdc-quomorphism f and g. By [12, Lem. 8] (and using the notations of Theorem 4 and §2.1), $\mathbf{A}_0^{(+)} \oplus \mathbf{B}_0^{(+)}$ is a closed subalgebra of $\mathbf{D}(f,g)$ and $\theta(f_0, g_0)$ is clearly contained in $\theta(f_D, g_D)$. Now, by [5, Lem. 3.1], the closedness of $\tilde{f}$ and $\tilde{g}$ implies that the congruence $\theta(f_D, g_D)$ is separately closed on $\mathbf{D}(f,g)$. Therefore, by [5, Lem. 3.2], $\theta(f_0, g_0)$ is separately closed on $\mathbf{A}_0^{(+)} \oplus \mathbf{B}_0^{(+)}$.

This shows that every pair of closed subsets of $\mathbf{A}$ and $\mathbf{B}$ satisfying property (i) in (CDCPO1) also satisfies property (ii) in it, and hence that f and g satisfy condition (CDPO1) in Theorem 3 if and only if they satisfy condition (CDCPO1), and, when they satisfy them, the closed subsets of $\mathbf{A}$ and $\mathbf{B}$ given by these two conditions are the same. This implies that f and g satisfy condition (CDPO2) in Theorem 3 if and only if they satisfy condition (CDCPO3) (because, in this case, these two conditions say exactly the same).

It remains to prove that if f and g satisfy conditions (CDPO1) and (CDPO2), then they also satisfy condition (CDCPO2), using that f and g have a pushout in CQ-Alg$_\Sigma$. So, assume that f and g have a pushout in CDQ-Alg$_\Sigma$, and let

$$(\mathbf{Q}, \hat{g} : \mathbf{A} \to \mathbf{Q}, \hat{f} : \mathbf{B} \to \mathbf{Q})$$

be the one described in Theorem 3: by the previous paragraph, it should be clear that (with the notations of Corollary 1) the carrier set Q of $\mathbf{Q}$ is equal to $(A_c \sqcup B_c)/\theta(f_c, g_c)$.

By the universal property of pushouts in Q-Alg$_\Sigma$, there exists a quomorphism $h : \mathbf{P} \to \mathbf{Q}$ such that $h \circ \tilde{g} = \hat{g}$ and $h \circ \tilde{f} = \hat{f}$. This h is surjective because, by construction, $Q = \hat{f}(A) \cup \hat{g}(B)$. But now, since, by assumption, f and g satisfy condition (CQPO2) in Theorem 4, we have that $P^{\omega(\varphi)} = \tilde{g}(A)^{\omega(\varphi)} \cup \tilde{f}(B)^{\omega(\varphi)}$, for every $\varphi \in \Omega^{(+)}$, and hence

$$Q^{\omega(\varphi)} = h(P^{\omega(\varphi)}) = h(\tilde{g}(A)^{\omega(\varphi)}) \cup h(\tilde{f}(B)^{\omega(\varphi)}) = \hat{g}(A)^{\omega(\varphi)} \cup \hat{f}(B)^{\omega(\varphi)},$$

which proves condition (CDCPO2). ∎

Conditions (CDCPO1), (CDCPO2) and (CDCPO3) are independent, as the following examples show.

EXAMPLE 1. Let Σ be a one-sorted signature with only one operation symbol φ, which is binary. Let $\mathbf{K}$ be the discrete Σ-algebra with carrier set

$K = \{k_1, k_2\}$, let $\mathbf{A}$ be the discrete Σ-algebra with carrier set $A = \{a\}$, and let $\mathbf{B}$ be the partial Σ-algebra with carrier set $B = \{b_1, b_2, c\}$ and the operation φ defined simply by $\varphi^{\mathbf{B}}(b_1, c) = c$. Let $f : \mathbf{K} \to \mathbf{A}$ be the constant mapping, which is a closed homomorphism, and let $g : \mathbf{K} \to \mathbf{B}$ be the closed homomorphism defined by $g(k_1) = b_1$ and $g(k_2) = b_2$.

Since f and g are totally defined, the carrier sets of $\mathbf{A}$ and $\mathbf{B}$ are the greatest closed subsets of these algebras with the same preimages under f and g, respectively. But there do not exist the greatest closed subsets of $\mathbf{A}$ and $\mathbf{B}$ with the same preimages under f and g and, moreover, satisfying condition (ii) in (CDCPO1).

Indeed, $A_1 = \emptyset$ and $B_1 = \{c\}$ are two closed subsets with this property, as well as $A_2 = A$ and $B_2 = \{b_1, b_2\}$, but $A_1 \cup A_2 = A$ and $B_1 \cup B_2 = B$, and there do not exist any pair of closed homomorphisms $p : \mathbf{A} \to \mathbf{C}$ and $q : \mathbf{B} \to \mathbf{C}$ to any algebra $\mathbf{C}$ such that $p \circ f = q \circ g$: should they exist, $q(b_1) = p(a) = q(b_2)$ and then $(b_1, c) \in \operatorname{dom} \varphi^{\mathbf{B}}$ would imply that $(q(b_2), q(c)) = (q(b_1), q(c)) \in \operatorname{dom} \varphi^{\mathbf{C}}$ but $(b_2, c) \notin \operatorname{dom} \varphi^{\mathbf{B}}$, and thus q would not be closed.

EXAMPLE 2. Let Σ, $\mathbf{K}$, $\mathbf{A}$, and $f : \mathbf{K} \to \mathbf{A}$ be as in the last example, and let now $\mathbf{B}$ be the partial Σ-algebra with carrier set $B = \{b_1, b_2, c, d_1, d_2\}$ and with the operation $\varphi^{\mathbf{B}}$ defined by $\varphi^{\mathbf{B}}(b_1, c) = d_1$ y $\varphi^{\mathbf{B}}(b_2, c) = d_2$, and let $g : \mathbf{K} \to \mathbf{B}$ be the closed homomorphisms defined by $g(k_1) = b_1, g(k_2) = b_2$.

It is easy to prove that the pushout in Alg_Σ of f and g is also their pushout in C-Alg_Σ, which implies that they satisfy conditions (CDCPO1), with $A_c = A$ and $B_c = B$, as well as (CDCPO2). But they do not satisfy condition (CDCPO3). Indeed, the congruence $\theta(f, g)$ identifies on the one hand a, b_1 and b_2, and on the other hand d_1 and d_2. Consider now the closed subsets $A_0 = \emptyset$ and $B_0 = \{d_1, d_2\}$ of $\mathbf{A}$ and $\mathbf{B}$, respectively. Then $f^{-1}(A_0) = \emptyset = g^{-1}(B_0)$, but the congruence θ_0 on $\mathbf{A}_0 \oplus \mathbf{B}_0$ generated by $E(f, g) \cap (A_0 \sqcup B_0)^2$ is the diagonal on $A_0 \sqcup B_0$, which does not identify d_1 with d_2.

EXAMPLE 3. Let Σ be the signature in the last examples. Let now $\mathbf{K}$ be the empty Σ-algebra, and let $\mathbf{A}$ and $\mathbf{B}$ be two discrete Σ-algebras with carrier sets $A = \{a\}$ and $B = \{b\}$, respectively. Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be the empty homomorphisms. It is straightforward to prove that these cdc-quomorphisms satisfy conditions (CDCPO1) (taking $A_c = A$ and $B_c = B$) and (CDCPO3) (because $\mathbf{A}_c^{(+)} \oplus \mathbf{B}_c^{(+)}$ is discrete), but they do not satisfy condition (CDCPO2): $(A \sqcup B)^2 \neq A^2 \cup B^2$.

To finish this paper, let us point out that we proved in [5, Thm. 5.2] that if two c-quomorphisms have a pushout in CQ-Alg_Σ, then they also have a pushout in Q-Alg_Σ, while in Corollary 2 we have proved that if two cdc-quomorphisms have a pushout in CQ-Alg$_\Sigma$ and in CDQ-Alg$_\Sigma$, then they also have a pushout in CDCQ-Alg$_\Sigma$, and that if two cdc-quomorphisms have a pushout in CQ-Alg$_\Sigma$ and CDCQ-Alg$_\Sigma$, then they also have a pushout in CDQ-Alg$_\Sigma$. It turns out that no other implication concerning the existence of pushouts of cdc-quomorphisms in these categories holds, except those obtained by combining these three implications. For instance, Example 1 shows that two cdc-quomorphisms (even two closed homomorphisms) with a pushout in CDQ-Alg$_\Sigma$ need not have a pushout in CDCQ-Alg$_\Sigma$. The full set of counterexamples can be found in [1]: see Table 9.1 therein.

If we restrict ourselves to pushouts of closed homomorphisms, then, besides the implications mentioned above, other implications also hold: by [4, Prop. 13] (and the fact that closed homomorphisms always satisfy condition (CDPO1)), if two closed homomorphisms have a pushout in Q-Alg$_\Sigma$, then they also have a pushout in CDQ-Alg$_\Sigma$; by [5, Cor. 5.3], two closed homomorphisms have a pushout in CQ-Alg$_\Sigma$ if and only if they have a pushout in Q-Alg$_\Sigma$ and in C-Alg$_\Sigma$; and by Proposition 2, if two closed homomorphisms have a pushout in C-Alg$_\Sigma$ and in CDCQ-Alg$_\Sigma$, then they also have a pushout in CDQ-Alg$_\Sigma$, and if two closed homomorphisms have a pushout in C-Alg$_\Sigma$ and in CDQ-Alg$_\Sigma$, then they also have a pushout in CDCQ-Alg$_\Sigma$. It can be proved again that no other implication concerning the existence of pushouts of closed homomorphisms in these categories holds, except those obtained by combining these implications. For instance, Example 2 shows that two closed homomorphisms with a pushout in C-Alg$_\Sigma$ need not have a pushout in CDCQ-Alg$_\Sigma$ or in CDQ-Alg$_\Sigma$. Again, the full set of counterexamples can be found in [1]: see Table 9.2 therein.

Acknowledgements. We thank M. Llabrés and P. Burmeister for their comments and suggestions on preliminary versions of this paper. This work has been partially supported by the DGES, grant BFM2000-1113-C02-01.

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Received May 15, 2003.