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ON A TOPOLOGICAL PRESENTATION OF GRAPHS

Abstract. In the paper a bitopological presentation of graphs has been described.

1. Introduction

Graphs are widely used as almost "standard" models of various dynamic systems, networks and many other devices. There exist a lot of presentations (definitions) of graphs. These presentations define graphs usually as two sorted relational systems i.e. sequences of the form $G = (A, V, \sigma)$ with A and V being sets (of *edges* and *vertices* resp.) and a structure σ which can be a function (e.g. of the form $A \to V \times V$), a relation (e.g. of the form $\sigma \subseteq V \times A \times V$) or e.g. a pair of functions of the form $A \to V$. In the last, perhaps the most popular, case by a graph it is meant an algebra of the form $\mathcal{G} = (V^{\mathcal{G}}, E^{\mathcal{G}}, d_0^{\mathcal{G}}, d_1^{\mathcal{G}})$ where $V^{\mathcal{G}}$ and $E^{\mathcal{G}}$ are sets of *vertices* and *edges* of the graph and $d_0^{\mathcal{G}}, d_1^{\mathcal{G}} : A \to V$ are functions called *incidence functions.* A disadvantage of the mostly used two sorted presentations of graphs is the fact that their (homo)morphisms are defined in a not "typically algebraic" way (see e.g. [4] for a comment and an interesting other definitions). The problem is here the fact that such a homomorphism should respect the partition of the set of elements (that means the set $A \cup V$) of a graph into edges and vertices. The presentation of graphs used in this paper defines graphs with disjoint sets of vertices and arrows as equationally definable algebras with one carrier-set only. By a *graph* we mean any triple $\mathcal{A} = (X, \mathbf{s}, \mathbf{t})$ with a set X and unary operations $\mathbf{s}, \mathbf{t} : X \longrightarrow X$ satisfying the conditions

$$\mathbf{s}(\mathbf{s}(x)) = \mathbf{t}(\mathbf{s}(x)) = \mathbf{s}(x) \ \& \ \mathbf{s}(\mathbf{t}(x)) = \mathbf{t}(\mathbf{t}(x)) = \mathbf{t}(x)$$

for each $x \in X$. The passage from this definition to the last of two-sorted presentations mentioned above and in the opposite direction is via the equations

$$V^{\mathcal{G}} = \{x \in X : \mathbf{s}(x) = x\}, E^{\mathcal{G}} = X \backslash V^{\mathcal{G}}, d_0^{\mathcal{G}} = \mathbf{s}_{|E^{\mathcal{G}}}, d_1^{\mathcal{G}} = \mathbf{t}_{|E^{\mathcal{G}}}$$

(from triples $\mathcal{A} = (X, \mathbf{s}, \mathbf{t})$ to quadruples $\mathcal{G} = (V^{\mathcal{G}}, E^{\mathcal{G}}, d_0^{\mathcal{G}}, d_1^{\mathcal{G}})$) and

$$X(\mathcal{G}) = V^{\mathcal{G}} \cup E^{\mathcal{G}}, \mathbf{s} = d_0^{\mathcal{G}} \cup id_{V^{\mathcal{G}}}, \mathbf{t} = d_1^{\mathcal{G}} \cup id_{V^{\mathcal{G}}}$$

(backwards). So the vertices of such (one sorted presentation of) graphs are the comon fix-point of the operations **s** and **t**. The class of homomorphisms of such graphs is richer than the class of "classical" homomorphisms of graphs. If one wants to use graph homomorphisms as a model of "aggregation of arrows into vertices" then one has to choose between the simple two-sorted definition of graphs and complicated definition of their homomorphisms, or a "non-typical" definition of graphs (in this case of the above one-sorted presentation) and typical algebraic definition of their homomorphisms. In this paper the second possibility has been chosen. It seems that in this case, in contrast e.g. to programming, it is easier to work with one-sorted than with two-sorted algebras.We show that the one-sorted presentation of graphs leads to a kind of "topological" presentation of them.

In the paper the standard mathematical notation and terminology (see e.g. [3]) is used. Exemption is the notation for the composition of relations and functions ($(R \circ S)(x) = S(R(x))$) and the fact that total relations are written in the form $R : X \rightarrow Y$ instead of more often used form $R \subseteq X \times Y$, that means if $R \subseteq X \times Y$ and for every $x \in X$ there exists $y \in Y$ with xRy then we write $R : X \rightarrow Y$ instead of $R \subseteq X \times Y$. For all unexplained concepts of category theory and notation used in the paper the reader is referred to [3].

2. Topologies induced by functions

Let X be a set and $g : X \longrightarrow X$ be a function satisfying the condition $g^2 = g \circ g = g$, i.e. g is (considered as a relation) transitive and dense. Let us note that in this case we have $g(X) = g_{fix} = \{x \in X : g(x) = x\}$, that means the set of all fix-points of g is the image of g. For any $A \subseteq X$ we define $C(A) = A \cup g(A)$.

FACT 2.1. *The operation $C : Pow(X) \longrightarrow Pow(X)$ is a topological closure operation in the set X.*

The topology and the closure operation determined in a set X by a transitive and dense function $g : X \longrightarrow X$ will be called *induced* by g and denoted by τ_g and C_g respectively. We will drop the index g if it does not cause any confusion. Let X be a set, $g : X \longrightarrow X$ be a transitive and dense function, C be the closure operation induced by g and g_{fix} be the set of all fix points of g.

FACT 2.2. *For every element x of X the one element set $\{x\}$ is closed in the topology induced by g iff $x \in g_{fix}$.*

Proof. Let $x \in X$ and $C(\{x\}) = \{x\}$. So we have $C(\{x\}) = \{x\} \cup g(\{x\}) = \{x\}$ which implies $g(\{x\}) = \{x\}$. The converse is straightforward. ∎

FACT 2.3. *For any set X, transitive and dense function $g : X \to X$ and $x \in X$ the set $\{x\}$ is open iff $x \in X \backslash g_{fix}$.*

Proof. Let $x \in X \backslash g(X)$ and let us consider the interior of $\{x\}$. Let $I : Pow(X) \longrightarrow Pow(X)$ be the interior operation in the topological space (X, τ_g). From the property $I(A) \subseteq A$ it follows that $I(\{x\}) = \emptyset$ or $I(\{x\}) = \{x\}$. If it was $I(\{x\}) = \emptyset$ then $\{x\}$ would be a frontier set, i.e. it would be closed in τ_g, which is impossible because $x \in X \backslash g_{fix}$ (c.f. fact 2.2 above). So it must be $I(\{x\}) = \{x\}$, i.e. $\{x\}$ is an open set. The converse is evident. ∎

The topological space (X, τ_g) may be very "irregular" and needn't satisfy even very weak separation conditions. On the other side, if τ_g is a T_1 topology, then g must be the identity relation in X and τ_g is the discrete topology. Some "between cases" are determined by the request that g has a non empty set of fix-points. These fix-points are exactly one element closed sets. All other one element sets are open. This property characterizes topological spaces induced by transitive and dense functions.

DEFINITION 2.1. *A topological space (X,τ) will be called $T_{1/2}$-space iff any one element set is either open or closed.*

The frontier of a one element set in a topological space induced by a function is always a one element set. In general in a $T_{1/2}$-space it needn't be the case.

EXAMPLE 2.1. *Let us consider the space (X, τ) with $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. This is evidently a $T_{1/2}$-space because the set $\{a\}$ is open as an element of the topology τ and sets $\{b\}$ and $\{c\}$ are closed because their complements $\{a, c\}$ and $\{a, b\}$ are open. Now we have closure($\{a\}$)=X and frontier($\{a\}$)=$\{b,c\}$. This space can be seen as an (undirected) graph consisting of one edge a and vertices (the end-points of a) b and c.*

DEFINITION 2.2. *For any $T_{1/2}$-space (X, τ) and a point $x \in X$ we define*

$$g_\tau(x) = y \Leftrightarrow y \in C(\{x\}) \ \& \ C(\{y\}) = \{y\}.$$

Let us note that $g_\tau : X \to X$ is a well defined function because the fontier set of $\{x\}$ is a one element set. If there were exist two elements y and y' satisfying the condition

$$y \in C(\{x\}) \ \& \ C(\{y\}) = \{y\} \ \& \ y' \in C(\{x\}) \ \& \ C(\{y'\}) = \{y'\},$$

then the set $\{y, y'\}$ would be included in the frontier of $\{x\}$. So this frontier would have more then one element. Let us also note that if the set $\{x\}$ is closed then we have $g_\tau(x) = \{x\}$ that is x is a fix-point of g_τ.

LEMMA 2.1. *For any $T_{1/2}$-space (X, τ) and any point $x \in X$ there holds $Fr(\{x\} = g_\tau(x)$.*

FACT 2.4. *The function g_τ is transitive and dense.*

Proof. Let $x, y \in X$ and let us assume that $\{x\}$ is an open set. We have
$g_\tau^2(x) = y \Leftrightarrow g_\tau(x) = z \ \&\ g_\tau(z) = y$ for some $z \in X$ ⌈def. of superposition⌉
$\Leftrightarrow z \in C(\{x\}) \ \&\ C(\{z\}) = \{z\} \ \&\ y \in C(\{z\}) \ \&\ C(\{y\}) = \{y\}$ ⌈def. of g_τ⌉
$\Rightarrow z \in C(\{x\}) \ \&\ y = z \ \&\ C(\{z\}) = \{z\}$ ⌈$y \in C(\{z\}) \ \&$ $C(\{z\}) = \{z\} \Rightarrow y = z$⌉
$\Rightarrow y \in C(\{x\}) \ \&\ C(\{y\}) = \{y\}$ ⌈$y = z$⌉
$\Rightarrow g_\tau(x) = y$.
So we have shown that $g_\tau^2 \subseteq g_\tau$. Now
$g_\tau(x) = y \Leftrightarrow y \in C(\{x\}) \ \&\ C(\{y\}) = \{y\}$ ⌈def. of g_τ^2⌉
$\Rightarrow y \in C(\{x\}) \ \&\ C(\{y\}) = \{y\} \ \&\ y \in C(\{y\}) \ \&\ C(\{y\}) = \{y\}$ ⌈$p \wedge q \Rightarrow p \wedge q \wedge q$⌉
$\Rightarrow g_\tau(x) = y \ \&\ g_\tau(y) = y$ ⌈def. of g_τ⌉
$\Rightarrow \exists_z x g_\tau z \ \&\ z g_\tau y$ ⌈$P(x_0) \Rightarrow \exists_x P(x)$⌉
$\Leftrightarrow (g_\tau \circ g_\tau)(x) = y$
which completes the proof of the first part of proposition in the case when $\{x\}$ is open. If $\{x\}$ is closed then the proof is straigthforward.

Let g and g' be transitive and dense functions in the sets X and X' respectively and $f : X \to X'$ be a function transforming g into g'. ■

FACT 2.5. *The triple $f : (X, \tau_g) \to (X', \tau_{g'})$ is a continuous mapping from the topological space (X, τ_g) into $(X', \tau_{g'})$.*

Proof. If $(f \times f)(g) \subseteq g\prime$, i.e. if $g \circ f \subseteq f \circ g'$ than we have
$f(C(A)) = f(A \cup g(A))$ ⌈def. of the closure operation⌉
$= f(A) \cup f(g(A))$ ⌈$f(A \cup B) = f(A) \cup f(B)$⌉
$= f(A) \cup (g \circ f)(A)$ ⌈$f(g(A)) = (g \circ f)(A)$⌉
$\subseteq f(A) \cup (f \circ g')(A)$ ⌈$g \circ f \subseteq f \circ (g' \cup id_{X'})$⌉
$= f(A) \cup g'(f(A))$
$= C'(f(A))$. ⌈def. of the closure operation⌉

■

LEMMA 2.2. *For any $T_{1/2}$-space (X, τ) and any points $x, y \in X$ there holds*

$$x \neq y \ \&\ \{x\} \in \tau \ \&\ y \in C(\{x\}) \Rightarrow C(\{y\}) = \{y\}.$$

Proof. Let us assume $x \neq y \ \&\ \{x\} \in \tau \ \&\ y \in C(\{x\})$. From the property $C(\{x\}) = \{x\} \cup Fr(\{x\})$ we infer that $y \in Fr(\{x\})$ that means for any $A \in \tau$ with $y \in A$ we have $A \cap \{x\} \neq \emptyset$. If the set $\{y\}$ would not be closed then it had to be open and from the fact that (X, τ) is of the type $T_{1/2}$ and $x \neq y$ we obtain $y \in \{y\} \in \tau \ \&\ \{y\} \cap \{x\} = \emptyset$ which contradicts the property $y \in Fr(\{x\})$. ■

LEMMA 2.3. *For any $T_{1/2}$-space $\Upsilon = (X, \tau)$ and points $x, y \in X$ there holds*

$$x \neq y \ \&\ y \in C(x) \Longrightarrow \{x\} \in \tau.$$

Proof. If it was $x \neq y$ & $y \in C(x)$ & $\{x\} \notin \tau$ then it would be $C(\{x\}) = \{x\}$ because Υ is $T_{1/2}$-space. Now it would be

$$y \in \{x\} = C(\{x\}) \ \& \ y \neq x$$

which is impossible. ∎

COROLLARY 2.1. *For any $T_{1/2}$-space (X, τ) and any points $x, y \in X$ there holds*

$$x \neq y \ \& \ y \in C(\{x\}) \Rightarrow C(\{y\}) = \{y\}.$$

Proof. Straightforward. ∎

FACT 2.6. *For any $T_{1/2}$-spaces $T = (X, \tau)$ and $T' = (X', \tau')$ and any continuous mapping $f : T \to T'$ we have*

$$g_\tau(x) = y \Rightarrow g_{\tau'}(f(x)) = f(y)$$

for all $x, y \in X$.

Proof. Let $x, y \in X$ and let $g_\tau(x) = y$. So we have

$$y \in C(\{x\}) \ \& \ C(\{y\}) = \{y\}.$$

From the continuity of f we obtain

$$f(y) \in f(C(\{x\})) \subseteq C'(f(\{x\})) \ \& \ f(C(\{y\})) = \{f(y)\}.$$

Let us consider the set $\{f(y)\}$. If it is closed then $C'(\{f(y)\}) = \{f(y)\}$ and $g_{\tau'}(f(x)) = f(y)$. If it is open then $f(x) = f(y)$ by Corollary 2.3 above (if it was $f(x) \neq f(y)$ then it would be $C'(\{f(y)\}) = \{f(y)\}$ because $f(y) \in C'(f(\{x\}))$). ∎

For any $T_{1/2}$ -space $\Upsilon = (X, \tau)$ let $\Gamma(\Upsilon) = (X, g_\tau)$ and for any continous mapping $f : \Upsilon \to \Upsilon'$ let $\Gamma(f) : \Gamma(\Upsilon) \to \Gamma(\Upsilon')$ be given by the assignment $\Gamma(f)(x) = f(x)$ for any $x \in X$. Then $\Gamma : \mathbf{Top}_{1/2} \to \mathbf{TDfun}$ is a functor from the category $\mathbf{Top}_{1/2}$ of $T_{1/2}$ -spaces into the category **TDfun** of transitive and dense functions. Analogously assigning to any pair $\gamma = (X, g)$ with X being a set and g a transitive and dense function from X into X the topological space (X, τ_g) and to any triple $f : (X, g) \to (X'g')$ with g and g' being transitive and dense funtions $g : X \to X$ and $g' : X' \to X'$ and $f : X \to X'$ being a fuction transforming g into g' the triple $f : (X, \tau_g) \to (X', \tau_{g'})$ we obtain a functor $\Delta : \mathbf{TDfun} \to \mathbf{Top}_{1/2}$ from the category **TDfun** into the category $\mathbf{Top}_{1/2}$.

COROLLARY 2.2. *Categories **TDfun** and $\mathbf{Top}_{1/2}$ are equivalent.*

Proof. It is an immediate consequence of the fact 2.6 and 2.7 above. ∎

The above corrollary allows to consider unary algebras of the form (A,ω) with the operation $\omega : A \to A$ satisfying the condition $\omega \circ \omega = \omega$ as $T_{1/2}$ topological spaces and vice versa. As we have seen in the introduction graphs can be treated as unary algebras of the form $\mathcal{A} = (X, \mathbf{s}, \mathbf{t})$ with a set X and unary operations $\mathbf{s}, \mathbf{t} : X \longrightarrow X$ satisfying the conditions

$$\mathbf{s}(\mathbf{s}(x)) = \mathbf{t}(\mathbf{s}(x)) = \mathbf{s}(x) \ \& \ \mathbf{s}(\mathbf{t}(x)) = \mathbf{t}(\mathbf{t}(x)) = \mathbf{t}(x)$$

that means a kind of "merging" of two algebras of the form (X,ω) with (seeing as relations) transitive and dense operations $\mathbf{s}$ and $\mathbf{t}$. The topological counterparts of such structures are triples of the form (X, τ_1, τ_2) with $\mathbf{T}_{1/2}$ topoplogies $\tau_1, \tau_2 \subseteq Pow(X)$. Such triples are known as *bitopological spaces* They will be considered in the next section.

Let us note some simply properties of $\mathrm{T}_{1/2}$-spaces.

COROLLARY 2.3. *For any $T_{1/2}$-space $\Upsilon = (X, \tau)$ and $x, y \in X$ it holds*

$$xR_\tau y \Leftrightarrow x = y \ \& \ C(x) = \{x\} \ \text{or} \ x \neq y \ \& \ y \in C(x).$$

Proof. If $xR_\tau y$ then

$xR_\tau y \Leftrightarrow (x = y \text{ or } x \neq y) \ \& \ (y \in C(x) \ \& \ C(y) = \{y\})$
$\Leftrightarrow x = y \ \& \ y \in C(x) \ \& \ C(y) = \{y\}$ or
$x \neq y \ \& \ y \in C(x) \ \& \ C(y) = \{y\}$.

Now

$x = y \ \& \ y \in C(x) \ \& \ C(y) = \{y\} \Leftrightarrow x = y \ \& \ C(x) = \{x\}$ ⌈evident⌉

and

$x \neq y \ \& \ y \in C(x) \ \& \ C(y) = \{y\} \Leftrightarrow x \neq y \ \& \ y \in C(x)$.⌈Corollary 2.1⌉ ∎

LEMMA 2.4. *For any set $A \subseteq X$ it holds*

$$C(A)\backslash A = R_\tau(A)\backslash A.$$

Proof. a) $C(A)\backslash A \subseteq R_\tau(A)\backslash A$.

Let $y \in C(A)\backslash A$. If it was $\sim(xR_\tau y)$ that means $y \notin C(x)$ for every $x \in A$ (see Corollary 2.3) then we would have

$$y \notin \bigcup_{x \in A} C(x) \subseteq C\Big(\bigcup_{x \in A} \{x\}\Big) = C(A)$$

which contradicts the assumption $x \in C(A)$. So it must exist $x \in A$ with $y \in C(x)$. Now let us assume that $\{y\} \neq C(y)$, i.e. that $\{y\}$ is an open set and let $x_0 \in A$ satisfy the condition $y \in C(x_0)$ (see the reasoning above). So $x_0 \neq y$ $(y \notin A)$ and we obtain $x_0 R_\tau y$ by Corollary 2.2. We have proved that

$$y \in C(A)\backslash A \Rightarrow \exists_{x \in A} \ xR_\tau y \ \& \ y \notin A \Leftrightarrow y \in (R_\tau(A)\backslash A)$$

which completes the proof.

b) $R_\tau(A)\backslash A \subseteq C(A)\backslash A$.

If $y \in (R_\tau(A)\backslash A)$ then there exists $x_0 \in A$ such that $xR_\tau y \ \& \ y \notin A$.
So we have

$$y \in C(x_0) \ \& \ C(y) = \{y\} \ \& \ x \neq y$$

(because $x_0 \in A$ & $y \notin A$). Now

$$y \in C(x_0) \subseteq C(A) \ \& \ y \notin A$$

(because $\{y\} \subseteq A$) and consequently

$$y \in C(A) \backslash A. \qquad \blacksquare$$

PROPOSITION 2.1. *For any set $A \subseteq X$ we have*

$$C(A) = C_R(A).$$

Proof. If $A \subseteq X$ then we have:

$$\begin{aligned} C(A) &= (C(A) \backslash A) \cup A \\ &= (R_\tau(A) \backslash A) \cup A && \lceil C(A) \backslash A = R_\tau(A) \backslash A \text{ by Lemma 2.4.} \rceil \\ &= A \cup R_\tau(A) \\ &= C_R(A). && \lceil \text{def. of } C_R \rceil \ \blacksquare \end{aligned}$$

PROPOSITION 2.2. *In any $T_{1/2}$-space $\Upsilon = (X, \tau)$ there holds*

$$C\Big(\bigcup_{t \in T} A_t\Big) = \bigcup_{t \in T} C(A_t)$$

for any family $(A_t)_{t \in T}$ of subsets of X.

Proof. For any $y \in X$ and any family $(A_t)_{t \in T}$ it holds

$$\begin{aligned} & y \in C\big(\bigcup_{t \in T} A_t\big) \Leftrightarrow y \in C_R\big(\bigcup_{t \in T} A_t\big) && \lceil \text{Proposition 2.1.} \rceil \\ & \Leftrightarrow y \in \bigcup_{t \in T} A_t \text{ or } \exists_{x \in \bigcup_{t \in T} A_t} x R_\tau y && \lceil \text{def. of } C_R \rceil \\ & y \in \bigcup_{t \in T} A_t \text{ or } \exists_{x \in X} \exists_{t \in T} \ x \in A_t \ \& \ x R_\tau y. \end{aligned}$$

If $y \in \bigcup_{t \in T} A_t$ then $y \in t_0$ for some $t_0 \in T$ and consequently

$$y \in \bigcup_{t \in T} C(A_t)$$

because $A_t \subseteq C(A_t)$ for any $t \in T$.

Let x_0 and $A_0 \in T$ satisfy the condition

$$y \notin \bigcup_{t \in T} A_t \ \& \ x_0 R_\tau y \ \& \ x_0 \in A_{t_0}.$$

Now

$$y \notin A_{t_0} \ \& \ x_0 R_\tau y \ \& \ x_0 \in A_{t_0}$$

that means $y \in R_\tau(A_{t_0})$, i.e. $y \in C_R(A_{t_0})$ which implies $y \in C(A_{t_0})$ by Proposition 2.2. So, there exists $t \in T$ such that $y \in C(A_t)$, i.e.

$$y \in \bigcup_{t \in T} C(A_t)$$

which completes the proof.

If $y \notin \bigcup_{t\in T} A_t$ then $y \notin A_t$ for every $t \in T$. So we have proved

$$C \bigcup_{t\in T} A_t \subseteq \bigcup_{t\in T} C(A).$$

The converse inclusion holds for any family of sets in any topology. ∎

COROLLARY 2.4. *In any $T_{1/2}$-space $\Upsilon = (X, \tau)$ and any family of subsets $(A_t)_{t\in T}$ of X there holds*

$$\forall_{t\in T} A_t \in \Upsilon \Rightarrow \Big(\bigcap_{t\in T} A_t\Big) \in \Upsilon.$$

Proof. Evident by De Morgan's laws. ∎

2.1. Bitopological spaces. By a *bitopological space* we mean a set endowed in two topologies, i.e. a triple of the form $(X, \tau_1, \tau_2), ((X, C_1, C_2) or$ (X, I_1, I_2) resp.) with τ_1, τ_2 being families of open sets (C_1, C_2 being topological closure operations and I_1, I_2 being topological interior operations resp.). Of course topologies in X may be defined in many other ways (e.g. by bases, subbases etc.). A triple $f : (X, \tau_1, \tau_2) \longrightarrow (X', \tau_1', \tau_2')$ with $(X, \tau_1, \tau_2), (X', \tau_1', \tau_2')$ being bitopological spaces and $f : X \longrightarrow X'$ being a function will be called *bicontinuous* mapping iff both $f : (X, \tau_1) \longrightarrow (X', \tau_1')$ and $f : (X, \tau_2) \longrightarrow (X', \tau_2')$ are continuous mappings. For more detailed description of bitopological spaces the reader is referred to [2].

$T_{1/2}$ bitopological spaces in which the frontier of any one element set is a one element set as well correspond to graphs. More precisely a graph can be seen as a bitopological $T_{1/2}$-space (X, τ_1, τ_2) in which the frontier of any one element set is (in both topologies) a one element set. In what follows we will frequently use such spaces. They will be called *d-spaces* (directed graph spaces). Let $\mathfrak{T} = (X, \tau_1, \tau_2)$ be such a d-space and $C_{\tau_1}, C_{\tau_2} : Pow(X) \to Pow(X)$ closure operations in $\mathfrak{T}$ corresponding to the topologies τ_1 and τ_2. If we will seen $\mathfrak{T}$ as a graph, then the operations C_{τ_1} and C_{τ_2} are our candidates for the source and target operation in the set X. Unfortunately in general they needn't have the common set of fix-poins.

EXAMPLE 2.2. *Let $\mathfrak{T} = (X, \tau_1, \tau_2)$ be the space with $X = \{a, b, c\}$ and topologies $\tau_1 = \{\emptyset, \{b\}, \{b, c\}, X\}$ and $\tau_2 = \{\emptyset, \{b\}, \{a, b\}, X\}$. Now we have $fix(C_{\tau_1}) = \{a\}$ and $fix(C_{\tau_2}) = \{c\}$. So if we would like to see the closure operations in the space as the source and target operations of a graph structure on the set X then the set $\{a\}$ would be a vertex in the sense of C_{τ_1} but not in the sense of C_{τ_2}. Consequently we have $(C_{\tau_1} \circ C_{\tau_2})(\{b\}) = C_{\tau_2}(\{a, b\}) = X$ and $(C_{\tau_1} \circ C_{\tau_1})(\{b\}) = C_{\tau_1}(\{a, b\}) = \{a, b\}$ that means $C_{\tau_1} \circ C_{\tau_2} \neq C_{\tau_1} \circ C_{\tau_1}$ which implies that the axioms defining graphs are not satisfied.*

The topological counterpart of the equations

$$\mathtt{t}(\mathtt{s}(x)) = \mathtt{s}(x) \ \& \ \mathtt{s}(\mathtt{t}(x)) = \mathtt{t}(x)$$

is the condition that both topologies have the same families of one element open sets (i.e. the restrictons of both topologies to one element sets equals) and the same families of one element closed sets. More precisely the counterpart of the equation $\mathtt{t}(\mathtt{s}(x)) = \mathtt{s}(x)$ is the condition that every one element closed set in τ_1 is closed in τ_2. The conterpart of the second equation is analogous.

DEFINITION 2.3. *Topologies τ_1 and τ_2 in a set X will be called open compatible on a family $F \subseteq Pow(X)$ iff for any set $A \in F$ $A \in \tau_1 \Longleftrightarrow A \in \tau_2$, closed compatible on F iff any for any set $A \in F$ A is closed in $\tau_1 \Longleftrightarrow A$ is closed in τ_2 and compatible iff they are open and closed compatible on F.*

Graphs are determined by d-spaces of the form $\mathfrak{X} = (X, \tau_1, \tau_2)$ with topologies compatible on the family of all one-element subsets of X. The bicontinous mappings correspond to graph homomorphisms.

PROPOSITION 2.3. *The category of graphs seeing as one sorted algebras $\mathcal{A} = (X, \mathtt{s}, \mathtt{t})$ with a set X and unary operations $\mathtt{s}, \mathtt{t} : X \longrightarrow X$ satisfying the conditions*

$$\mathtt{s}(\mathtt{s}(x)) = \mathtt{t}(\mathtt{s}(x)) = \mathtt{s}(x) \ \& \ \mathtt{s}(\mathtt{t}(x)) = \mathtt{t}(\mathtt{t}(x)) = \mathtt{t}(x)$$

is equivalent to the category of d-spaces and bicontinous mappings.

Proof. Straigthforward. ∎

2.1.1. Modal logic

In the paper [5] a new approach to the modal operators of necessity and possibility has been introduced. It bases on some "negation - operations" in the so called *bi-Heyting lattices.* The main examples of these operators are given by means of graphs. We show how these "negations" can be easily defined by means of topologies introduced in the paper. Let us recall some notions from [5].

DEFINITION 2.4. *A Heyting algebra is a bounded distributive lattice L with an "implication" operation $\rightarrow: L \times L \rightarrow L$ with the following property*

$$x \leq y \rightarrow z \ iff \ x \wedge y \leq z$$

for all $x, y, z \in L$. A co-Heyting algebra is a bounded distributive lattice L with a "subtraction" operation $\backslash : L \times L \rightarrow L$ with the following property

$$x \backslash y \leq z \ iff \ x \leq y \vee z$$

for all $x, y, z \in L$. Notice that L is a co-Heyting algebra iff the dual lattice L^0, obtained by reversing the order relation of L, is a Heyting algebra. The operation $\backslash$ in L is simply $\rightarrow$ in L^0. A bi-Heyting algebra is a bounded distributive lattice that is both a Heyting and a co-Heyting algebra.

In Gentzen's formalism the defining properties for $\rightarrow$ and $\backslash$ may be written in the form

$$\frac{x \leq y \rightarrow z}{x \wedge y \leq z} \qquad \frac{x \backslash y \leq z}{x \leq y \vee z}.$$

Having these operations one can define two "negation": $\neg x = x \rightarrow 0$ (the usual intuitionistic negation) and $\sim x = 1 \backslash x$, called in [5] the *supplement*, where 0 and 1 are the bottom and top elements of the lattice, respectively. They have the following defining properties

$$\frac{x \leq \neg y}{x \wedge y = 0} \qquad \frac{\sim x \leq y}{1 = x \vee y}.$$

So $\neg x$ is the largest element disjoint from x and $\sim x$ is the smallest element whose join with x gives the top element 1.

PROPOSITION 2.4. *(see [5]) In a Heyting algebra the negation operation $\neg$ is order reversing and satisfies $x \leq \neg\neg x$. In a co-Heyting algebra the supplementary operation $\sim$ is also order reversing and $\sim\sim x \leq x$.*

EXAMPLES. (see [5]) (1) A Boolean algebra is a bi-Heyting algebra. Define $x \rightarrow y = c(x) \vee y$ and $x \backslash y = x \wedge c(y)$, where $c(\ )$ is the operation of Boolean complement. Notice that in this case $\neg x = \sim x = c(x)$. Conversely, a bi-Heyting algebra such that $\neg x = \sim x$ for all x is automatically a Boolean algebra.

(2) Let X be a topological space. It is well-known that the lattice of open sets of X constitutes a Heyting algebra. We define $U \rightarrow V$ (for U and V open sets of X) to be the interior of $c(U) \cup V$, where $c(\ )$ is the usual Boolean complement.

Dually, the closed sets of X constitutes a co-Heyting algebra by defining $F \backslash G$ (for F and G closed sets of X) to be the closure of $F \cap c(G)$.

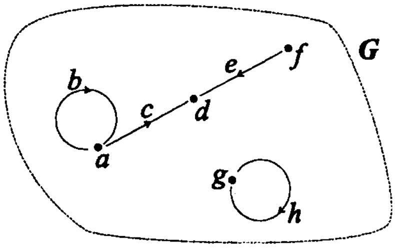
The third example considered by the authors comes from the theory of graphs. It is written in a form based on two sorted presentation of graphs[1]. We present it in a little different form based on one sorted graphs.

(3) Let $G = (X, s, t)$ be a graph. Then the lattice $P(G)$ of subgraphs of G is a bi-Heyting algebra[2]. The following text is a citation from [5] (p. 29).

"Take, for example, the following graph:

[1]The authors used the name "irreflexive multigraph" instead of "graph".

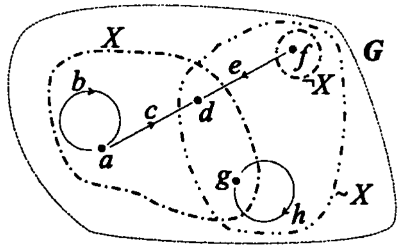
[2]It is exactly the set of all closed sets of the topology (closure operation) determined by the corresponding graph.



We write $G = \{a, b, c, d, e, f, g, h\}$ keeping in mind the relations $\delta_0(b) = a, \delta_1(b) = a, \delta_0(c) = a, \delta_1(c) = d$, etc. In this notation, a subgraph of a graph G is simply a subset of G closed under the operations of taking source and target of its arrows.

We can clearly take unions and intersections of a subgraphs but what about complements? Taking the set-theoretical complement $c(X)$ of X will not do, since it is not a graph in general. We may get "problematic" edges, i.e. edges whose sources or targets are missing in $c(X)$. To make a graph we have two options: either disregard problematic edges or, alternatively, keep them and add their sources and targets. The first option leads to the Heyting negation $\neg X$, whereas the second leads to the co-Heyting supplement $\sim X$.

Take, for example, that subgraph $X = \{a, b, c, d, g\}$ of the graph above. The set-theoretical complement is $\{e, f, h\}$ which is not a subgraph, the problematic arrows being e and h. If we disregard them, we obtain $\neg X = \{f\}$, the largest subgraph disjoint from X. On the other hand, if we keep them and add the missing sources and targets, namely d and g we obtain $\sim X = \{d, e, f, g, h\}$, the smallest subgraph whose union with X gives the whole graph G."



The problem with the construction of the Heyting negation is to find the greatest subgraph of a graph disjoint to another subgraph of this graph. In the "topological" presentation of graphs it can be solved by Proposition 2.2. Other considerations from [5] can be translated into the topological formalism proposed in part 2. So instead about some very special operations on graphs one can apply simply the standard constructions to the topological spaces corresponding to graphs. Of course the reasoning can be immediately extended to hypergraphs.

3. Concluding remarks

There exists a lot of various "translations" of the language of relations (functions) into that of (families of) sets. Perhaps the most well known examples are the principle of abstraction or the relationships between tolerance relations and covering families of a set[3]. There are also known some connections between relational systems and topology, e.g. topological spaces generated by semi- or partial orders. The result presented in the paper is of the same kind. The only essential difference here is the type of considered spaces; they are not "similar" to the "classical" topological spaces with a very "geometrical" origin. On the another side the origin of graphs is of a geometric character. In this sense the result of the paper can be seen as an illustration of the fact that various generalizations of some geometrical ideas may leads to "non-geometric" notions. It may also be interesting how the properties of n-topologies (or at least bitopologies) are related to the properties of n-graphs and vice versa, e.g. how one can characterize the convergence in the language of graphs.

The origins of the one-sorted definition of graphs come probably from the "French school of category theory" (cf [3] or [1] where some references can be found).

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[3]This relationship play an important role in various theories of concurency, e.g. in the theory of Petri nets. The sets of concurent (independent) actions are exactly classes of a special tolerancy relation (the relation of mutual independence).