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PRICING OF EUROPEAN AND AMERICAN CLAIMS IN THE CRR MODEL WITH FIXED PLUS - CONCAVE TRANSACTION COSTS

Abstract. In the paper CRR model with fixed + concave transaction costs is studied. Pricing of European and American claims is considered. The paper is a generalization of [4], where only concave transaction cost were investigated.

1. Introduction

In the paper we consider a discrete time financial market where two assets are given for trading, a riskless bond and a risky stock whose price is characterized by the so-called Cox-Ross-Rubinstein (CRR) model (see [3]). Transfers of wealth from one asset to another take place only at the discrete moments and the fixed + concave transaction costs for these transfers are incurred. A fixed costs are paid obligatory at each time moment even if there are no transactions. The case of nonobligatory costs i.e. the case when we pay fixed costs only after transaction is more complicated (due to discontinuity of the cost function) and will be studied independently.

We show that under some mild assumptions a replicating strategy is optimal for a special class of European claims. Next, we prove that if the transaction costs are sufficiently small, a replicating strategy is optimal for any European claim. Moreover, for both European and American claims the sets of capitals which are sufficient, starting from a given moment to hedge contingent claims, are characterized.

The paper extends [1], [2], [6], [7] where the CRR model with proportional transaction costs was studied and [4] where the model with only concave transaction costs was considered.

The CRR model is convenient from calculation point of view. As is shown in [5], a number of discrete time models with random rate of return

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can be reduced to certain CRR models. Fixed transactions costs together with proportional or concave appear on various financial market, however frequently in mathematical modeling are neglected.

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2. The model

Let (Ω, F, P) be a probability space with $\Omega = \{a, b\}^T$ where $-1 < a < 0$ and $b > 0$. We consider a market with two assets, a risky stock and a riskless bond with the constant price assumed for simplicity to be equal to one. Throughout this paper (in)equalities or other statements depending on $\omega \in \Omega$ if not stated otherwise will be understood in the P almost sure sense.

Let s_t be the price of the stock at time $t = 0, 1, \ldots, T$. We assume that s_t satisfies the following formula: $s_{t+1} = (1 + \eta_{t+1})s_t$, $t = 0, 1, \ldots, T-1$, $s_0 \in \mathbf{R}^+ \backslash \{0\}$, where $\eta_t, t = 1, \ldots, T$ is a sequence of i.i.d. random variables such that $P(\eta_t = a) + P(\eta_t = b) = 1$ and $0 < P(\eta_t = a) < 1$ for each $t = 1, \ldots, T$.

The above recursive formula for the price of the stock characterize so called Cox-Ross-Rubinstein model.

For any $\omega = (\omega_1, \ldots, \omega_T) \in \Omega$, we put $\omega_0^e = (e, \omega_2, \ldots, \omega_T)$ and $\omega_t^e = (\omega_1, \ldots, \omega_t, e, \omega_{t+2}, \ldots, \omega_T)$ for $t = 1, \ldots, T-1$, and $e = a, b$.

Let $F = \{F_t, t = 0, 1, \ldots, T\}$ be a family of increasing sub-σ-fields such that $F_t = \sigma(s_u, 0 \le u \le t), t = 0, 1, \ldots, T$. We assume that $F = F_T$.

In our model we consider fixed + concave transaction costs. We define two functions $c : \mathbf{R}^+ \cup \{0\} \to \mathbf{R}^+ \cup \{0\}$ and $d : \mathbf{R}^+ \cup \{0\} \to \mathbf{R}^+ \cup \{0\}$ which satisfy the following conditions:

1. $c(x)$- is convex and increasing function,
2. $c(0) = 0$ and $(1 - \mu)x \le c(x) \le x$,
3. $\lim_{x \to +\infty} c'(x) = 1$, $c'(x) < 1$ and $c'(0) = 1 - \mu$,
4. $d(x)$- is concave and increasing function,
5. $d(0) = 0$ and $x \le d(x) \le (1 + \lambda)x$,
6. $\lim_{x \to +\infty} d'(x) = 1$, $d'(x) > 1$ and $d'(0) = 1 + \lambda$,
   with λ being the proportional transaction cost rate for purchasing the asset and μ being the proportional transaction cost rate for selling the asset.

Define the functions $\tau_1 : \mathbf{R} \to \mathbf{R}^+ \cup \{0\}$ and $\tau_2 : \mathbf{R} \to \mathbf{R}^+ \cup \{0\}$:

$$\tau_1(x) = \begin{cases} d(x) + c, & x \ge 0 \\ 0, & x < 0 \end{cases} \tag{2.1}$$

and

$$\tau_2(x) = \begin{cases} c(x) - c, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{2.2}$$

where $c > 0$ is a fixed cost for purchasing and selling assets. Notice that we assume in this paper that fixed costs for purchasing and selling assets are equal. Let

$$\tau(x) = \begin{cases} \tau_1(x), & x \geq 0 \\ -\tau_2(-x) & x < 0. \end{cases} \tag{2.3}$$

The function $\tau(x)$ can be interpreted as the cost of getting the stock position worth x (x negative means that we sell $|x|$ stocks).

A trading strategy (x, y) is a pair of processes $\{(x_t, y_t), t = 0, \ldots, T\}$, where x_t, y_t are F_t measurable for each $t = 0, \ldots, T$, and x_t is an amount of money located at time t on the banking account and y_t is the number of assets in our portfolio at time t after possible transactions. In what follows we shall assume that at any time t we can make at almost one transaction and even when we don't change our portfolio (keep the same number of assets) we have to pay a constant transaction cost equal to c. A trading strategy (x, y) is said to be self-financial if

$$x_t - x_{t-1} + \tau((y_t - y_{t-1})s_t) = 0, \quad t = 0, \ldots, T. \tag{2.4}$$

We shall denote the set of all self-financing, trading strategies by $\mathbb{A}$.

3. European claims

A European claim φ is a pair $\varphi = (\varphi_1, \varphi_2)$ of F_T measurable random variables. Here φ_1, φ_2 denote number of units of bonds and stocks respectively, that are paid to the buyer of the option at time T.

We say that a trading strategy $(x, y) \in \mathbb{A}$ hedges a European claim φ if

$$\varphi_1 - x_{T-1} + \tau((\varphi_2 - y_{T-1})s_T) \leq 0. \tag{3.1}$$

We say that a trading strategy $(x, y) \in \mathbb{A}$ is replicating for a European claim φ if

$$\varphi_1 - x_{T-1} + \tau((\varphi_2 - y_{T-1})s_T) = 0. \tag{3.2}$$

For any claim φ, if $(x, y) \in \mathbb{A}$ hedges φ then we put $x_T = \varphi_1(s_T)$ and $y_T = \varphi_2(s_T)$. For CRR model the replication means:

$$x_{T-1} - \varphi_1(s_{T-1}(1+e)) = \tau((\varphi_2(s_{T-1}(1+e)) - y_{T-1})s_{T-1}(1+e)), \tag{3.3}$$

for $e = a, b$. For $e = a$ we have

$$x_{T-1} - \varphi_1^a = \tau((\varphi_2^a - y_{T-1})s_{T-1}^a), \tag{3.4}$$

where $s_{T-1}^a := s_{T-1}(1+a)$, $\varphi_1^a := \varphi_1(s_{T-1}(1+a))$, $\varphi_2^a := \varphi_2(s_{T-1}(1+a))$. For $e = b$ we have

$$x_{T-1} - \varphi_1^b = \tau((\varphi_2^b - y_{T-1})s_{T-1}^b), \tag{3.5}$$

where $s_{T-1}^b := s_{T-1}(1+b)$, $\varphi_1^b := \varphi_1(s_{T-1}(1+b))$, $\varphi_2^b := \varphi_2(s_{T-1}(1+b))$.

Define function ϕ as follows

$$\phi_{t+1}(z) := x_t^a - x_t^b + \tau((y_t^a - z)s_t^a) - \tau((y_t^b - z)s_t^b). \tag{3.6}$$

Subtracting (3.4) from (3.5) we conclude that the replication condition is satisfied when

$$\phi_T(y_{T-1}) = 0. \tag{3.7}$$

Notice that $\tau(x)$ and $\phi_{t+1}(z)$ are continuous functions. Furthermore taking into account that

$$d(x) - d(y) < x - y \quad \text{for} \quad x < y$$

we have

$$\begin{aligned}\lim_{z\to-\infty} \phi_{t+1}(z) &= \lim_{z\to-\infty}[x_t^a - x_t^b + \tau((y_t^a - z)s_t^a) - \tau((y_t^b - z)s_t^b)] = \\ &= x_t^a - x_t^b + \lim_{z\to-\infty}[d((y_t^a - z)s_t^a) - d((y_t^b - z)s_t^b)] \leq \\ &\leq x_t^a - x_t^b + \lim_{z\to-\infty}[(y_t^b - z)s_t^b - (y_t^a - z)s_t^a] = -\infty.\end{aligned}$$

Similarly using the fact that

$$c(x) - c(y) > x - y \quad \text{for} \quad x < y$$

we obtain

$$\lim_{z\to+\infty} \phi_{t+1}(z) = +\infty.$$

Consequently the range of $\phi_{t+1}(z)$ is equal to $\mathbf{R}$.

We have:

THEOREM 3.1. *If $\frac{1+b}{1+a} > \frac{1+\lambda}{1-\mu}$ then for each European claim φ and CRR model there exists a unique replicating strategy $(\widehat{x}_t, \widehat{y}_t)$. Moreover, if*

$$\widehat{y_{T-1}^a} \leq \widehat{y_{T-1}} \leq \widehat{y_{T-1}^b} \tag{1}$$

then for each $0 \leq t \leq T-1$

$$\widehat{y_t^a} \leq \widehat{y_t} \leq \widehat{y_t^b}. \tag{2}$$

Proof. Notice that under $\frac{1+b}{1+a} > \frac{1+\lambda}{1-\mu}$ the mapping $\phi_{t+1}(z)$ is strictly increasing. Since its range equal to $\mathbf{R}$ there is the unique solution $\widehat{y_{T-1}}$ of the equation $\phi_T(\widehat{y_{T-1}}) = 0$.

It remains to show inequality *(2)*. We use backward induction. For $t = T-1$ inequality *(2)* follows from *(1)*. Assume *(2)* holds for t. Then:

$$\widehat{y_{t-1}^{aa}} \leq \widehat{y_{t-1}^a} \leq \widehat{y_{t-1}^{ab}}$$

and

$$\widehat{y_{t-1}^{ba}} \leq \widehat{y_{t-1}^b} \leq \widehat{y_{t-1}^{bb}}.$$

Since be the uniqueness of the solutions to $\phi_t(\widehat{y_{t-1}}) = 0$ we have $\widehat{y^{ba}_{t-1}} = \widehat{y^{ab}_{t-1}}$ and we obtain $\widehat{y^a_{t-1}} \leq \widehat{y^b_{t-1}}$.

We know, that for any $t \leq T$ the system of equations is satisfied:

$$\left\{\begin{array}{l} \widehat{x_t} - \widehat{x^a_t} = \tau((\widehat{y^a_t} - \widehat{y_t})s^a_t) \\ \widehat{x_t} - \widehat{x^b_t} = \tau((\widehat{y^b_t} - \widehat{y_t})s^b_t). \end{array}\right. \tag{3.8}$$

Therefore

$$\left\{\begin{array}{l} \widehat{x^a_{t-1}} - \widehat{x^{ba}_{t-1}} = \tau((\widehat{y^{ba}_{t-1}} - \widehat{y^a_{t-y1}})s^{ba}_{t-1}) \\ \widehat{x^b_{t-1}} - \widehat{x^{ab}_{t-1}} = \tau((\widehat{y^{ab}_{t-1}} - \widehat{y^b_{t-1}})s^{ab}_{t-1}). \end{array}\right.$$

Subtracting equations and taking account that $\widehat{x^{ba}_{t-1}} = \widehat{x^{ab}_{t-1}}$ (which follows from the fact that $\widehat{y^{ba}_{t-1}} = \widehat{y^{ab}_{t-1}}$) we obtain:

$$\widehat{x^a_{t-1}} - \widehat{x^b_{t-1}} = \tau((\widehat{y^{ba}_{t-1}} - \widehat{y^a_{t-1}})s^{ba}_{t-1}) - \tau((\widehat{y^{ab}_{t-1}} - \widehat{y^b_{t-1}})s^{ab}_{t-1}).$$

Notice that

$$\tau((\widehat{y^{ba}_{t-1}} - \widehat{y^a_{t-1}})s^{ba}_{t-1}) - \tau((\widehat{y^{ab}_{t-1}} - \widehat{y^b_{t-1}})s^{ab}_{t-1}) \geq -\tau((\widehat{y^a_{t-1}} - \widehat{y^b_{t-1}})s^{ab}_{t-1}) + c$$

and

$$\tau((\widehat{y^{ba}_{t-1}} - \widehat{y^a_{t-1}})s^{ba}_{t-1}) - \tau((\widehat{y^{ab}_{t-1}} - \widehat{y^b_{t-1}})s^{ab}_{t-1}) \leq \tau((\widehat{y^b_{t-1}} - \widehat{y^a_{t-1}})s^{ba}_{t-1}) - c.$$

Therefore

$$-\tau((\widehat{y^a_{t-1}} - \widehat{y^b_{t-1}})s^{ab}_{t-1}) + c \leq \widehat{x^a_{t-1}} - \widehat{x^b_{t-1}} \leq \tau((\widehat{y^b_{t-1}} - \widehat{y^a_{t-1}})s^{ba}_{t-1}) - c.$$

Next, we observe that

$$\begin{aligned} \phi_t(\widehat{y^a_{t-1}}) &= \widehat{x^a_{t-1}} - \widehat{x^b_{t-1}} + \tau((\widehat{y^a_{t-1}} - \widehat{y^a_{t-1}})s^a_{t-1}) - \tau((\widehat{y^b_{t-1}} - \widehat{y^a_{t-1}})s^b_{t-1}) = \\ &= \widehat{x^a_{t-1}} - \widehat{x^b_{t-1}} + c - \tau((\widehat{y^b_{t-1}} - \widehat{y^a_{t-1}})s^{ab}_{t-1}) \leq \\ &\leq \tau((\widehat{y^b_{t-1}} - \widehat{y^a_{t-1}})s^{ba}_{t-1}) - c + c - \tau((\widehat{y^b_{t-1}} - \widehat{y^a_{t-1}})s^{ab}_{t-1}) = 0 \end{aligned}$$

and

$$\phi_t(\widehat{y^b_{t-1}}) \geq -\tau((\widehat{y^a_{t-1}} - \widehat{y^b_{t-1}})s^{ab}_{t-1}) + c + \tau((\widehat{y^a_{t-1}} - \widehat{y^b_{t-1}})s^{ab}_{t-1}) - c = 0.$$

By monotonicity and continuity of the function $\phi_t(z)$ there exists $\widehat{y_{t-1}}$ such that $\phi_t(\widehat{y_{t-1}}) = 0$ and should lie between $\widehat{y^a_{t-1}}$ and $\widehat{y^b_{t-1}}$ which by induction completes the proof. □

Now, for any $(p, q) \in \mathbf{R}^2$ we define the sets:

$$C_{(p,qs)} = \{(u, v) \in \mathbf{R}^2 : p - u + \tau((q - v)s) \leq 0\} \tag{3.9}$$

and

$$C^0_{(p,qs)} = \{(u, v) \in \mathbf{R}^2 : p - u + \tau'((q - v)s) \leq 0\}, \tag{3.10}$$

where

$$\tau'(x) = \begin{cases} d(x), & x \geq 0 \\ -c(-x) & x < 0. \end{cases} \tag{3.11}$$

Given an option (φ_1, φ_2), we say that a hedging strategy $(x, y) \in \mathbb{A}$ is optimal if for any other hedging strategy $(\overline{x}, \overline{y}) \in \mathbb{A}$ we have $C_{(\overline{x_t}, \overline{y_t} s_t)} \subseteq C_{(x_t, y_t s_t)}$.

The following theorem describes a relation between replicating strategies for concave and fixed + concave transaction costs.

THEOREM 3.2. *Let $\frac{1+b}{1+a} > \frac{1+\lambda}{1-\mu}$ and $(\widehat{x_t}, \widehat{y_t})$ be a replicating strategy for any European claim (φ_1, φ_2) with fixed + concave transaction costs, and $(\overline{x_t}, \overline{y_t})$ be a replicating strategy for any European claim (φ_1, φ_2) with concave transaction costs. Then*

$$\begin{cases} \widehat{y_t} = \overline{y_t} \\ \widehat{x_t} = (T-t)c + \overline{x_t}, \end{cases}$$

where c denotes a fixed transaction costs.

Proof. We use backward induction.

The strategy $(\widehat{x_t}, \widehat{y_t})$ replicates the claim (φ_1, φ_2) with fixed + concave transaction costs when

$$\varphi_1^a - \varphi_1^b + \tau((\varphi_2^a - \widehat{y_{T-1}})s_{T-1}^a) - \tau((\varphi_2^b - \widehat{y_{T-1}})s_{T-1}^b) = 0$$

and the strategy $(\overline{x_t}, \overline{y_t})$ replicates the claim (φ_1, φ_2) with concave transaction costs when

$$\varphi_1^a - \varphi_1^b + \tau'((\varphi_2^a - \overline{y_{T-1}})s_{T-1}^a) - \tau'((\varphi_2^b - \overline{y_{T-1}})s_{T-1}^b) = 0.$$

Since $\tau(x) = \tau'(x) + c$ by uniqueness of the solution to (3.7) we have that

$$\widehat{y_{T-1}} = \overline{y_{T-1}}.$$

Next

$$\widehat{x_{T-1}} = \varphi_1^a + \tau((\varphi_2^a - \widehat{y_{T-1}})s_{T-1}^a)$$

and

$$\overline{x_{T-1}} = \varphi_1^a + \tau'((\varphi_2^a - \overline{y_{T-1}})s_{T-1}^a).$$

Subtracting $\overline{x_{T-1}}$ from $\widehat{x_{T-1}}$ we obtain

$$\widehat{x_{T-1}} - \overline{x_{T-1}} = \tau((\varphi_2^a - \widehat{y_{T-1}})s_{T-1}^a) - \tau'((\varphi_2^a - \overline{y_{T-1}})s_{T-1}^a).$$

Since $\tau(x) = \tau'(x) + c$ we have

$$\widehat{x_{T-1}} = \overline{x_{T-1}} + c.$$

Assume that in the moment $t+1$ conditions hold

$$(3.12)\qquad \begin{cases} \widehat{y_{t+1}} = \overline{y_{t+1}} \\ \widehat{x_{t+1}} = (T-t-1)c + \overline{x_{t+1}}. \end{cases}$$

By definition of replication and by inductively assuming we have

$$\widehat{x_t^a} - \widehat{x_t^b} + \tau((\widehat{y_t^a} - \widehat{y}_t)s_t^a) - \tau((\widehat{y_t^b} - \widehat{y}_t)s_t^b) = 0$$

for the strategy $(\widehat{x}_t, \widehat{y}_t)$ and for the strategy $(\overline{x_t}, \overline{y_t})$:

$$\overline{x_t^a} - \overline{x_t^b} + \tau'((\overline{y_t^a} - \overline{y_t})s_t^a) - \tau'((\overline{y_t^b} - \overline{y_t})s_t^b) = 0.$$

Therefore by uniqueness of the solution to $\phi_{t+1}(y_t) = 0$ and using again the fact that $\tau(x) = \tau'(x) + c$ we obtain that

$$\widehat{y}_t = \overline{y_t}.$$

We observe that

$$\widehat{x}_t = \widehat{x_t^a} + \tau((\widehat{y_t^a} - \widehat{y}_t)s_t^a)$$

and

$$\overline{x_t} = \overline{x_t^a} + \tau'((\overline{y_t^a} - \overline{y_t})s_t^a).$$

For $e = b$ by analogy we obtain

$$\overline{x_t} = \overline{x_t^b} + \tau'((\overline{y_t^b} - \overline{y_t})s_t^b).$$

Subtracting $\overline{x_t}$ from $\widehat{x}_t$ and by $\widehat{x_{t+1}} = (T-t-1)c + \overline{x_{t+1}}$ we have:

$$\widehat{x}_t - \overline{x_t} = (T-t)c,$$

which completes the proof. □

Now, for each $t = 1, \dots, T$ we define the sets $\mathbf{\Delta}'_t, \mathbf{\Delta}_t$ consisting of a special type of pairs of random variables.

Let $\mathbf{\Delta}'_t, t = 1, \dots, T$ denote a set of all pairs of random variables $(p_1(s_t), p_2(s_t))$ such that p_2 is a nondecreasing real function and there exists a random variable $q(s_{t-1})$ such that

$$\tau'((p_2(s_{t-1}^b) - q(s_{t-1}))s_{t-1}^b) - \tau'((p_2(s_{t-1}^a) - q(s_{t-1}))s_{t-1}^a) = p_1(s_{t-1}^a) - p_1(s_{t-1}^b)$$

and

$$p_2(s_{t-1}^a) \le q(s_{t-1}) \le p_2(s_{t-1}^b),$$

and $\mathbf{\Delta}_t, t = 1, \dots, T$ denote a set of all pairs of random variables $(p_1(s_t), p_2(s_t))$ such that p_2 is a nondecreasing real function and there exists a random variable $q(s_{t-1})$ such that

$$\tau((p_2(s_{t-1}^b) - q(s_{t-1}))s_{t-1}^b) - \tau((p_2(s_{t-1}^a) - q(s_{t-1}))s_{t-1}^a) = p_1(s_{t-1}^a) - p_1(s_{t-1}^b)$$

and

$$p_2(s_{t-1}^a) \le q(s_{t-1}) \le p_2(s_{t-1}^b).$$

The class $\mathbf{\Delta}'_t$ and $\mathbf{\Delta}_t$ is quite natural and contains the following claims:

1) put option with $p_1(s) = M\mathbf{1}_{s \le M}$ and $p_2(s) = -\mathbf{1}_{s \le M}$.
2) call option with $p_1(s) = -M\mathbf{1}_{s \ge M}$ and $p_2(s) = \mathbf{1}_{s \ge M}$.

Using [4] we have the following fact:

THEOREM 3.3. *If the strategy* $(\overline{x_t}, \overline{y_t})$ *is replicating strategy for European claim* $(\varphi_1, \varphi_2) \in \mathbf{\Delta}'_T$ *with only concave transaction costs, then for any strategy* $(x_t, y_t) \in \mathbb{A}$ *we have: if* $(x_{t+1}, y_{t+1}) \in C^0_{(\overline{x_{t+1}}, \overline{y_{t+1}} s_{t+1})}$ *then* $(x_t, y_t) \in C^0_{(\overline{x_t}, \overline{y_t} s_t)}$.

We want to prove the optimality of the strategy $(\widehat{x_t}, \widehat{y_t})$ with fixed + concave transaction costs, the existence of which we proved in Theorem 3.1.

THEOREM 3.4. *Under* $\frac{1+b}{1+a} > \frac{1+\lambda}{1-\mu}$ *for any European claim* $\varphi \in \mathbf{\Delta}_T$ *the replicating strategy* $(\widehat{x_t}, \widehat{y_t}) \in \mathbb{A}$ *is optimal.*

Proof. Let φ be a European claim. By Theorem 3.1 there exists a unique strategy $(\widehat{x_t}, \widehat{y_t}) \in \mathbb{A}$ and $\varphi \in \mathbf{\Delta}_T$ and inequalities $\widehat{y_t^a} \leq \widehat{y_t} \leq \widehat{y_t^b}$, $t = 0, \dots, T-1$ hold.

Let $\varphi_1 = \widehat{x_T}$ and $\varphi_2 = \widehat{y_T}$.

Let (x_t, y_t) be a hedging strategy for the claim φ, i.e.

1^a) $\widehat{x_{T-1}^a} - x_{T-1} + \tau((\widehat{y_{T-1}^a} - y_{T-1}) s_{T-1}^a) \leq 0$

2^b) $\widehat{x_{T-1}^b} - x_{T-1} + \tau((\widehat{y_{T-1}^b} - y_{T-1}) s_{T-1}^b) \leq 0$.

Since the strategy $(\widehat{x_t}, \widehat{y_t})$ is replicating we have:

3^a) $\widehat{x_{T-1}^a} - \widehat{x_{T-1}} + \tau((\widehat{y_{T-1}^a} - \widehat{y_{T-1}}) s_{T-1}^a) = 0$

3^b) $\widehat{x_{T-1}^b} - \widehat{x_{T-1}} + \tau((\widehat{y_{T-1}^b} - \widehat{y_{T-1}}) s_{T-1}^b) = 0$.

We want to show that

$$\widehat{x_{T-1}} - x_{T-1} + \tau'((\widehat{y_{T-1}} - y_{T-1}) s_{T-1}) \leq 0, \tag{3.13}$$

i.e. $(x_{T-1}, y_{T-1}) \in C^0_{(\widehat{x_{T-1}}, \widehat{y_{T-1}} s_{T-1})}$. We know that $\widehat{x_{T-1}^e} = \varphi_1(s_{T-1}^e)$ and $\widehat{y_{T-1}^e} = \varphi_2(s_{T-1}^e)$ for $e = a, b$.

There are two cases:

1. $y_{T-1} \leq \widehat{y_{T-1}}$.

From 2^b) and 3^b) we have:

$$\widehat{x_{T-1}} - x_{T-1} + \tau((\widehat{y_{T-1}^b} - y_{T-1}) s_{T-1}^b) - \tau((\widehat{y_{T-1}^b} - \widehat{y_{T-1}}) s_{T-1}^b) \leq 0.$$

Because $\widehat{y_{T-1}} \leq \widehat{y_{T-1}^b}$ we obtain:

$$\begin{aligned} &\tau((\widehat{y_{T-1}^b} - y_{T-1}) s_{T-1}^b) - \tau((\widehat{y_{T-1}^b} - \widehat{y_{T-1}}) s_{T-1}^b) \geq \\ &\geq d((\widehat{y_{T-1}} - y_{T-1}) s_{T-1}^b) \geq (\widehat{y_{T-1}} - y_{T-1}) s_{T-1}^b \end{aligned}$$

and

$$\begin{aligned} &\tau((\widehat{y_{T-1}} - y_{T-1}) s_{T-1}) = d((\widehat{y_{T-1}} - y_{T-1}) s_{T-1}) + c \leq \\ &(1+\lambda)(\widehat{y_{T-1}} - y_{T-1}) s_{T-1} + c = \tfrac{1+\lambda}{1+b}(\widehat{y_{T-1}} - y_{T-1}) s_{T-1}^b + c. \end{aligned}$$

Therefore

$$(\widehat{y_{T-1}} - y_{T-1})s_{T-1}^b \geq \frac{1+b}{(1+\lambda)}(\tau((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) - c)$$

and

$$\tau((\widehat{y_{T-1}^b} - y_{T-1})s_{T-1}^b) - \tau((\widehat{y_{T-1}^b} - \widehat{y_{T-1}})s_{T-1}^b) \geq (\widehat{y_{T-1}} - y_{T-1})s_{T-1}^b \geq$$
$$\geq \tfrac{1+b}{(1+\lambda)}(\tau((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) - c) \geq \tau((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) - c.$$

Consequently,

$$\widehat{x_{T-1}} - x_{T-1} + \tau'((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) \leq 0,$$

which means that $(x_{T-1}, y_{T-1}) \in C^0_{(\widehat{x_{T-1}},\widehat{y_{T-1}}s_{T-1})}$.

2. $y_{T-1} \geq \widehat{y_{T-1}}$.

From 1^a) and 3^a) we have

$$\widehat{x_{T-1}} - x_{T-1} + \tau((\widehat{y_{T-1}^a} - y_{T-1})s_{T-1}^a) - \tau((\widehat{y_{T-1}^a} - \widehat{y_{T-1}})s_{T-1}^a) \leq 0.$$

Moreover

$$\tau((\widehat{y_{T-1}^a} - \widehat{y_{T-1}})s_{T-1}^a) - \tau((\widehat{y_{T-1}^a} - y_{T-1})s_{T-1}^a) \leq$$
$$\leq c((y_{T-1} - \widehat{y_{T-1}})s_{T-1}^a) \leq (y_{T-1} - \widehat{y_{T-1}})s_{T-1}^a$$

and

$$-\tau((y_{T-1} - \widehat{y_{T-1}})s_{T-1}) = c((y_{T-1} - \widehat{y_{T-1}})s_{T-1}) - c \geq$$
$$\geq (1-\mu)(y_{T-1} - \widehat{y_{T-1}})s_{T-1} - c = \tfrac{1-\mu}{1+a}(y_{T-1} - \widehat{y_{T-1}})s_{T-1}^a - c.$$

Therefore

$$(y_{T-1} - \widehat{y_{T-1}})s_{T-1}^a \leq \frac{1+a}{1-\mu}(c - \tau((y_{T-1} - \widehat{y_{T-1}})s_{T-1})).$$

Finally

$$\tau((\widehat{y_{T-1}^a} - y_{T-1})s_{T-1}^a) - \tau((\widehat{y_{T-1}^a} - \widehat{y_{T-1}})s_{T-1}^a) \geq$$
$$\geq \tfrac{1+a}{1-\mu}(\tau((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) - c) \geq \tau((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) - c.$$

Summarizing

$$\widehat{x_{T-1}} - x_{T-1} + \tau'((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) \leq 0$$

so that (3.13) is satisfied.

Assume inductively now that $(x_{t+1}, y_{t+1}) \in C^0_{(\widehat{x_{t+1}},\widehat{y_{t+1}}s_{t+1})}$ i.e.

$$\widehat{x_{t+1}} - x_{t+1} + \tau'((\widehat{y_{t+1}} - y_{t+1})s_{t+1}) \leq 0.$$

We want to show that in the moment t $(x_t, y_t) \in C^0_{(\widehat{x}_t,\widehat{y}_t s_t)}$, i.e.

$$\widehat{x}_t - x_t + \tau'((\widehat{y}_t - y_t)s_t) \leq 0.$$

Since $(x_{t+1}, y_{t+1}) \in C^0_{(\widehat{x_{t+1}},\widehat{y_{t+1}}s_{t+1})}$ and by Theorem 3.2 we have

$$(3.14)\qquad \begin{cases} \widehat{y_{t+1}} = \overline{y_{t+1}} \\ \widehat{x_{t+1}} = (T-t-1)c + \overline{x_{t+1}}, \end{cases}$$

$$(3.15)\qquad \begin{cases} y'_{t+1} = y_{t+1} \\ x'_{t+1} = x_{t+1} - (T-t-1)c, \end{cases}$$

where (x'_{t+1}, y'_{t+1}) is a hedging strategy for the claim (φ_1, φ_2) without fixed transaction costs. From this we obtain

$$(x'_{t+1} + (T-t-1)c, y'_{t+1}) \in C^0_{((T-t-1)c+\overline{x_{t+1}}, \overline{y_{t+1}} s_{t+1})}.$$

Therefore

$$(x'_{t+1}, y'_{t+1}) \in C^0_{(\overline{x_{t+1}}, \overline{y_{t+1}} s_{t+1})}.$$

By Theorem 3.3 we get

$$(x'_t, y'_t) \in C^0_{(\overline{x_t}, \overline{y_t} s_t)}.$$

Again, by Theorem 3.2 we obtain

$$(x_t - (T-t)c, y_t) \in C^0_{(\widehat{x}_t - (T-t)c, \widehat{y}_t s_t)},$$

which means that

$$(x_t, y_t) \in C^0_{(\widehat{x}_t, \widehat{y}_t s_t)}.$$

By backward induction the proof is therefore complete. □

3.1. Small transaction costs. In this subsection we consider small transaction costs, i.e. costs which satisfy the following inequality:

$$(3.16)\qquad \min\{1+b, \tfrac{1}{1+a}\} > \tfrac{1+\lambda}{1-\mu}.$$

Under (3.16) clearly $\frac{1+b}{1+a} > \frac{1+\lambda}{1-\mu}$ so that by Theorem 3.1 for any European claim φ there exists unique optimal, self-financing, trading strategy $(\widehat{x}_t, \widehat{y}_t)$ which replicates the portfolio (φ_1, φ_2) at time T.

By [4] we have the following fact:

THEOREM 3.5. *Under* (3.16) *if the strategy* $(\overline{x_t}, \overline{y_t})$ *is a replicating, then for any strategy* $(x_t, y_t) \in \mathbb{A}$ *we have: if* $(x_{t+1}, y_{t+1}) \in C^0_{(\overline{x_{t+1}}, \overline{y_{t+1}} s_{t+1})}$ *then* $(x_t, y_t) \in C^0_{(\overline{x_t}, \overline{y_t} s_t)}$.

Now, we want to prove the optimality of the strategy $(\widehat{x}_t, \widehat{y}_t)$.

THEOREM 3.6. *If the condition* (3.16) *is satisfied, then for any European claim* φ *a unique replicating strategy* $(\widehat{x}_t, \widehat{y}_t) \in \mathbb{A}$ *is optimal.*

Proof. Let φ be a given European claim.

Let $\varphi_1 = \widehat{x_T}$ and $\varphi_2 = \widehat{y_T}$. Let (x_t, y_t) be any strategy hedges claim φ, i.e.

$1^a)$ $\widehat{x^a_{T-1}} - x_{T-1} + \tau((\widehat{y^a_{T-1}} - y_{T-1})s^a_{T-1}) \le 0$
$2^b)$ $\widehat{x^b_{T-1}} - x_{T-1} + \tau((\widehat{y^b_{T-1}} - y_{T-1})s^b_{T-1}) \le 0.$
Since the strategy $(\widehat{x}_t, \widehat{y}_t)$ is replicating then:
$3^a)$ $\widehat{x^a_{T-1}} - \widehat{x_{T-1}} + \tau((\widehat{y^a_{T-1}} - \widehat{y_{T-1}})s^a_{T-1}) = 0$
$3^b)$ $\widehat{x^b_{T-1}} - \widehat{x_{T-1}} + \tau((\widehat{y^b_{T-1}} - \widehat{y_{T-1}})s^b_{T-1}) = 0.$
We want to show that

$$\widehat{x_{T-1}} - x_{T-1} + \tau'((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) \le 0,$$

i.e. $(x_{T-1}, y_{T-1}) \in C^0_{(\widehat{x_{T-1}}, \widehat{y_{T-1}} s_{T-1})}$.

There are six cases:

1. $y_{T-1} \le \widehat{y_{T-1}} \le \widehat{y^b_{T-1}}$.
This case is analogous to the proof of case **1.** in Theorem 3.3.

2. $\widehat{y^b_{T-1}} \le y_{T-1} \le \widehat{y_{T-1}}$.

From $2^b)$ and $3^b)$ we have

$$\widehat{x_{T-1}} - x_{T-1} + \tau((\widehat{y^b_{T-1}} - y_{T-1})s^b_{T-1}) - \tau((\widehat{y^b_{T-1}} - \widehat{y_{T-1}})s^b_{T-1}) \le 0.$$

From the properties of the functions c and d we obtain

$$\begin{aligned}&\tau((\widehat{y^b_{T-1}} - y_{T-1})s^b_{T-1}) - \tau((\widehat{y^b_{T-1}} - \widehat{y_{T-1}})s^b_{T-1}) \ge \\ &\ge (1-\mu)(\widehat{y_{T-1}} - y_{T-1})s^b_{T-1}\end{aligned}$$

and

$$\begin{aligned}&\tau((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) = \tau_1((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) = \\ &= d((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) + c \le \tfrac{1+\lambda}{1+b}(\widehat{y_{T-1}} - y_{T-1})s^b_{T-1} + c.\end{aligned}$$

From this

$$(\widehat{y_{T-1}} - y_{T-1})s^b_{T-1} \ge \frac{1+b}{1+\lambda}(\tau((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) - c).$$

Finally we have

$$\begin{aligned}&\tau((\widehat{y^b_{T-1}} - y_{T-1})s^b_{T-1}) - \tau((\widehat{y^b_{T-1}} - \widehat{y_{T-1}})s^b_{T-1}) \ge \\ &\ge (1-\mu)\tfrac{1+b}{1+\lambda}(\tau((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) - c) = \tau'((\widehat{y_{T-1}} - y_{T-1})s_{T-1}).\end{aligned}$$

And

$$\widehat{x_{T-1}} - x_{T-1} + \tau'((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) \le 0,$$

which means that $(x_{T-1}, y_{T-1}) \in C^0_{(\widehat{x_{T-1}}, \widehat{y_{T-1}} s_{T-1})}$.

3. $y_{T-1} \le \widehat{y^b_{T-1}} \le \widehat{y_{T-1}}$.

From $2^b)$, $3^b)$ and from (3.16) we have

$$(b-\lambda)(\widehat{y^b_{T-1}} - y_{T-1}) \ge 0 \ge ((1+\lambda) - (1-\mu)(1+b))(\widehat{y_{T-1}} - \widehat{y^b_{T-1}}).$$

Therefore
$(1+b)(\widehat{y^b_{T-1}}-y_{T-1})-(1-\mu)(1+b)(\widehat{y^b_{T-1}}-\widehat{y_{T-1}})\geq(1+\lambda)(\widehat{y_{T-1}}-y_{T-1})$.
From the properties of the function c and d we obtain
1) $\tau((\widehat{y^b_{T-1}}-y_{T-1})s^b_{T-1})\geq(1+b)(\widehat{y^b_{T-1}}-y_{T-1})s_{T-1}+c$,
2) $\tau((\widehat{y^b_{T-1}}-\widehat{y_{T-1}})s^b_{T-1})\leq(1-\mu)(1+b)(\widehat{y^b_{T-1}}-\widehat{y_{T-1}})s_{T-1}+c$,
3) $\tau((\widehat{y_{T-1}}-y_{T-1})s_{T-1})\leq(1+\lambda)(\widehat{y_{T-1}}-y_{T-1})+c$.
From the last three inequalities we obtain:

$$\begin{aligned}&\tau((\widehat{y^b_{T-1}}-y_{T-1})s^b_{T-1})-c-\tau((\widehat{y^b_{T-1}}-\widehat{y_{T-1}})s^b_{T-1})+c\geq\\&\geq\tau((\widehat{y_{T-1}}-y_{T-1})s_{T-1})-c=\tau'((\widehat{y_{T-1}}-y_{T-1})s_{T-1}).\end{aligned}$$

Since
$\widehat{x_{T-1}}-x_{T-1}+\tau((\widehat{y^b_{T-1}}-y_{T-1})s^b_{T-1})-\tau((\widehat{y^b_{T-1}}-\widehat{y_{T-1}})s^b_{T-1})\leq 0$
we therefore have

$$\widehat{x_{T-1}}-x_{T-1}+\tau'((\widehat{y_{T-1}}-y_{T-1})s_{T-1})\leq 0.$$

4. $\widehat{y^a_{T-1}}\leq\widehat{y_{T-1}}\leq y_{T-1}$.
This case is analogous to the proof of case **2.** in Theorem 3.3.

5. $\widehat{y_{T-1}}\leq y_{T-1}\leq\widehat{y^a_{T-1}}$.
From 1^a) and 3^a) we have

$$\widehat{x_{T-1}}-x_{T-1}+\tau((\widehat{y^a_{T-1}}-y_{T-1})s^a_{T-1})-\tau((\widehat{y^a_{T-1}}-\widehat{y_{T-1}})s^a_{T-1})\leq 0.$$

From the properties of the functions c and d we get

$$\begin{aligned}&\tau((\widehat{y^a_{T-1}}-y_{T-1})s^a_{T-1})-\tau((\widehat{y^a_{T-1}}-\widehat{y_{T-1}})s^a_{T-1})=\\&=d((\widehat{y^a_{T-1}}-y_{T-1})s^a_{T-1})-d((\widehat{y^a_{T-1}}-\widehat{y_{T-1}})s^a_{T-1})\geq\\&(1+\lambda)(\widehat{y_{T-1}}-y_{T-1})s^a_{T-1}\end{aligned}$$

and

$$\begin{aligned}&\tau((\widehat{y_{T-1}}-y_{T-1})s_{T-1})=-c((y_{T-1}-\widehat{y_{T-1}})s_{T-1})+c\leq\\&\leq-(1-\mu)(y_{T-1}-\widehat{y_{T-1}})s_{T-1}+c=\tfrac{1-\mu}{1+a}(\widehat{y_{T-1}}-y_{T-1})s^a_{T-1}+c.\end{aligned}$$

Therefore

$$(\widehat{y_{T-1}}-y_{T-1})s^a_{T-1}\geq\frac{1+a}{1-\mu}\tau'((\widehat{y_{T-1}}-y_{T-1})s_{T-1}).$$

Finally

$$\widehat{x_{T-1}}-x_{T-1}+\tau'((\widehat{y_{T-1}}-y_{T-1})s_{T-1})\leq 0.$$

6. $\widehat{y_{T-1}}\leq\widehat{y^a_{T-1}}\leq y_{T-1}$.
From 1^a), 3^a) and from (3.16) we obtain

$$(a+\mu)(y_{T-1}-\widehat{y^a_{T-1}})\leq 0\leq((1-\mu)-(1+\lambda)(1+a))(\widehat{y^a_{T-1}}-\widehat{y_{T-1}}).$$

From this we get
$(1+a)(\widehat{y^a_{T-1}} - y_{T-1}) - (1+\lambda)(1+a)(\widehat{y^a_{T-1}} - \widehat{y_{T-1}}) \geq (1-\mu)(\widehat{y^a_{T-1}} - y_{T-1})$.
From the properties of the functions c and d we have

1) $\tau((\widehat{y^a_{T-1}} - y_{T-1})s^a_{T-1}) = -\tau_2((y_{T-1} - \widehat{y^a_{T-1}})s^a_{T-1}) \geq$
$\geq (\widehat{y^a_{T-1}} - y_{T-1})s^a_{T-1} + c = (1+a)(\widehat{y^a_{T-1}} - y_{T-1})s_{T-1} + c,$
2) $\tau((\widehat{y^a_{T-1}} - \widehat{y_{T-1}})s^a_{T-1}) = d((\widehat{y^a_{T-1}} - \widehat{y_{T-1}})s^a_{T-1}) + c \leq$
$\leq (1+a)(1+\lambda)(\widehat{y^a_{T-1}} - \widehat{y_{T-1}})s_{T-1} + c,$
3) $\tau((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) = -c((y_{T-1} - \widehat{y_{T-1}})s_{T-1}) + c \leq$
$\leq (1-\mu)(\widehat{y_{T-1}} - y_{T-1})s_{T-1} + c.$

The last three inequalities imply

$$\tau((\widehat{y^a_{T-1}} - y_{T-1})s^a_{T-1}) - c - \tau((\widehat{y^a_{T-1}} - \widehat{y_{T-1}})s^a_{T-1}) + c \geq$$
$$\geq \tau((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) - c = \tau'((\widehat{y_{T-1}} - y_{T-1})s_{T-1}),$$

which together with
$\widehat{x_{T-1}} - x_{T-1} + \tau((\widehat{y^a_{T-1}} - y_{T-1})s^a_{T-1}) - \tau((\widehat{y^a_{T-1}} - \widehat{y_{T-1}})s^a_{T-1}) \leq 0$
lead us to the inequality

$$\widehat{x_{T-1}} - x_{T-1} + \tau'((\widehat{y_{T-1}} - y_{T-1})s_{T-1}) \leq 0.$$

Therefore $(x_{T-1}, y_{T-1}) \in C^0_{(\widehat{x_{T-1}}, \widehat{y_{T-1}} s_{T-1})}$.

Assume inductively now that $(x_{t+1}, y_{t+1}) \in C^0_{(\widehat{x_{t+1}}, \widehat{y_{t+1}} s_{t+1})}$ i.e.

$$\widehat{x_{t+1}} - x_{t+1} + \tau'((\widehat{y_{t+1}} - y_{t+1})s_{t+1}) \leq 0.$$

We want to show that in the moment t the strategy $(x_t, y_t) \in C^0_{(\widehat{x_t}, \widehat{y_t} s_t)}$ i.e.

$$\widehat{x_t} - x_t + \tau'((\widehat{y_t} - y_t)s_t) \leq 0.$$

Since $(x_{t+1}, y_{t+1}) \in C^0_{(\widehat{x_{t+1}}, \widehat{y_{t+1}} s_{t+1})}$ then by Theorem 3.2 we obtain

$$\begin{cases} \widehat{y_{t+1}} = \overline{y_{t+1}} \\ \widehat{x_{t+1}} = (T-t-1)c + \overline{x_{t+1}} \end{cases}$$

and

$$\begin{cases} y'_{t+1} = y_{t+1} \\ x'_{t+1} = x_{t+1} - (T-t-1)c, \end{cases}$$

where (x'_{t+1}, y'_{t+1}) is the hedging strategy for the claim (φ_1, φ_2) without fixed transaction costs. From this we have

$$(x'_{t+1} + (T-t-1)c, y'_{t+1}) \in C^0_{((T-t-1)c + \overline{x_{t+1}}, \overline{y_{t+1}} s_{t+1})}.$$

Therefore

$$(x'_{t+1}, y'_{t+1}) \in C^0_{(\overline{x_{t+1}}, \overline{y_{t+1}} s_{t+1})}.$$

By Theorem 3.3 we have

$$(x'_t, y'_t) \in C^0_{(\overline{x_t}, \overline{y_t} s_t)}.$$

Consequently by Theorem 3.2 we obtain

$$(x_t - (T-t)c, y_t) \in C^0_{(\widehat{x}_t - (T-t)c, \widehat{y}_t s_t)},$$

which means that

$$(x_t, y_t) \in C^0_{(\widehat{x}_t, \widehat{y}_t s_t)}$$

and by backward induction the proof is completed. □

4. American claims

We define an American claim f as a pair $\{f(t) = (f_1(t), f_2(t)), t \in 0, 1, \ldots, T\}$ of F adapted processes. Here, $f_1(t), f_2(t)$ denote quantities of units of bonds and stocks respectively, that are paid to the option's buyer assuming he exercises the option at time t.

We say that a strategy $(x, y) \in \mathbb{A}$ hedges an American claim f if

$$(4.1) \qquad f_1(t) - x_{t-1} + \tau((f_2(t) - y_{t-1})s_t) \leq 0 \qquad \text{for each } t = 0, 1, \ldots, T.$$

Given an claim f, we say that a hedging strategy $(x, y) \in \mathbb{A}$ is optimal if for any other hedging strategy $(\overline{x}, \overline{y}) \in \mathbb{A}$ we have $C_{(\overline{x_0}, \overline{y_0} s_0)} \subseteq C_{(x_0, y_0 s_0)}$.

In the moment T for replicating strategy $(q_1(T-1), q_2(T-1))$ in one step we have

$$f_1(T) - q_1(T-1) + \tau((f_2(T) - q_2(T-1))s_T) = 0.$$

Therefore under the assumptions of Theorem 3.3 or Theorem 3.5 any strategy (η_1, η_2) which hedges $(f_1(T), f_2(T))$ should be in $C^0_{(q_1(T-1), q_2(T-1)s_{T-1})}$. Consequently a strategy (η'_1, η'_2) which we chose at time $T-2$ after subtraction an obligatory cost c such be in $C^0_{(q_1(T-1), q_2(T-1)s_{T-1})}$. On the other hand that strategy we choose at time $T-2$ should also hedge $(f_1(T-1), f_2(T-1))$ at time $T-1$ i.e. we should have

$$f_1(T-1) - \eta'_1 + \tau((f_2(T-1) - \eta'_2)s_{T-1}) \leq 0.$$

Therefore $(\eta'_1 - c, \eta'_2) \in C^0_{(f_1(T-1), f_2(T-1)s_{T-1})} \cap C^0_{(q_1(T-1), q_2(T-1)s_{T-1})}$. Consequently an optimal strategy would be the one which replicates the peak of $C^0_{(f_1(T-1), f_2(T-1)s_{T-1})} \cap C^0_{(q_1(T-1), q_2(T-1)s_{T-1})}$.

Notice that the intersection of sets $C^0_{(f_1(T-1), f_2(T-1)s_{T-1})} \cap C^0_{(q_1(T-1), q_2(T-1)s_{T-1})}$ is under concave + fixed transaction costs no longer of C^0 sets form. To describe its form we need more general class of function than only the function τ or τ'.

Let Γ denote a set of all functions γ which satisfy the following conditions:

$(C1)$ $\forall z_1, z_2 \geq 0$ and $z_1 \leq z_2$, $\quad z_2 - z_1 \leq \gamma(z_2) - \gamma(z_1) \leq (1+\lambda)(z_2 - z_1)$,

$(C2)$ $\forall z_1, z_2 \leq 0$ and $z_1 \leq z_2$, $\quad z_2 - z_1 \geq \gamma(z_2) - \gamma(z_1) \geq (1-\mu)(z_2 - z_1)$,

$(C3)$ $\forall z \in \mathbf{R}, \quad \gamma(z) \le \tau(z)$,

$(C4)$ $\gamma(0) = 0$.

For any $(p_1, p_2) \in \mathbf{R}^2$ we define sets $\overline{\partial} C_{(p_1,p_2)}$ and $\underline{\partial} C_{(p_1,p_2)}$ as follows:

$$\overline{\partial} C^0_{(p_1,p_2)} = \{(u,v) \in \mathbf{R}^2 : p_1 - u + \tau'(p_2 - v) = 0 \wedge v > p_2\},$$

$$\underline{\partial} C^0_{(p_1,p_2)} = \{(u,v) \in \mathbf{R}^2 : p_1 - u + \tau'(p_2 - v) = 0 \wedge v < p_2\}.$$

For any $p = (p_1, p_2)$ and $q = (q_1, q_2)$ we define a set $V(p,q)$ as follows:

$$V(p,q) = \big\{(c,d) \in \partial(C^0_p \cap C^0_q) : \forall (u,v) \in \partial(C^0_p \cap C^0_q) \text{ if } v > d \text{ then } (u,v) \in \overline{\partial} C^0_p \cup \overline{\partial} C^0_q, \text{ and if } v < d \text{ then } (u,v) \in \underline{\partial} C^0_p \cup \underline{\partial} C^0_q\big\}.$$

Using [4] we have the following facts:

LEMMA 4.1. *For any $(c,d) \in V(p,q)$ there exists $\gamma \in \Gamma$ such that*

$$C^0_p \cap C^0_q = \{(u,v) \in \mathbf{R}^2 : c - u + \gamma(d - v) \le 0\}.$$

LEMMA 4.2. *Let $(p_1(s_t), p_2(s_t))$, $(q_1(s_t), q_2(s_t)) \in \mathbf{\Delta}_t$. There exist random variables $c(s_t), d(s_t)$ such that for any $\omega \in \Omega$ there is a function $\gamma_{s_t(\omega)} \in \Gamma$, with the equality*

$$C^0_{(p_1, p_2 s_t)(\omega)} \cap C^0_{(q_1, q_2 s_t)(\omega)} = \{(u,v) \in \mathbf{R}^2 \colon c(s_t)(\omega) - u + \gamma_{s_t}(d(s_t)s_t - v)(\omega) \le 0\}.$$

Moreover, there exists a unique random variable $w(s_{t-1})$ such that $\gamma_{s^b_{t-1}}((d^b_{t-1} - w(s_{t-1}))s^b_{t-1}) - \gamma_{s^a_{t-1}}((d^a_{t-1} - w(s_{t-1}))s^a_{t-1}) = c^a_{t-1} - c^b_{t-1}$ and $d^a_{t-1} \le w(s_{t-1}) \le d^b_{t-1}$.

Now, we show auxiliary lemma.

LEMMA 4.3. *For each $k \in \mathbf{R}$ the following inclusions are equivalent:*

$$(C_{(x,ys)} \subset C_{(x',y's)}) \Longleftrightarrow (C^0_{(x-k,ys)} \subset C^0_{(x'-k,y's)}).$$

Proof. ($\Rightarrow$) Assume that $C_{(x,ys)} \subset C_{(x',y's)}$. Let $(z_1, z_2) \in C^0_{(x-k,ys)}$, i.e.

$$x - k - z_1 + \tau'((y - z_2)s) \le 0.$$

From this

$$x - k - c - z_1 + \tau((y - z_2)s) \le 0,$$

$$x - (z_1 + (c + k)) + \tau((y - z_2)s) \le 0.$$

That means that $(z_1 + (c+k), z_2) \in C_{(x,ys)}$. Since $C_{(x,ys)} \subset C_{(x',y's)}$ we have $(z_1 + (c+k), z_2) \in C_{(x',y's)}$, i.e.

$$x' - (z_1 + (c + k)) + \tau((y' - z_2)s) \le 0,$$

$$x' - k - z_1 + \tau'((y' - z_2)s) \le 0.$$

Therefore $(z_1, z_2) \in C^0_{(x'-k,y's)}$.

($\Leftarrow$) Assume that $C^0_{(x-k,ys)} \subset C^0_{(x'-k,y's)}$. Let $(z_1, z_2) \in C_{(x,ys)}$, i.e.

$$x - z_1 + \tau((y - z_2)s) \leq 0.$$

From this

$$x + c - z_1 + \tau'((y - z_2)s) \leq 0,$$
$$(x - k) - (z_1 - c - k) + \tau'((y - z_2)s) \leq 0.$$

That means that $(z_1 - c - k, z_2) \in C^0_{(x-k,ys)}$. Since $C^0_{(x-k,ys)} \subset C^0_{(x'-k,y's)}$ we have $(z_1 - c - k, z_2) \in C^0_{(x'-k,y's)}$, i.e.

$$x' - k - (z_1 - c - k) + \tau'((y' - z_2)s) \leq 0,$$
$$x' - z_1 + \tau((y' - z_2)s) \leq 0.$$

Finally, we obtain $(z_1, z_2) \in C_{(x',y's)}$, which completes the proof. □

THEOREM 4.4. *If $(x'_t, y'_t) \in \mathbb{A}$ is a hedging strategy for an American claim $(f_1(t), f_2(t))$ with fixed + concave transaction costs then*

$$(\overline{x_t}, \overline{y_t}) = (x'_t - (T - t)c, y'_t) \in \mathbb{A}$$

is a hedging strategy for an American claim $(f_1(t) - (T - t)c, f_2(t))$ without fixed transaction costs.

Proof. Let $(x'_t, y'_t) \in \mathbb{A}$ be a hedging strategy with fixed transaction costs. In the moment $t = T$ the following conditions are satisfied:

$$f_1(T) - x'_{T-1} + \tau((f_2(T) - y'_{T-1})s_T) \leq 0 \tag{4.2}$$

and

$$x'_{T-1} - x'_{T-2} + \tau((y'_{T-1} - y'_{T-2})s_{T-1}) = 0. \tag{4.3}$$

Therefore

$$f_1(T) - (x'_{T-1} - c) + \tau'((f_2(T) - y'_{T-1})s_T) \leq 0,$$
$$x'_{T-1} - c - (x'_{T-2} - 2c) + \tau'((y'_{T-1} - y'_{T-2})s_{T-1}) = 0.$$

Let $(\overline{x_T}, \overline{y_T}) = (x'_T, y'_T)$, $(\overline{x_{T-1}}, \overline{y_{T-1}}) = (x'_{T-1} - c, y'_{T-1})$ and $(\overline{x_{T-2}}, \overline{y_{T-2}}) = (x'_{T-2} - 2c, y'_{T-2})$. We obtain

$$f_1(T) - \overline{x_{T-1}} + \tau'((f_2(T) - \overline{y_{T-1}})s_T) \leq 0,$$
$$\overline{x_{T-1}} - \overline{x_{T-2}} + \tau'((\overline{y_{T-1}} - \overline{y_{T-2}})s_{T-1}) = 0.$$

We assume that in the moment $t + 1$ the strategy $(x'_{t+1}, y'_{t+1}) \in \mathbb{A}$ hedges an American claim $(f_1(t+1), f_2(t+1))$ with fixed transaction costs and the strategy $(\overline{x_{t+1}}, \overline{y_{t+1}}) = (x'_{t+1} - (T - t - 1)c, y'_{t+1}) \in \mathbb{A}$ hedges an American claim $(f_1(t+1) - (T - t - 1)c, f_2(t+1))$ without fixed transaction costs and

$(\overline{x_t}, \overline{y_t}) = (x'_t - (T-t)c, y'_t)$, $(\overline{x_{t-1}}, \overline{y_{t-1}}) = (x'_{t-1} - (T-t-1)c, y'_{t-1})$. Then

$$f_1(t+1) - x'_{t+1} + \tau((f_2(t+1) - y'_{t+1})s_{t+1}) \leq 0,$$
$$x'_t - x'_{t-1} + \tau((y'_t - y'_{t-1})s_t) = 0.$$

Moreover

$$f_1(t+1) - (T-t-1)c - \overline{x_{t+1}} + \tau'((f_2(t+1) - \overline{y_{t+1}})s_{t+1}) \leq 0,$$
$$\overline{x_t} - \overline{x_{t-1}} + \tau'((\overline{y_t} - \overline{y_{t-1}})s_t) = 0.$$

In the moment t the strategy $(x'_t, y'_t) \in \mathbb{A}$ hedges an American claim $(f_1(t), f_2(t))$ with fixed transaction costs, i.e.

$$f_1(t) - x'_t + \tau((f_2(t) - y'_t)s_t) \leq 0,$$
$$x'_{t-1} - x'_{t-2} + \tau((y'_{t-1} - y'_{t-2})s_{t-1}) = 0.$$

From this we get

$$(f_1(t) - (T-t)c) - (x'_t - (T-t)c - c) + \tau'((f_2(t) - y'_t)s_t) \leq 0,$$
$$(x'_{t-1} - (T-t-1)c) - (x'_{t-2} - (T-t-1)c - c) + \tau'((y'_{t-1} - y'_{t-2})s_{t-1}) = 0.$$

By inductively assuming we obtain

$$(f_1(t) - (T-t)c) - (\overline{x_t} - c) + \tau'((f_2(t) - \overline{y_t})s_t) \leq 0,$$
$$\overline{x_{t-1}} - (x'_{t-2} - (T-t-2)c) + \tau'((\overline{y_{t-1}} - y'_{t-2})s_{t-1}) = 0.$$

Finally, we have $(\overline{x_{t-2}}, \overline{y_{t-2}}) = (x'_{t-2} - (T-t-2)c, y'_{t-2})$. By backward induction we obtain that the strategy $(\overline{x_t}, \overline{y_t})$ hedges an American claim $(f_1(t) - (T-t)c, f_2(t))$ without fixed transaction costs. The proof is completed. □

Using [4] we have the following fact:

THEOREM 4.5. *Let f be an American claim such that $f \in \mathbf{\Delta}_t$, for each $t = 1, \ldots, T$ and $\frac{1+b}{1+a} > \frac{1+\lambda}{1-\mu}$ or f be any claim but condition* (3.16) *is satisfied. Then there exists a strategy $(\widehat{x}_t, \widehat{y}_t) \in \mathbb{A}$ which is optimal with concave transaction costs, i.e. for any strategy $(x_t, y_t) \in \mathbb{A}$ hedges claim (f_1, f_2) we have*

$$(x_t, y_t) \in C^0_{(\widehat{x}_t, \widehat{y}_t)}$$

for each $t = 0, 1, \ldots, T-1$.

THEOREM 4.6. *Let f be an American claim such that $f \in \mathbf{\Delta}_t$, for each $t = 1, \ldots, T$ and $\frac{1+b}{1+a} > \frac{1+\lambda}{1-\mu}$ or f be any claim but condition* (3.16) *is satisfied. Let $(\widehat{x}_t, \widehat{y}_t)$ be an optimal strategy for American claim $(f_1(t) - (T-t)c, f_2(t))$ with concave transaction costs. Then the strategy $(\widehat{x}_t + (T-t)c, \widehat{y}_t)$ is optimal for American claim $(f_1(t), f_2(t))$ with fixed + concave transaction costs.*

Proof. Notice that if $(f_1(t), f_2(t)) \in \Delta_t$ then $(f_1(t) - (T-t)c, f_2(t)) \in \Delta_t$.

Let $(\overline{x_t}, \overline{y_t}) = (x'_t - (T-t)c, y'_t) \in \mathbb{A}$ be any hedging strategy for an American claim $(f_1(t) - (T-t)c, f_2(t))$ without fixed transaction costs. By Theorem 4.5. we obtain that $C^0_{(\overline{x_t}, \overline{y_t} s_t)} \subset C^0_{(\widehat{x}_t, \widehat{y}_t s_t)}$. From this $C^0_{(x'_t - (T-t)c, y'_t s_t)} \subset C^0_{((\widehat{x}_t + (T-t)c) - (T-t)c, \widehat{y}_t s_t)}$. By Lemma 4.3 for $k = (T-t)c$ we get that the condition $C^0_{(x'_t - (T-t)c, y'_t s_t)} \subset C^0_{(\widehat{x}_t, \widehat{y}_t s_t)}$ is equivalent to the condition $C_{(x'_t, y'_t s_t)} \subset C_{(\widehat{x}_t + (T-t)c, \widehat{y}_t s_t)}$, which completes the proof of optimality of the strategy $(\widehat{x}_t + (T-t)c, \widehat{y}_t)$. □

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