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ON THE SOLUTION OF SOME MAXIMIZATION PROBLEMS BASED ON A JENSEN INEQUALITY

Abstract. Models of fluid mechanics phenomena like the deflector of maximal drag or maximal lifting lead to maximization problems whose solutions are obtained using a method based on a Jensen inequality. The purpose of this paper is to point out the united character of the given solutions. A scheme is derived for an unconstrained and for a constrained maximization problem, which is applied to four examples.

1. Introduction

Models of fluid mechanics phenomena like the deflector of maximal drag [3]–[6] or maximal lifting [9], lead to maximization problems whose solutions are obtained using a method based on a Jensen inequality. The unknown of these optimization problems is a function, which in our examples represents the velocity distribution. The involved object functional is nonlinear.

The purpose of this paper is to point out the united character of the given solutions.

The main idea to maximize a functional $I(u)$ is based on the use of the Jensen inequality, to majorate $I(u)$ to a functional $J(u)$, whose maximal point may be easily computed and which is a constant function u_*. If the Jensen inequality is applied to a constant function then there is the equality case. From the relations

$$I(u) \leq J(u) \leq J(u_*) = I(u_*),$$

we deduce that u_* maximizes the functional $I(u)$, too.

Maklakov D.V. is among the first who used this method [5], [6].

Key words and phrases: nonlinear maximization problems, Jensen's inequality, deflector of maximal drag, maximal lifting.

2. The Jensen inequality

We recall the used Jensen's inequality [2]. It may be obtained from the general Jensen inequality [1]. The presented proof of this theorem is from [8].

THEOREM 2.1. *Let S be a nonempty set, V a linear space of real functions defined over S. Assume that the constant function 1 belongs to V and let L be a linear and positive functional such that $L(1) = 1$. If $\varphi \in V$ and $F \in C[\alpha, \beta]$ is a convex function such that $F \circ \varphi \in V$ then:*

(1) (*i*) $L(\varphi) \in [\alpha, \beta]$,

(2) (*ii*) $F(L(\varphi)) \leq L(F \circ \varphi)$.

Proof. (i) The condition $F \circ \varphi \in V$ implies that $\alpha \leq \varphi(x) \leq \beta$ for any $x \in S$. Using the properties of L we obtain $\alpha \leq L(\varphi) \leq \beta$.

(ii) For any $\varepsilon > 0$, separating the convex sets $\{(t, y) : y \geq F(t), t \in [\alpha, \beta]\}$ and $\{(L(\varphi), F(L(\varphi)) - \varepsilon)\}$ (or from the existence of lateral derivatives in the point $L(\varphi)$ of the convex function F), there exists a first degree polynomial $p(t) = u + vt$ such that

$$p(t) \leq F(t), \quad \forall t \in [\alpha, \beta], \tag{3}$$

and

$$p(L(\varphi)) \geq F(L(\varphi)) - \varepsilon. \tag{4}$$

From (3) it results that $p \circ \varphi \leq F \circ \varphi$ and then

$$L(F \circ \varphi) \geq L(p \circ \varphi) = L(u + v\varphi) = u + vL(\varphi) = p(L(\varphi)) \geq F(L(\varphi)) - \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, we have (ii). ∎

As a consequence we have:

THEOREM 2.2. *If $f(x) \geq 0, g(x)$ are continuous functions in $[a, b]$, then*

$$\int_a^b f(x) \exp(g(x)) \mathrm{d}x \geq \int_a^b f(x) \ \exp\left(\frac{\int_a^b f(x) g(x) \mathrm{d}x}{\int_a^b f(x) \mathrm{d}x}\right) \mathrm{d}x,$$

where the equality case holds if $g(x)$ is a constant function.

Proof. Let $S = [a, b], V = C[a, b], L(u) = \frac{\int_a^b f(x) u(x) \mathrm{d}x}{\int_a^b f(x) \mathrm{d}x}, \varphi = g$ and $[\alpha, \beta] = g([a, b])$. From (2) we obtain

$$F\left(\frac{\int_a^b f(x) g(x) \mathrm{d}x}{\int_a^b f(x) \mathrm{d}x}\right) \leq \frac{\int_a^b F(g(x)) f(x) \mathrm{d}x}{\int_a^b f(x) \mathrm{d}x}.$$

For $F = \exp$, it results the desired inequality. ∎

3. An unconstrained maximization problem

Let us consider the maximization problem of the functional

$$I(u) = \frac{F(\int_a^b f(x)u(x)\mathrm{d}x)}{\int_a^b f(x)\exp(u(x))\mathrm{d}x} \to \max \tag{5}$$

where f and F are given functions. We suppose that $f(x) > 0$ in (a, b). The unknown function $u(x)$ is searched in the set of functions which assure the existence of the integrals. We shall use the notations:

$$\xi = \int_a^b f(x)\mathrm{d}x; \qquad y(u) = \int_a^b f(x)u(x)\mathrm{d}x. \tag{6}$$

Applying Jensen's inequality to the denominator, we find

$$I(u) \le \frac{F(\int_a^b f(x)u(x)\mathrm{d}x)}{\int_a^b f(x)\exp\left(\frac{\int_a^b f(x)u(x)\mathrm{d}x}{\int_a^b f(x)\mathrm{d}x}\right)\mathrm{d}x} = \frac{F(y(u))}{\xi\exp(\frac{y(u)}{\xi})} = J(u).$$

Let y_* be a point which maximizes the function $y \to \frac{F(y)}{\xi\exp(\frac{y}{\xi})}$. Then, for the function $u_*(x) = \frac{y_*}{\xi}, \forall x \in [a, b]$ we have

$$I(u_*) = J(u_*) = \frac{F(y(u_*))}{\xi\exp(\frac{y(u_*)}{\xi})} = \frac{F(\int_a^b f(x)u_*(x)\mathrm{d}x)}{\xi\exp\left(\frac{\int_a^b f(x)u_*(x)\mathrm{d}x}{\xi}\right)} =$$

$$= \frac{F(y_*)}{\xi\exp(\frac{y_*}{\xi})} = \max_y \frac{F(y)}{\xi\exp\left(\frac{y}{\xi}\right)} \ge I(u),$$

hence u_* is the solution of the maximization problem (5).

EXAMPLE 3.1. In [9] it is studied the maximal lifting for the optimal profile on sprayless planning surface and the following maximization problem is obtained:

$$I(u) = \frac{2\int_{-1}^1 u(x)\mathrm{d}x}{\int_{-1}^1 \exp(u(x))\mathrm{d}x} \to \max. \tag{7}$$

By our scheme, we have $\xi = 2$, $F(y) = 2y$ and $\frac{F(y)}{\xi\exp(\frac{y}{\xi})} = \frac{y}{\exp(\frac{y}{2})}$, whose maximum point is $y_* = 2$. The function which maximizes the functional $I(u)$ is then $u_*(x) = \frac{y_*}{\xi} = 1$.

EXAMPLE 3.2. In [3] the deflector of maximal drag is modeled for the plane potential flow of an inviscid, incompressible and unlimited fluid jet which encounters a symmetrical, curvilinear obstacle. The derived maximization problem is

$$\frac{(\int_{-1}^{1}\frac{t(x)}{\sqrt{1+x}}\mathrm{d}x)^2}{\int_{-1}^{1}\exp(t(x))\mathrm{d}x} \longrightarrow \max. \tag{8}$$

If we put $t(x) = u(x) - \ln\sqrt{\frac{x+1}{2}}$, the above problem becomes

$$I(u) = \frac{(\int_{-1}^{1}\frac{u(x)}{\sqrt{1+x}}\mathrm{d}x + 2\sqrt{2})^2}{\sqrt{2}\int_{-1}^{1}\frac{\exp(u(x))}{\sqrt{1+x}}\mathrm{d}x} \longrightarrow \max. \tag{9}$$

Thus $\xi = 2\sqrt{2}$, $F(y) = \frac{(y+2\sqrt{2})^2}{\sqrt{2}}$ and $\frac{F(y)}{\xi\exp(\frac{y}{\xi})} = \frac{(y+2\sqrt{2})^2}{4\exp(\frac{y}{2\sqrt{2}})}$, which has the maximum point at $y_* = 2\sqrt{2}$. Hence $u_*(x) = \frac{y_*}{\xi} = 1$ maximizes the functional (8) and consequently the solution of (9) is $t_*(x) = 1 - \ln\sqrt{\frac{x+1}{2}}$.

4. A constrained maximization problem

Given a non-decreasing function F, a non-negative function f and a positive real number k, find the function $u(x)$ which maximize the functional

$$I(u) = F(\int_a^b f(x)u(x)\mathrm{d}x) \tag{10}$$

subject to the constraint

$$\int_a^b f(x)\exp(u(x))\mathrm{d}x = k. \tag{11}$$

Using the notations (6), if we apply Jensen's inequality to the constraint (11), then we obtain

$$k = \int_a^b f(x)\exp(u(x))\mathrm{d}x \geq \int_a^b f(x)\exp\left(\frac{\int_a^b f(x)u(x)\mathrm{d}x}{\int_a^b f(x)\mathrm{d}x}\right)\mathrm{d}x = \xi\exp\left(\frac{y(u)}{\xi}\right).$$

Consequently

$$y(u) \leq \xi\ln\frac{k}{\xi} \quad \text{and} \quad F(y(u)) \leq F\left(\xi\ln\frac{k}{\xi}\right).$$

If $u_*(x) = \ln\frac{k}{\xi}$, $\forall x \in [a,b]$ then $y(u_*) = \int_a^b f(x)u_*(x)\mathrm{d}x = \xi\ln\frac{k}{\xi}$. Thus the constrain (11) is fullfield and u_* maximizes the functional (10). The maximum value of the functional (10) is $F(\xi\ln\frac{k}{\xi})$.

In the next two examples the deflector of maximal drag is modeled for the potential flow of an inviscid, incompressible and limited fluid jet, which encounters a symmetrical, curvilinear obstacle. As the canonical domain, in [5] the Levi-Civita circle is used , while in [4] the half plane is used.

EXAMPLE 4.1. In [4] it is considered the optimization problem

$$\frac{\sqrt{a+1}}{\pi}\int_{-1}^{1}\frac{t(x)}{\sqrt{x+1}}\frac{\mathrm{d}x}{a-x}\to\max \qquad (a>1) \tag{12}$$

subject to the constraint

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\exp(t(x))}{a-x}\mathrm{d}x=\tilde{k}. \tag{13}$$

If we put $t(x)=u(x)+\ln\sqrt{\frac{2}{x+1}}$, the objective functional becomes (see the next section)

$$I(u)=\mathrm{T}(\sqrt{\frac{2}{a+1}})+\frac{\sqrt{a+1}}{\pi}\int_{-1}^{1}\frac{u(x)}{\sqrt{x+1}(a-x)}\mathrm{d}x \tag{14}$$

where

$$\mathrm{T}(\alpha)=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\alpha^{2n+1}}{(2n+1)^2}=\frac{1}{\pi}[\mathrm{Li}_2(\alpha)-\mathrm{Li}_2(-\alpha)], \qquad 0<\alpha<1,$$

and $\mathrm{Li}_2(\alpha)=\sum_{n=0}^{\infty}\frac{\alpha^n}{n^2}$ is Euler's dilogarithm series [7], [5], [6]. The constraint (14) becomes

$$\int_{-1}^{1}\frac{\exp(u(x))}{\sqrt{x+1}(a-x)}\mathrm{d}x=\frac{\pi\tilde{k}}{\sqrt{2}}\stackrel{\mathrm{def}}{=}k. \tag{15}$$

Thus

$$\xi=\int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt{x+1}(a-x)}=\frac{1}{\sqrt{a+1}}\ln\frac{\sqrt{a+1}+\sqrt{2}}{\sqrt{a+1}-\sqrt{2}}\stackrel{\mathrm{def}}{=}g(a)$$

and

$$F(y)=\mathrm{T}(\sqrt{\frac{2}{a+1}})+\frac{\sqrt{a+1}}{\pi}y.$$

The solution of the problem (14)-(15) is then

$$u_*(x)=\ln\frac{k}{\xi}=\ln\frac{\pi\tilde{k}}{\sqrt{2}g(a)},$$

while the maximum value of the objective functional is

$$I(u_*)=\mathrm{T}(\sqrt{\frac{2}{a+1}})+\frac{\sqrt{a+1}}{\pi}g(a)\ln\frac{\pi\tilde{k}}{\sqrt{2}g(a)}=\Phi(a).$$

Finally, in [4], the parameter a is determined to maximize the function $\Phi(a)$.

EXAMPLE 4.2. In [5] (p.38) the maximization problem is

$$\frac{4a(a^2+1)}{\pi}\int_0^{\frac{\pi}{2}}\frac{\nu(x)\sin x}{a^4+1-2a^2\cos 2x}\mathrm{d}x \to \max \qquad (0<a<1), \tag{16}$$

subject to

$$\frac{8a^2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\exp(u(x))\sin x}{a^4+1-2a^2\cos 2x}\mathrm{d}x = \tilde{k}, \tag{17}$$

$$u(x) = \nu(x) + \ln\cos x. \tag{18}$$

Eliminating $\nu(x)$ the maximization functional will be

$$I(u) = \mathrm{T}\left(\frac{2a}{a^2+1}\right) + \frac{4a(a^2+1)}{\pi}\int_0^{\frac{\pi}{2}}\frac{u(x)\sin x}{a^4+1-2a^2\cos 2x}\mathrm{d}x \tag{19}$$

with the restriction rewritten as

$$\int_0^{\frac{\pi}{2}}\frac{\exp(u(x))\sin x}{a^4+1-2a^2\cos 2x}\mathrm{d}x = \frac{\pi\tilde{k}}{8a^2} \stackrel{\text{def}}{=} k. \tag{20}$$

In this case

$$\xi = \int_0^{\frac{\pi}{2}}\frac{\sin x}{a^4+1-2a^2\cos 2x}\mathrm{d}x = \frac{1}{2(a^2+1)}\ln\frac{1-a}{1+a} \stackrel{\text{def}}{=} h(a)$$

and

$$F(y) = \mathrm{T}\left(\frac{2a}{a^2+1}\right) + \frac{4a(a^2+1)}{\pi}y.$$

Then, the solution of the problem (19)-(20) is

$$u_*(x) = \ln\frac{k}{h(a)} = \ln\frac{\pi\tilde{k}}{8a^2h(a)},$$

and the maximum value of the objective functional is

$$I(u_*) = \mathrm{T}\left(\frac{2a}{a^2+1}\right) + \frac{2}{\pi}\ln\frac{1-a}{1+a}\ln\frac{\pi\tilde{k}}{8a^2h(a)} \stackrel{\text{def}}{=} \Psi(a).$$

Next, in [5], the parameter a is computed to maximize the function $\Psi(a)$.

5. Used integrals

1. $I_1 = \int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt{x+1}(a-x)}$ $(a>1)$. Changing successively the variables $x=\cos t$ and $\cos\frac{t}{2} = s$ we obtain

$$I_1 = \sqrt{2}\int_0^{\pi}\frac{\sin\frac{t}{2}}{a-\cos t}\mathrm{d}t = -\sqrt{2}\int_0^{1}\frac{\mathrm{d}s}{s^2-\frac{a+1}{2}} = \frac{1}{\sqrt{a+1}}\ln\frac{\sqrt{a+1}+\sqrt{2}}{\sqrt{a+1}-\sqrt{2}} = g(a).$$

2. $I_2=\int_{-1}^{1}\frac{\ln\sqrt{x+1}}{\sqrt{x+1}(a-x)}\mathrm{d}x \quad (a>1)$. If we put $x=\cos t$, then we obtain

$$I_2=\frac{\sqrt{2}\ln 2}{2}\int_0^{\pi}\frac{\sin\frac{t}{2}}{a-\cos t}\mathrm{d}t+\sqrt{2}\int_0^{\pi}\frac{\sin\frac{t}{2}\ln(\cos\frac{t}{2})}{a-\cos t}\mathrm{d}t=\frac{\ln 2}{2}g(a)+I_{21}.$$

For the second term, changing $\cos\frac{t}{2}=s$, we have

$$I_{21}=-\sqrt{2}\int_0^1\frac{\ln s}{s^2-\alpha^2}=\sqrt{2}\sum_{k=0}^{\infty}\frac{1}{\alpha^{2k+2}}\int_0^1 s^{2k}\ln s\mathrm{d}s=-\frac{\pi}{\alpha\sqrt{2}}\mathrm{T}\left(\frac{1}{\alpha}\right),$$

where $\alpha^2=\frac{a+1}{2}$. Thus

$$I_2=\frac{\ln 2}{2}g(a)-\frac{\pi}{\sqrt{a+1}}\mathrm{T}\left(\sqrt{\frac{2}{a+1}}\right).$$

3. $I_3=\int_0^{\frac{\pi}{2}}\frac{\sin x}{a^4+1-2a^2\cos 2x}\mathrm{d}x \quad (0<a<1)$. For $\cos x=t$ we have

$$I_3=-\frac{1}{4a^2}\int_0^1\frac{dt}{t^2-(\frac{a^2+1}{2a})^2}=\frac{1}{2a(a^2+1)}\ln\frac{1+a}{1-a}=h(a).$$

4. $I_4=\int_0^{\frac{\pi}{2}}\frac{\sin x\ln(\cos x)}{a^4+1-2a^2\cos 2x}\mathrm{d}x \quad (0<a<1)$. If $\cos x=t$ and $\alpha^2=\frac{a^2+1}{2a}$ then

$$I_4=-\frac{1}{4a^2}\int_0^1\frac{\ln t}{t^2-\alpha^2}\mathrm{d}t=\frac{1}{4a^2}\sum_{k=0}^{\infty}\frac{1}{\alpha^{2k+2}}\int_0^1 t^{2k}\ln t\mathrm{d}t=$$

$$=-\frac{\pi}{8a^2\alpha}\mathrm{T}\left(\frac{1}{\alpha}\right)=-\frac{\pi}{4a(a^2+1)}\mathrm{T}\left(\frac{2a}{a^2+1}\right).$$

Using a Computer Algebra System, I_1 is computed by *Derive 5.0* while *Mathematica 4.0, Maple 6.0, Mathcad 2000* gives $\frac{2}{\sqrt{a+1}}\mathrm{atanh}\sqrt{\frac{2}{a+1}}$. I_3 is computed by *Mathematica 4.0, Maple 6.0* and *Mathcad 2000.* For I_2 *Maple 6.0* gives an improper result. I_4 isn't computed by any of the above CAS.

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