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NATURAL OPERATORS LIFTING PROJECTABLE-PROJECTABLE VECTOR FIELDS TO PRODUCT PRESERVING BUNDLE FUNCTORS ON FIBERED-FIBERED MANIFOLDS

Abstract. For any product-preserving bundle functor F defined on the category $\mathcal{F}^2\mathcal{M}$ of fibered-fibered manifolds, we determine all natural operators transforming projectable-projectable vector fields on $Y \in Ob(\mathcal{F}^2\mathcal{M})$ to vector fields on FY. We also determine all natural affinors on FY and prove a composition property analogous to that concerning Weil bundles.

0. Preliminaries

The classical results by Kainz and Michor, [2], Eck, [1] and Luciano, [5] read that the product preserving bundle functors on the category $\mathcal{M}f$ of manifolds are just Weil bundles, [4].

Let us remind the result by Kolář, [3]. For a bundle functor $F : \mathcal{M}f \to \mathcal{FM}$, denote by $\mathcal{F}$ the flow operator associated to F which is defined by $\mathcal{F}X = \frac{d}{dt}|_0 F(Fl_t^X)$ for any vector field X on M. Further, consider an element c of a Weil algebra A and let $L(c)_M : TT^AM \to TT^AM$ denote the natural affinor by Koszul, [3] and [4]. Then we have a natural operator $L(c)_M \circ \mathcal{T}^A$: $TM \to TT^AM$ lifting vector fields on a manifold M to a Weil bundle T^AM. As for the absolute natural operators $T \to TT^A$, i.e. independent on a vector field X, they are of the form Λ_D for a derivation $D \in \operatorname{Der} A$. They are defined as follows.

The Lie algebra $\mathcal{A}ut(A)$ associated to the Lie group of all algebra automorphisms of A is identified with the algebra of derivations $\operatorname{Der} A$ of A. For

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any $D \in \operatorname{Der} A$ consider its one-parameter subgroup $\delta(t) \in \operatorname{Aut} A$. It determines the vector field $D_M = \frac{d}{dt}_{|0} \widetilde{\delta}(t)_M$ on $T^A M$ where $\widetilde{}$ indicates the natural transformation determined by a Weil algebra homomorphis. Finally, we obtain natural operators $\Lambda_{D,M} : TM \to TT^A M$ defined by $\Lambda_{D,M} X = D_M$ for any vector field X on M. Then the result of Kolář reads

All natural operators $TM \to TT^A M$ are of the form $L(c)_M \circ \mathcal{T}^A + \Lambda_{D,M}$ for some $c \in A$ and $D \in \operatorname{Der} A$.

Let us remind some results concerning product preserving bundle functors F on the category $\mathcal{FM}$ of fibered manifolds, [8] and [6]. They are just of the form T^μ for a homomorphism $\mu : A \to B$ of Weil algebras. Bundle functors T^μ are defined as follows. Let $i, j : \mathcal{M}f \to \mathcal{FM}$ be functors defined by $i_M = \mathrm{id}_M : M \to M$ and $j(M) : M \to pt$ for a manifold M and the single-point manifold pt. If $F : \mathcal{FM} \to \mathcal{FM}$ preserves products then so do $G^F = F \circ i$ and $H^F = F \circ j$. It follows the existence of Weil algebras A and B such that $G^F = T^A$ and $H^F = T^B$. Further, there is an obvious identity natural transformation $\tau_M : i(M) \to j(M)$ and thus we have a natural transformation $\mu_M = F\tau_M$ corresponding to a Weil algebra homomorphism $\mu : A \to B$. Thus the functor T^μ can be defined as the pull-back $T^A M \times_{T^B M} T^B Y$ in respect to μ and $T^B p$ for a fibered manifold $p : Y \to M$. Then $F = T^\mu$ (modulo a natural equivalence).

Let $\bar{F}$ be another product preserving bundle functor on $\mathcal{FM}$. Then the results of [6] also yield natural transformations $\eta : F \to \bar{F}$ in the form of couples of $(\mu, \bar{\mu})$-related natural transformations $\nu = \eta \circ i : T^A \to T^{\bar{A}}$ and $\rho = \eta \circ j : T^B \to T^{\bar{B}}$ for Weil algebra homomorphisms $\nu : A \to \bar{A}$ and $\rho : B \to \bar{B}$.

In [8], the second author published the extension of the mentioned above Kolář's result and obtained the complete description of natural operators lifting projectable vector fields from a fibered manifold Y to vector fields on $T^\mu Y$.

Let us recall the concept of a fibered-fibered manifold. It is a surjective submersion $\pi : Y \to \underline{Y}$ between fibered manifolds $p : Y \to M$ and $\underline{p} : \underline{Y} \to \underline{M}$ which is fibered and transforms submersically fibers of Y into fibers of $\underline{Y}$. A smooth fibered map $f : Y \to Y'$ is said to be a morphism of fibered-fibered manifolds Y and Y' if there is the so-called base map $\underline{f} : \underline{Y} \to \underline{Y}'$ which is also fibered. In formulas, we have $\pi' \circ f = \underline{f} \circ \pi$. Thus we have the category $\mathcal{F}^2\mathcal{M}$ of fibered-fibered manifolds which is local and admissible in the sense of [4].

In [7] we have proved that product preserving bundle functors on $\mathcal{F}^2\mathcal{M}$ correspond bijectively to commutative diagrams

(1)
$$\begin{array}{ccccc} & & A_2 & & \\ & \overset{12}{\Theta}\nearrow & & \searrow\overset{24}{\Theta} & \\ \Theta = A_1 & & & & A_4 \\ & \underset{13}{\Theta}\searrow & & \nearrow\underset{34}{\Theta} & \\ & & A_3 & & \end{array}$$

formed by Weil algebra homomorphisms. The main result of [7] gives the bijective correspondence between the category $pp\mathcal{F}^2\mathcal{M}$ of product preserving bundle functors on $\mathcal{F}^2\mathcal{M}$ with natural transformations among them and the category $\mathcal{K}_0$ of commutative diagrams Θ with their morphisms, i.e. $f_p : A_p \to B_p$ commuting with all $\overset{pq}{\Theta} : A_p \to A_q$ and $\overset{pq}{\Theta}{}' : B_p \to B_q$ for all $(p, q) \in S = \{(1,2), (1,3), (2,4), (3,4)\}$. Weil algebras $A_1, \dots, A_4$ and homomorphisms $\overset{pq}{\Theta} : A_p \to A_q$ are obtained as follows.

Let $j_1, \dots, j_4 : \mathcal{M}f \to \mathcal{F}^2\mathcal{M}$ be functors defined as follows. For $\mathcal{FM}$-objects $i_1(M) = (M \xrightarrow{\mathrm{id}_M} M)$, $i_2(M) = (M \to \mathrm{pt})$ and $\mathrm{Pt} = (\mathrm{pt} \to \mathrm{pt})$ put $j_1(M) = \ (i_1(M) \xrightarrow{\mathrm{id}_M} i_1(M))$, $j_2(M) = (i_1(M) \xrightarrow{\mathrm{pt}_M} \mathrm{Pt})$, $j_3(M) =$
$(i_2(M) \xrightarrow{\mathrm{id}_M} \ i_2(M))$, $j_4(M) = (i_2(M) \xrightarrow{\mathrm{pt}_M} \mathrm{Pt})$ and $j_p(f) = f : j_p(M) \to j_p(N)$. If $F : \mathcal{F}^2\mathcal{M} \to \mathcal{FM}$ is a product preserving bundle functor, then so are $\overset{(p)}{G} = F \circ j_p$. It follows the existence of Weil algebras A_p such that $\overset{(p)}{G} = T^{A_p}$ The composition of F with the obvious identity natural transformations $\overset{pq}{\tau}_M = \mathrm{id}_M : j_p(M) \to j_q(M)$ yield natural transformations $\overset{pq}{\Theta} : T^{A_p} \to T^{A_q}$ determined by Weil algebra homomorphisms $\overset{pq}{\Theta} : A_p \to A_q$ for all $(p, q) \in S$.

For a commutative diagram Θ, a bundle functor F^Θ is defined as follows. For a fibered-fibered manifold

(2)
$$\begin{array}{ccc} Y & \xrightarrow{\pi} & \underline{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \underline{p}} \\ M & \xrightarrow[\underline{\pi}]{} & \underline{M} \end{array}$$

we have

$$F^\Theta Y = \{(x_1, x_2, x_3, x_4) \in T^{A_1}\underline{M} \times T^{A_2}M \times T^{A_3}\underline{Y} \times T^{A_4}Y;\ \overset{12}{\Theta}_{\underline{M}}(x_1) = T^{A_2}\underline{\pi}(x_2),$$

(3) $$\overset{13}{\Theta}_{\underline{M}}(x_1) = T^{A_3}\underline{p}(x_3),\ \overset{24}{\Theta}_M(x_2) = T^{A_4}p(x_4),\ \overset{34}{\Theta}_{\underline{Y}}(x_3) = T^{A_4}\pi(x_4)\}.$$

Then $F = F^\Theta$ (modulo a natural equivalence).

The following remark (which we will not use later) gives an information about another expression of $F^{\Theta}Y$ in the form of the pull-back $((T^{A_2}M \times_{T^{A_2}\underline{M}} T^{A_1}\underline{M}) \times_{T^{A_1}\underline{M}} (T^{A_3}\underline{Y} \times_{T^{A_3}\underline{M}} T^{A_1}\underline{M})) \times_{T^{A_4}M \times_{T^{A_4}\underline{M}} T^{A_4}\underline{Y}} T^{A_4}Y$ considered in respect to $(\overset{24}{\Theta}_M \circ p_1 \circ \mathrm{pr}_1, \overset{34}{\Theta}_{\underline{Y}} \circ p_1 \circ \mathrm{pr}_2)$ and $T^{A_4}p$, $T^{A_4}\pi$ for the obvious projections $\mathrm{pr}_1, \mathrm{pr}_2$ and p_1. It is easy to see that the pull-backs on the left-hand side are in fact $T^{\overset{12}{\Theta}}(M \to \underline{M})$ and $T^{\overset{13}{\Theta}}(\underline{Y} \to \underline{M})$ and so we have

$$(4)\quad F^{\Theta}Y = (T^{\overset{12}{\Theta}}(M \to \underline{M}) \times_{T^{A_1}\underline{M}} T^{\overset{13}{\Theta}}(\underline{Y} \to \underline{M})) \times_{T^{A_4}M \times_{T^{A_4}\underline{M}} T^{A_4}\underline{Y}} T^{A_4}Y.$$

We shall investigate projectable-projectable vector fields. A vector field on a fibered-fibered manifold $\pi : Y \to \underline{Y}$ is said to be projectable-projectable if it is projectable in respect to both of the projections π and p. The flow Fl_t^X of X is formed by local $\mathcal{F}^2\mathcal{M}$-isomorphisms. The space of all projectable-projectable vector fields on Y will be denoted by $\chi_{proj,proj}(Y)$.

1. Some property of product preserving bundle functors defined on $\mathcal{F}^2\mathcal{M}$

According to the Weil theory, [2], for Weil algebras A, B we have the canonical identification $T^B \circ T^A \simeq T^{A\otimes B}$. We note that bundle functors F^{Θ} take values in the category $\mathcal{F}^2\mathcal{M}$ and prove the analogous composition property.

Consider $F^{\Theta}Y$ in the form (3). It is easy to see that $F^{\Theta}Y$ is a fibered-fibered manifold if we consider it in the form

$$(5)\qquad \begin{array}{ccc} F^{\Theta}Y & \longrightarrow & T^{\overset{13}{\Theta}}\underline{Y} \\ \downarrow & & \downarrow \\ T^{\overset{12}{\Theta}}M & \longrightarrow & T^{A_1}\underline{M} \end{array}$$

where all the arrows are the obvious projections. One can easily verify using $\mathcal{F}^2\mathcal{M}$-coordinates on Y that $F^{\Theta}Y$ is an $\mathcal{F}^2\mathcal{M}$-object.

PROPOSITION 1. *Let F^{Θ}, F^{η} be product preserving bundle functors on $\mathcal{F}^2\mathcal{M}$ corresponding to commutative diagrams Θ and η of the form* (1). *Then it holds $F^{\eta} \circ F^{\Theta} = F^{\Theta\otimes\eta}$.*

Proof. Let Θ and η be commutative diagrams

$$\Theta = \begin{array}{ccccc} & & A_2 & & \\ & \overset{12}{\Theta}\nearrow & & \searrow\overset{24}{\Theta} & \\ A_1 & & & & A_4 \\ & \underset{\Theta}{\overset{13}{}}\searrow & & \nearrow\underset{\Theta}{\overset{34}{}} & \\ & & A_3 & & \end{array} \qquad \eta = \begin{array}{ccccc} & & B_2 & & \\ & \overset{12}{\eta}\nearrow & & \searrow\overset{24}{\eta} & \\ B_1 & & & & B_4 \\ & \overset{13}{\eta}\searrow & & \nearrow\overset{34}{\eta} & \\ & & B_3 & & \end{array}$$

of Weil algebra homomorphisms. Further, let

$$\gamma = \begin{array}{ccccc} & & C_2 & & \\ & \overset{12}{\gamma}\nearrow & & \searrow\overset{24}{\gamma} & \\ C_1 & & & & C_4 \\ & \overset{13}{\gamma}\searrow & & \nearrow\overset{34}{\gamma} & \\ & & C_3 & & \end{array}$$

be the diagram corresponding to $F^\eta \circ F^\Theta$. By the definition of bundle functors of this kind and by the already mentioned composition property of Weil functors, [2], we have $C_p = A_p \otimes B_p$ for $p = 1, \dots, 4$. For, $C_p = T^{B_p} \circ T^{A_p}(\mathbb{R}) = T^{A_p \otimes B_p}(\mathbb{R}) = A_p \otimes B_p$. The quadriple of isomorphisms $C_p \simeq A_p \otimes B_p$ yields the isomorphism between γ and $\Theta \otimes \eta$ which is easy to verify. ∎

We describe some special case of F^Θ. Let $A_1 = A_2 = A_3 = A_4 = A$ and $\overset{pq}{\Theta} = \mathrm{id}_A$ for $(p,q) \in S$. Then $F^\Theta Y = T^A Y$. This property will be usefull to describe the composition of the tangent bundle with bundle functors F^Θ.

In the very end of the section, we remark some special cases of functors F^Θ.

(a) Let $A_3 = A_2 = A_1 = A$ and $\overset{12}{\Theta} = \overset{13}{\Theta} = \mathrm{id}_A$. Further, let $\overset{14}{\Theta} = \overset{12}{\Theta} = \mu : A \to B$ if we put $A_4 = B$. Then we have $F^\Theta Y = T^\mu(Y \to M \times_{\underline{M}} \underline{Y})$. If $\overset{14}{\Theta} = \overset{24}{\Theta} = \mu = \mathrm{id}_A$, we have $F^\Theta Y = T^A Y$.

(b) Let $A_1 = A_2 = A_3 = \mathbb{R}$, the maps $\overset{12}{\Theta} = \overset{13}{\Theta}$ be the identities on $\mathbb{R}$ and $\overset{23}{\Theta} = \overset{24}{\Theta} = i : \mathbb{R} \to A = A_4$ be the canonical inclusions of reals into A. Then we have $F^\Theta Y = V^A(Y \to M \times_{\underline{M}} \underline{Y})$ for the vertical Weil bundle V^A associated to the Weil algebra A.

2. Natural vector fields on bundle functors F^Θ

In this section, we generalize the definition of an absolute natural operator $\Lambda_D : T \to TT^A$ recalled in Preliminaries from Weil functors to bundle functors F^{Θ}. Further, we give all absolute natural operators to those bundle functors.

Let $\operatorname{Aut} A$ be the Lie group of algebra automorphisms of A, [4] and $\mathcal{A}utA$ be its Lie algebra. Further, let $\operatorname{Der} A$ be the algebra of derivations of A. We will show that the coincidence $\mathcal{A}utA = \operatorname{Der} A$ proved in [3] and [4] can be modified to F^{Θ} as well as it has been done for bundle functors $T^{\mu} : \mathcal{FM} \to \mathcal{FM}$ in [8]. In [8] we have also defined $\operatorname{Aut}(A, \mu, B)$ as the set of all μ-related pairs of automorphisms $(\nu, \rho) \in \operatorname{Aut} A \times \operatorname{Aut} B$ forming a Lie group. Analogously we have defined $\operatorname{Der}(A, \mu, B) = \{(D_A, D_B) \in \operatorname{Der} A \times \operatorname{Der} B; \mu \circ D_A = D_B \circ \mu\}$ and proved $\mathcal{A}ut(A, \mu, B) = \operatorname{Der}(A, \mu, B)$.

As for bundle functors F^{Θ}, consider the commutative diagram (1). Denote by $\operatorname{Aut}(A, \Theta)$ the set of all Θ-related automorphisms $f_p \in \operatorname{Aut}(A_p)$, i.e. $\operatorname{Aut}(A, \Theta) = \{(f_1, f_2, f_3, f_4) \in \operatorname{Aut} A_1 \times \operatorname{Aut} A_2 \times \operatorname{Aut} A_3 \times \operatorname{Aut} A_4; f_q \circ \overset{pq}{\Theta} = \overset{pq}{\Theta} \circ f_p \text{ for all } (p, q) \in S\}$. Clearly, $\operatorname{Aut}(A, \Theta)$ is an subgroup of $\operatorname{Aut} A_1 \times \operatorname{Aut} A_2 \times \operatorname{Aut} A_3 \times \operatorname{Aut} A_4$. To prove it is a Lie subgroup we show it is closed. Consider maps $F^{p,q} : \operatorname{Aut} A_p \to \operatorname{Hom}(A_p, A_q)$ defined by $F^{p,q}(f_p) = \overset{pq}{\Theta} \circ f_p$ and $F_{p,q} : \operatorname{Aut} A_q \to \operatorname{Hom}(A_p, A_q)$ defined by $F_{p,q}(f_q) = f_q \circ \overset{pq}{\Theta}$ for all $(p, q) \in S$. Denote $\Delta_{p,q} = \{(f_1, f_2, f_3, f_4) \in \operatorname{Aut} A_1 \times \operatorname{Aut} A_2 \times \operatorname{Aut} A_3 \times \operatorname{Aut} A_4; F^{p,q}(f_p) = F_{p,q}(f_q)\}$. Then $\operatorname{Aut}(A, \Theta) = \bigcap_{(p,q)\in S} \Delta_{p,q}$ which is obviously closed. It proves our claim.

Analogously denote by $\operatorname{Der}(A, \Theta)$ the set of all Θ-related derivations $D_p \in \operatorname{Der}(A_p)$, i.e. $\operatorname{Der}(A, \Theta) = \{(D_1, D_2, D_3, D_4) \in \operatorname{Der} A_1 \times \operatorname{Der} A_2 \times \operatorname{Der} A_3 \times \operatorname{Der} A_4;\ D_q \circ \overset{pq}{\Theta} = \overset{pq}{\Theta} \circ D_p \text{ for all } (p, q) \in S\}$.

We state the following assertion giving the description of a Lie algebra $\mathcal{A}ut(A, \Theta)$.

PROPOSITION 2. *Let $\mathcal{A}ut(A, \Theta)$ be the Lie algebra associated to a Lie group* $\operatorname{Aut}(A, \Theta)$ *for any commutative diagram* (1). *Then it holds $\mathcal{A}ut(A, \Theta) =$* $\operatorname{Der}(A, \Theta)$.

Proof. It is done directly using the concept of the exponencial mapping. ∎

Proposition 2 enables us to modify the definition of an absolute natural operator Λ_D to bundle functors F^{Θ} as follows. Let $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t))$ be a one-parameter subgroup in $\operatorname{Aut}(A, \Theta)$ tangent to $D = (D_{A_1}, D_{A_2}, D_{A_3}, D_{A_4}) \in \operatorname{Der}(A, \Theta)$. By restriction we have a vector field $D_Y = \frac{d}{dt}|_0(\widetilde{\gamma}_{1,\underline{M}}(t), \widetilde{\gamma}_{2,M}(t), \widetilde{\gamma}_{3,\underline{Y}}(t), \widetilde{\gamma}_{4,Y}(t))$ on $F^{\Theta}Y$. Thus we have an absolute natural operator $\Lambda_{D,Y} : T_{proj,proj}Y \to TF^{\Theta}Y$ determined by $X \to D_Y$ for any projectable-projectable vector field X on Y.

Let us recall that a natural operator $\Lambda_Y : \chi_{proj,proj}(Y) \to \chi(F^{\Theta}Y)$ is a system of regular $\mathcal{F}^2\mathcal{M}$-invariant operators $\Lambda_Y : \chi_{proj,proj}(Y) \to \chi(F^{\Theta}Y)$

for any $\mathcal{F}^2\mathcal{M}$-object Y. The condition of $\mathcal{F}^2\mathcal{M}$-invariance reads as follows. For any $\mathcal{F}^2\mathcal{M}$-objects Y_1, Y_2, and any couple of f-related projectable-projectable vector fields $X_1 \in \chi_{proj,proj}(Y_1)$, $X_2 \in \chi_{proj,proj}(Y_2)$ (i.e. $Tf \circ X_1 = X_2 \circ f$) under an $\mathcal{F}^2\mathcal{M}$-map $f : Y_1 \to Y_2$ it holds $TF^{\Theta}f \circ \Lambda_{Y_1}(X_1) = \Lambda_{Y_2}(X_2) \circ F^{\Theta}f$, i.e. $\Lambda_{Y_1}(X_1)$ and $\Lambda_{Y_2}(X_2)$ are $F^{\Theta}f$-related. The regularity of Λ_Y means that Λ_Y transforms smoothly parametrized families of projectable-projectable vector fields into smoothly parametrized families of vector fields.

A natural operator $\Lambda_Y : T_{proj,proj}Y \to TF^{\Theta}Y$ is called absolute if Λ_Y is constant on $\chi_{proj,proj}(Y)$ for any $Y \in \mathrm{Obj}(\mathcal{F}^2\mathcal{M})$.

We state the main result of this section

PROPOSITION 3. *Let F be a product preserving bundle functor on $\mathcal{F}^2\mathcal{M}$. Then every absolute natural operator $\mathcal{A}_Y : T_{proj,proj}Y \to TFY$ is of the form $\Lambda_{D,Y}$ for some $D \in \mathrm{Der}(A,\Theta)$, where Θ is a commutative diagram of the form* (1) *corresponding to F.*

Proof. Let Θ be a commutative diagram associated to F. By Proposition 1 and the remark following it we have $TT^{\Theta} = T^{\Theta\otimes \mathrm{id}}$ for the identity diagram all vertices of which being the Weil algebra $\mathbb{D}$ of dual numbers. Clearly, any natural operator $\mathcal{A}_Y$ in question is a natural transformation $\mathcal{A}_Y : T^{\Theta}Y \to T^{\Theta\otimes\mathrm{id}}Y$. It is identified with a Θ-related quadriple of Weil algebra homomorphisms $D = (D_1, D_2, D_3, D_4)$ as follows. For every $(p,q) \in S$, the following diagram of Weil algebra homomorphisms commutes

(6)
$$\begin{array}{ccc} A_p \otimes \mathbb{D} & \xrightarrow{\overset{pq}{\Theta}\otimes \mathrm{id}_{\mathbb{D}}} & A_q \otimes \mathbb{D} \\ {\scriptstyle D_p}\uparrow & & \uparrow{\scriptstyle D_q} \\ A_p & \xrightarrow[\overset{pq}{\Theta}]{} & A_q \end{array}$$

By [4], Chapter X a Weil algebra $A \otimes \mathbb{D}$ is identified with $A \times A$ endowed with the multiplication defined by

(7)
$$(a,b)(c,d) = (ac, ad + bc).$$

The fact that D_p and D_q cover the identity maps id_{A_p}, id_{A_q} and $\overset{pq}{\Theta} \otimes \mathrm{id}_{\mathbb{D}} \simeq (\overset{pq}{\Theta}, \overset{pq}{\Theta})$ in the identification (7) yield for any $(p,q) \in S$ commutative diagrams

$$\begin{array}{ccc} A_p & \xrightarrow{\overset{pq}{\Theta}} & A_q \\ {\scriptstyle pr_2\circ D_p}\uparrow & & \uparrow{\scriptstyle pr_2\circ D_q} \\ A_p & \xrightarrow[\overset{pq}{\Theta}]{} & A_q \end{array} \tag{8}$$

The multiplication (7) yields that $D_p = (\mathrm{id}, D_{A_p})$ for a derivation $D_{A_p} \in \mathrm{Der}\, A_p$ which can be immediately verified. The fact that $D_{A_p} \in \mathrm{Der}\, A_p$ and $D_{A_q} \in \mathrm{Der}\, A_q$ are related in the sense of diagrams (8) for any $(p,q) \in S$ yields $D \in \mathrm{Der}(A,\Theta)$. Cleraly $\mathcal{A}_Y = \Lambda_Y$, which completes the proof. ∎

Another proof. The flow $Fl_t^{\mathcal{A}_Y}$ of $\mathcal{A}_Y$ is $\mathcal{F}^2\mathcal{M}$-invariant and global because $\mathcal{A}_Y$ is $\mathcal{F}^2\mathcal{M}$-invariant. So $Fl_t^{\mathcal{A}_Y} : FY \to FY$ is a natural transformation. Let $\eta_t \in \mathrm{Aut}(A,\Theta)$ correspond to $Fl_t^{\mathcal{A}_Y}$. Then $D = \frac{d}{dt}_{|t=0}\eta_t \in \mathrm{Der}(A,\Theta)$ and $\Lambda_{D,Y} = \mathcal{A}_Y$. ∎

3. Natural operators $T_{proj,proj} \to TF^{\Theta}$

In the beginning of this section, we remind the concept of a natural affinor by Koszul, [3] and [4]. For the multiplication $m : \mathbb{R} \times TM \to TM$ of tangent vectors by reals and applying a Weil functor T^A we obtain $T^A m : T^A\mathbb{R} \times T^ATM \to T^ATM$. Applying the exchange isomorphism we have $T^A m : A \times TT^AM \to TT^AM$. Thus we have an action $L_M : A \times TT^AM \to TT^AM$ and for any $c \in A$ we have a map $L(c)_M : TT^AM \to TT^AM$ which is a natural affinor. The coordinate expression of $L(c)_M$ is of the form $c(a_1,\dots,a_m,b_1,\dots,b_m) = (a_1,\dots,a_m,cb_1,\dots,cb_m)$. In [8], the second author generalized this action to T^{μ}. For $(y_1,y_2) \in T^AM \times_{T^BM} T^BY = T^{\mu}Y$ and $c \in A$ we have $L(c)_Y(y_1,y_2) = (L(c)_M(y_1), L(\mu(c))_Y(y_2))$.

We generalize this action to F^{Θ} as follows: For $(y_1,y_2,y_3,y_4) \in T(T^{A_1}\underline{M} \times T^{A_2}M \times T^{A_3}\underline{Y} \times T^{A_4}Y) \cap TF^{\Theta}Y$ and $c \in A_1$ put

$$L(c)_Y(y_1,y_2,y_3,y_4) = \tag{9}$$
$$(L(c)_{\underline{M}}(y_1), L(\overset{12}{\Theta}(c))_M(y_2), L(\overset{13}{\Theta}(c))_{\underline{Y}}(y_3), L(\overset{24}{\Theta}\circ\overset{12}{\Theta}(c))_Y(y_4)).$$

Analogously to [8] T^{μ} we state the assertion giving all natural affinors on F^{Θ}.

THEOREM 1. *Let Θ be a commutative diagram of the form (1) and $Y \in Ob(\mathcal{F}^2\mathcal{M})$ be of the form (2). Then every natural affinor on $F^{\Theta}Y$ is of the form $L(c)_Y$ for some $c \in A_1$.*

Let us recall that a natural affinor L_Y on $F^{\Theta}Y$ is a system of $\mathcal{F}^2\mathcal{M}$-invariant affinors (i.e. tensor fields of type $(1,1)$) $L_Y : TF^{\Theta}Y \to TF^{\Theta}Y$ on $F^{\Theta}Y$ for any $\mathcal{F}^2\mathcal{M}$-object Y. The $\mathcal{F}^2\mathcal{M}$-invariance means that for any $\mathcal{F}^2\mathcal{M}$-map $f : Y_1 \to Y_2$ we have $L_{Y_2} \circ TF^{\Theta}f = TF^{\Theta}f \circ L_{Y_1}$.

A natural affinor on $F^\Theta Y$ is in particular a natural transformation $T^{\Theta\otimes\mathrm{id}}Y \to T^{\Theta\otimes\mathrm{id}}Y$ satisfying respective property. Then the proof is very similar to that of Proposition 3 and so it is omitted. An independent proof will be presented after the proof of Theorem 2.

In [8] we have proved that for product preserving bundle functors on $\mathcal{FM}$, i.e. T^μ, all natural operators lifting projectable vector fields are of the form $L(c) \circ T^\mu + \Lambda_D$ for some $c \in A$ and $D \in \mathrm{Der}(A, \mu, B)$. For a projectable-projectable vector field X on a fibered-fibered manifold Y one can verify the following formula of Kolář

$$\mathcal{F}X = \eta_Y \circ FX \tag{10}$$

where $\mathcal{F}X$ is the flow lifting of X, i.e. $\mathcal{F}X = \frac{d}{dt}_{|t=0} F(Fl_t^X)$ and $\eta_Y : F \circ TY = T^{\mathrm{id}\otimes\Theta}Y \simeq T^{\Theta\otimes\mathrm{id}}Y = TFY$ is the exchange isomorphism.

THEOREM 2. *Let F be a product preserving bundle functor on $\mathcal{F}^2\mathcal{M}$. Further, let X be a projectable-projectable vector field on a fibered-fibered manifold Y. Then any natural operator $\mathcal{A}_Y : T_{proj,proj}Y \to TFY$ is of the form*

$$L(c)_Y \circ \mathcal{F}X + \Lambda_{D,Y}$$

for some $c \in A_1$ and $D \in \mathrm{Der}(A, \Theta)$ where Θ is the commutative diagram (1) associated to F.

Proof. Without loss of generality one can suppose $\mathcal{A}_Y(0) = 0$ since $\mathcal{A}_Y(0)$ is an absolute operator, the description of which is given by Proposition 3 and since $\mathcal{A}_Y$ can be replaced by $\mathcal{A}_Y - \mathcal{A}_Y(0)$.

Let us consider Y in the form of a finite product of $j_1(\mathbb{R}), j_2(\mathbb{R}), j_3(\mathbb{R})$ and $j_4(\mathbb{R})$. Then $FY \simeq \mathbb{R}^{\dim FY}$, and therefore vector fields on FY can be treated as maps $FY \to FY$. Let $\frac{\partial}{\partial x}$ be the usual canonical vector field on the fibered-fibered manifold $j_1(\mathbb{R})$. Since any projectable-projectable vector field X on Y covering a non-vanishing vector field on $\underline{M}$ is $\mathcal{F}^2\mathcal{M}$-isomorphism related with $\frac{\partial}{\partial x} = \frac{\partial}{\partial x} \times 0$ on Y then $\mathcal{A}_Y$ is uniquely determined by the maps $\alpha_k : FY \to FY$, $k \in \mathbb{R}$ defined by $\alpha_k(a) = \mathcal{A}_Y(k\frac{\partial}{\partial x})(a)$. Using the invariance with respect to homotheties on Y, we obtain the homogenity condition $\alpha_{tk}(ta) = t\alpha_k(a)$ for $t \neq 0$.

Then the homogenous function theorem and $\mathcal{A}_Y(0) = 0$ follow that α_k are constant maps linearly depending on k. Then using the invariance with respect to $\mathrm{id}_\mathbb{R} \times t\,\mathrm{id}$ one deduces that the vector space of all natural operators $\mathcal{A}_Y$ in question satisfying $\mathcal{A}_Y(0) = 0$ is at most $\dim(A_1)$-dimensional. But all operators of the form $X \mapsto (L(c))_Y \circ \mathcal{F}X$ form $\dim(A_1)$-dimensional vector space. So $\mathcal{A}_Y(X) = (L(c))_Y \circ \mathcal{F}X$ for some $c \in A_1$. It completes the proof. ∎

In what follows, we present an independent proof of Theorem 1. In the first step, we show that the vectors $\mathcal{F}(X)(v)$ for $X \in \chi_{proj,proj}(Y)$ and $v \in FY$ form a dense subset in $TF^{\Theta}Y$ for sufficiently dimensional Y, $\underline{Y}$, M and $\underline{M}$. Let A be a Weil algebra of width k and $\mathbb{D}$ be the algebra of dual numbers corresponding to the tangent bundle. Further, consider the immersion element $j^A(t_1,\dots,t_k,0,\dots,0) \in T^A\mathbb{R}^m$. Then we have $\mathcal{T}^A(\frac{\partial}{\partial x^1})(j^A(t_1,\dots,t_k,0,\dots,0)) = j^{A\otimes\mathbb{D}}(t_1,\dots,t_k,t,0,\dots,0) \in TT^A\mathbb{R}^m$. It follows that $\mathcal{T}^A(\frac{\partial}{\partial x^1})(j^A(t_1,\dots,t_k,0\dots,0))$ has a dense orbit in TT^AM whenever $m \geq k+1$ by the rank theorem.

To extend it to a fibered-fibered manifold Y, suppose $Y = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Then we have $F^{\Theta}Y = T^{A_1}\mathbb{R}^{m_1} \times T^{A_2}\mathbb{R}^{m_2} \times T^{A_3}\mathbb{R}^{n_1} \times T^{A_4}\mathbb{R}^{n_2}$ and it follows from the manifold case that the vectors $F^{\Theta}(X)(v) = T^{A_1}(X_1)(v_1) \times T^{A_2}(X_2)(v_2) \times T^{A_3}(X_3)(v_3) \times T^{A_4}(X_4)(v_4)$ form a dense subset in $TT^{\Theta}Y$ for constant vector fields X on Y and points $v \in Y$. In addition to that, such a dense subset is obtained even in respect to $\mathcal{F}^2\mathcal{M}$-isomorphisms $f = f_1 \times f_2 \times f_3 \times f_4$. It proves our claim.

The assertion of Theorem 1 is obtained by the fact that a natural affinor Λ_Y on $F^{\Theta}Y$ is determined by $\Lambda_Y \circ \mathcal{F}Y(X)$, $X \in \chi_{projproj}(Y)$. But the map $\mathcal{A}_Y : X \mapsto \Lambda_Y \circ \mathcal{F}X$ determines a natural operator satisfying $\mathcal{A}_Y(0) = 0$. By the proof of Theorem 2 we have $\mathcal{A}_Y(X) = L(c)_Y \circ \mathcal{F}X$ for some $c \in A_1$. Then $\Lambda_Y = L(c)_Y$ as well. ∎

4. Final remarks

A fibered-fibered manifold Y is of dimension (m_1, m_2, n_1, n_2) if $\dim \underline{M} = m_1$, $\dim \underline{Y} = m_1 + m_2$, $\dim M = m_1 + n_1$, $\dim Y = m_1 + m_2 + n_1 + n_2$. All fibered-fibered manifolds of dimension (m_1, m_2, n_1, n_2) and their local isomorphisms form a category which we will denote by $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$.

Let F^{Θ}, $F^{\bar{\Theta}} : \mathcal{F}^2\mathcal{M} \to \mathcal{F}\mathcal{M}$ be product preserving bundle functor and let $\eta : F^{\Theta}_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to F^{\bar{\Theta}}_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}}$ be a $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural transformation. Assume m_1, m_2, n_1, n_2 to be positive integers. Then by similar methods as for Weil bundles on $\mathcal{M}f$ one can show that there exists exactly one natural transformation extending η. So, modifying a little the proofs one can obtain the $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-versions of Theorem 1 and Theorem 2.

THEOREM 1′. *Let Θ be a commutative diagram of the form* (1) *and Y be an object of the category $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ of the form* (2), *$m_1, m_2, n_1, n_2 \geq 1$. Then every $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural affinor on $F^{\Theta}Y$ is of the form $L(c)_Y$ for some $c \in A_1$.*

THEOREM 2′. *Let* Θ *be a commutative diagram of the form* (1), $m_1, m_2, n_1, n_2 \geq 1$, Y *be as in Theorem 1'. Then every* $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$*-natural operator* $\mathcal{A}_Y : T_{proj,proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} Y \to TF^\Theta Y$ *is of the form*

$$L(c)_Y \circ \mathcal{F}^\Theta X + \Lambda_{D,Y}$$

for some $c \in A_1$ *and* $D \in \mathrm{Der}(A, \Theta)$.

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