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ON THE LEAVES OF A PREFOLIATION OF A $\mathbb{K}$-DIFFERENTIAL SPACE

Abstract. The definition of a prefoliation (M, F) of a $\mathbb{K}$-differential space and the theorem about regularity of the inclusion of a leaf of a prefoliation are reminded. An example of a pair (M, F) of $\mathbb{K}$-differential spaces with the same set of points, which shows that even if the identity of $.M$ is an immersion $F \to M$ and $(\operatorname{top} M, \operatorname{top} F)$ is a topological foliation in the sense of Ehresmann then (M, F) has not to be a prefoliation, is given. In the end, we show that if L is a proper leaf of a prefoliation (M, F) then the both structures of a $\mathbb{K}$-differential spaces coincide on L.

1. Introduction

In [10] the notion of a $\mathbb{K}$-differential space was introduced. This notion is a generalization of the notion of the differential space in the sense of Sikorski ([6]). The definition of the $\mathbb{K}$-differential space seems to be specially interesting from the point of view of the theory of foliations. Namely, in [10] it was proved that if R is an arbitrary equivalence relation on the set $.M$ of points of a $\mathbb{K}$-differential space M then one can define a structure of a $\mathbb{K}$-differential space in the set of equivalence classes. Moreover, the topology determined by this structure is equal to the quotient topology of the topology determined by the $\mathbb{K}$-differential space M.

In [3] we have introduced the definition of a prefoliation of a $\mathbb{K}$-differential space. This definition is based on the definition of a topological foliation ([1],[2]). The notion of a prefoliation of a $\mathbb{K}$-differential space is a particular case of the notion of a topological foliation and, at the same time, it is a generalization of such notions as a prefoliation of a manifold ([4],[9]), a Stefan foliation ([7],[8]) and a regular foliation ([5]). In [3] we have shown that, for a prefoliation (M, F), the identity mapping of the set of points of M is a smooth mapping of the $\mathbb{K}$-differential space F into M. Moreover, there is a proof of the regularity of this mapping. As a corollary, we have obtained the regularity of the natural inclusion of a leaf into M.

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In the present paper, we construct an example which shows that for two $\mathbb{K}$-differential spaces F and M with the same set of points, even if the identity mapping is an immersion $F \to M$ and the pair $(\operatorname{top} M, \operatorname{top} F)$ is a topological foliation then (M, F) has not to be a prefoliation. We also prove that if L is a proper leaf of a prefoliation (M, F), i.e. both topologies $\operatorname{top} M$ and $\operatorname{top} F$ coincide on L then the structures of $\mathbb{K}$-differential spaces M and F coincide on L. More precisely, the equality $M_L = F_L = F|L$ holds.

The paper consists of six sections. In section 2 we remind the notion of a $\mathbb{K}$-differential space and smooth mapping of $\mathbb{K}$-differential spaces. Section 3 contains the definition of a tangent space and a tangent mapping. Some elementary properties of a tangent mapping are proved. In section 4 we define a prefoliation of a $\mathbb{K}$-differential space and present some elementary properties of the prefoliation which are proved in [3]. Section 5 is devoted to the constuction of the example mentioned above. In the end, in section 6 we prove the main theorem of the present paper. We show there that if topologies of M and F coincide on the leaf L then the structures of $\mathbb{K}$-differential spaces also coincide on L.

2. $\mathbb{K}$-differential space

Let $\mathbb{K}$ be an arbitrary field with a non-trivial norm (we can think here about $\mathbb{R}$ or $\mathbb{C}$). Let M be a family of functions with values in $\mathbb{K}$. Define the set

$$.M := \bigcup_{\alpha \in M} D_\alpha$$

which will be called *the set of points of* M. In the set of points of M define a topology $\operatorname{top} M$ as the weakest topology containing the family

$$\{\alpha^{-1}(B) : B \text{ is open in } \mathbb{K} \text{ and } \alpha \in M\}.$$

Next set

$$\begin{aligned} \operatorname{an} M := \{\varphi \circ (\alpha_1, \ldots \alpha_m) : m \in \mathbb{N} \text{ and } \alpha_1, \ldots, \alpha_m \in M \\ \text{and } \varphi \text{ is an analytical function defined on an open} \\ \text{set in } \mathbb{K}^m \text{ with values in } \mathbb{K}\}. \end{aligned}$$

If $A \subset .M$ then

$$M|A := \{\alpha|A \cap D_\alpha : \alpha \in M\}$$

and

$$\begin{aligned} M_A := \{\beta : \forall_{p \in D_\beta} \exists_{U \in \operatorname{top} M} \exists_{\alpha \in M} (p \in U \cap A \subset D_\beta \wedge U \subset D_\alpha \\ \wedge \beta|U \cap A = \alpha|U \cap A)\}. \end{aligned}$$

It is easy to see that

$$M|A \subset M_A. \tag{2.1}$$

The family M_A will be called *a localization of* M *to the set* A.

From the above definition we have

PROPOSITION 1. *Let M be a family of functions with values in $\mathbb{K}$. If $B \subset A \subset .M$ then*

$$(M_A)_B = M_B.$$

Define now a $\mathbb{K}$-differential space.

DEFINITION 1. *The family M of functions with its values in $\mathbb{K}$ is called a $\mathbb{K}$-differential space, if the condition*

$$\operatorname{an} M = M = M_{.M}$$

is fulfilled.

Let M be an arbitrary family of functions with its values in $\mathbb{K}$. One can prove the following

PROPOSITION 2. *The family $(\operatorname{an} M)_{.M}$ is the smallest $\mathbb{K}$-differential space with the set $.M$ as the set of points, containing M.*

DEFINITION 2. *The $\mathbb{K}$-differential space $(\operatorname{an} M)_{.M}$ is called the $\mathbb{K}$-differential space generated by a family M.*

It was proved in [11] the following

PROPOSITION 3. *Let M be a $\mathbb{K}$-differential space and $A \subset .M$. Then $M_A \subset M$ if and only if $A \in \operatorname{top} M$.*

The above proposition gives

COROLLARY 1. *Let M be a $\mathbb{K}$-differential space and $A \in \operatorname{top} M$. Then $M|A = M_A$.*

Let M, N be $\mathbb{K}$-differential spaces.

DEFINITION 3. *The mapping $f : .M \to .N$ is said to be smooth if for each $\beta \in N$ we have $\beta \circ f \in M$.*

If the above condition holds then we write $f : M \to N$.

It is obvious that if $f : M \to N$ then $f : \operatorname{top} M \to \operatorname{top} N$ i.e. f is a continuous mapping respective to the topologies $\operatorname{top} M$ and $\operatorname{top} N$.

3. Tangent space, tangent mapping

Let M be a $\mathbb{K}$-differential space and $p \in .M$. Define $M(p) := \{\alpha \in M : p \in D_\alpha\}$.

DEFINITION 4. *Any $\mathbb{K}$-linear mapping $v : M(p) \to \mathbb{K}$ such that*

$$v(\alpha \cdot \beta) = v(\alpha) \cdot \beta(p) + \alpha(p) \cdot v(\beta)$$

for $\alpha, \beta \in M(p)$ is called a vector tangent to M at p. The family of all vectors tangent to M at p forms a vector space. This vector space is said to be tangent to M at p. It is denoted by $\mathrm{T}_p M$.

It is easy to see that if $v \in \mathrm{T}_p M$, $U \in \operatorname{top} M$ and $\alpha \in M$ then

$$v(\alpha) = v(\alpha|U). \tag{3.1}$$

Let M, N be $\mathbb{K}$-differential spaces and $f : M \to N$. For each $p \in .M$, the mapping f determines a linear mapping $(f_*)_p : \mathrm{T}_p M \to \mathrm{T}_{f(p)} N$ called *a tangent mapping*. Namely, for $v \in \mathrm{T}_p M$ and $\beta \in N(f(p))$ we have

$$((f_*)_p v)(\beta) = v(\beta \circ f).$$

DEFINITION 5. *The mapping $f : M \to N$ is called an immersion, if for each $p \in .M$ the tangent mapping $(f_*)_p$ is a monomorphism.*

Let $A \subset .M$ and $\iota : A \hookrightarrow .M$. We have the following

LEMMA 1. *The mapping ι is smooth $M_A \to M$ and ι is an immersion. Moreover, if $A \in \operatorname{top} M$ then for any $p \in A$ the mapping $(\iota_*)_p$ is an isomorphism.*

Proof. Let $\beta \in M$, then $\beta \circ \iota = \beta|A \in M|A \subset M_A$ by (2.1). Thus $\iota : M_A \to M$.

Let $p \in A$ and $v \in \mathrm{T}_p M_A$. Assume that $(\iota_*)_p v = 0$, so for any $\beta \in M$

$$v(\beta|A) = \left((\iota_*)_p v\right)(\beta) = 0. \tag{3.2}$$

Let $\alpha \in M_A(p)$. Then there exist $U \in \operatorname{top} M$ and $\beta \in M$ such that $p \in U \cap A \subset D_\alpha$ and $U \subset D_\beta$ and $\alpha|U \cap A = \beta|U \cap A$. Remark that $U \cap A \in \operatorname{top} M_A$ ([11],[3]). Therefore, by (3.1) and (3.2) we get

$$v(\alpha) = v(\alpha|U \cap A) = v(\beta|U \cap A) = v((\beta|A)|U \cap A) = v(\beta|A) = 0.$$

Thus $v = 0$.

Assume now that $A \in \operatorname{top} M$. Let $w \in \mathrm{T}_p M$. Define $v : M_A(p) \to \mathbb{K}$ by the formula

$$v(\beta) = w(\beta).$$

The definition is correct since $\beta \in M_A \subset M$ by Proposition 3. It is obvious that v is an element of $\mathrm{T}_p M_A$. Moreover, by (3.1)

$$\left((\iota_*)_p v\right)(\alpha) = v(\alpha|A) = v(\alpha) = w(\alpha)$$

which means that $(\iota_*)_p v = w$. Thus ι_* is an epimorphism. ∎

It is easy to prove

LEMMA 2. *Let M, N, P be $\mathbb{K}$-differential spaces. If $f : M \to N$ and $g : N \to P$ then $g \circ f : M \to P$ and for any $p \in .M$ the equality*

$$((g \circ f)_*)_p = (g_*)_{f(p)} \circ (f_*)_p$$

holds. Moreover, if f, g are immersions then $g \circ f$ is an immersion.

We have

PROPOSITION 4. *Let M, N be $\mathbb{K}$-differential spaces and $A \subset .M$. If $f : M \to N$ is an immersion then $f|A : M_A \to N$ and this mapping is an immersion.*

Proof. Remark that $f|A = f \circ \iota$ with $\iota : A \hookrightarrow .M$. Obviously, by Lemma 1 and Lemma 2 we have the assertions. ■

4. Prefoliation of a $\mathbb{K}$-differential space

Let M be a $\mathbb{K}$-differential space.

DEFINITION 6. *A pair (M, F) of $\mathbb{K}$-differential spaces is called a prefoliation of M iff*

1) $.F = .M$,

2) $\operatorname{top} F$ *is locally connected,*

3) $\forall_{p \in .M} \exists_{U \in \operatorname{top} F} (p \in U \wedge F_U = M_U)$.

Connected components of $\operatorname{top} F$ *are called leaves of* (M, F).

It is easy to see that the notion of a prefoliation is a generalization of the notion of the regular foliation, and of the Stefan foliation. Moreover, if (M, F) is a prefoliation of a $\mathbb{K}$-differential space M then $(\operatorname{top} M, \operatorname{top} F)$ is a topological foliation in the sense of Ehresmann ([1],[2]).

It is not difficult to prove the following

THEOREM 1. *If (M, F) is a prefoliation of a $\mathbb{K}$-differential space M then for the mapping $f = \mathrm{id}_{.M}$ we have $f : F \to M$.*

Proof. If $\beta \in M$ then for each $p \in .M$ denote by U_p such a neighbourhood of p in $\operatorname{top} F$ for which

$$F_{U_p} = M_{U_p}.$$

Thus we get an open covering $\{U_p\}_{p \in .M}$ of the set $.M$ respective to $\operatorname{top} F$ with

$$\beta|U_p \in M|U_p \subset M_{U_p} = F_{U_p}.$$

Therefore $\beta \in F$ since F is a $\mathbb{K}$-differential space. ■

From the theorem we get

COROLLARY 2. *Let (M, F) be a prefoliation of a $\mathbb{K}$-differential space M. Then* $\operatorname{top} M \subset \operatorname{top} F$ *and* $M \subset F$.

We also have

THEOREM 2. *If (M, F) is a prefoliation of a $\mathbb{K}$-differential space M then the mapping $f = \mathrm{id}_{.M}$ is an immersion $F \to M$.*

Proof. Suppose that $(f_*)_p(v) = 0$ for some $v \in \mathrm{T}_p F$. Then for each $\beta \in M(p)$ we have $v(\beta \circ f) = 0$, i.e. for each $G \in \operatorname{top} F$ with $p \in G$ by (3.1) we have

$$v((\beta \circ f)|G) = 0 \tag{4.1}$$

Let $\alpha \in F(p)$. There exists $U \in \operatorname{top} F$ such that $p \in U$ and $F_U = M_U$ by the definition of a prefoliation. Therefore, $\alpha|D_\alpha \cap U \in F|U = F_U = M_U$ by Corollary 1. Thus there exist $V \in \operatorname{top} M$ and $\gamma \in M$ such that $p \in V \cap U \subset D_\alpha$ and $V \subset D_\gamma$ and $\gamma|V \cap U = \alpha|V \cap U$. Obviously $\gamma \in M(p)$ and because of Corollary 2, $V \cap U \in \operatorname{top} F$. Consequently,

$$v(\alpha) = v(\alpha|V \cap U) = v(\gamma|V \cap U) = v(\gamma \circ f|V \cap U) = 0$$

by (4.1). Thus $v = 0$. ∎

Using Proposition 4 we simply get

COROLLARY 3. *If L is a leaf of a prefoliation (M, F) then* $\mathrm{id}_L : F_L \to M$ *is an immersion.*

COROLLARY 4. *If L is a leaf of a prefoliation (M, F) then $M_L \subset F|L$.*

Proof. The assertion follows directly from Corollary 2 and the simple fact: if $M \subset F$, $.M = .F$ and $A \subset .M$ then $M_A \subset F_A$. ∎

5. Example

In the present section we construct a pair (M, F) of $\mathbb{K}$-differential spaces such that

1° $.M = .F$;

2° $\operatorname{top} F$ is locally connected;

3° $\mathrm{id}_{.M} : F \to M$ is an immersion;

4° $(\operatorname{top} M, \operatorname{top} F)$ is a topological foliation;

5° (M, F) is not a prefoliation of a $\mathbb{K}$-differential space.

To this end, we introduce a new definition of the tangent space $\mathrm{T}_p M$. Let M be an arbitrary $\mathbb{K}$-differential space and $p \in .M$. In the set $M(p)$ define an equivalence relation in the following way:

$$\alpha \equiv \beta \Longleftrightarrow \exists_{U \in \operatorname{top} M} (p \in U \wedge \alpha|U = \beta|U),$$

for $\alpha, \beta \in M(p)$. The equivalence class of a function $\alpha \in M(p)$ will be called *a germ of α at p* and will be denoted by $\xi = [\alpha]_p$. Obviously, the value of a germ ξ at the point p is well defined. The set of all germs at p with naturally defined addition and multiplication forms a ring. It follows from the equality $\operatorname{an} M = M$ and from the fact that M contains all constant functions on open sets in $\operatorname{top} M$ ([11]). Denote this ring by R_p. It is easy to check that the set

$$\mathbf{m}_p = \{\xi \in R_p : \xi(p) = 0\}$$

is an ideal in R_p. One can prove the following

PROPOSITION 5. *The* $\mathbb{K}$*-vector spaces* $\mathrm{T}_p M$ *and* $(\mathbf{m}_p/\mathbf{m}_p^2)^*$ *are isomorphic.*

Let F be the $\mathbb{R}$-differential space generated by the family of all continuous functions defined in $\mathbb{R}$. For each $p \in \mathbb{R}$ the vector space $\mathrm{T}_p F$ is zero-dimensional.

Let M be the $\mathbb{R}$-differential space generated by the family of all C^∞ functions defined in $\mathbb{R}$. Analogously as in [6] for differential spaces in the sense of Sikorski, one can prove that for each $p \in \mathbb{R}$ the tangent space $\mathrm{T}_p M$ is one-dimensional.

Remark that $\operatorname{top} M$ also is the natural topology in $\mathbb{R}$.

For the pair (M, F) we have:

1) $.M = .F = \mathbb{R}$;

2) $\operatorname{top} F$ is locally connected as the natural topology in $\mathbb{R}$;

3) $\mathrm{id}_{\mathbb{R}} : F \to M$ since $M \subset F$ and this mapping is an immersion since $\dim \mathrm{T}_p F = 0$ for each $p \in \mathbb{R}$;

4) $(\operatorname{top} M, \operatorname{top} F)$ is the trivial topological foliation since $\operatorname{top} M = \operatorname{top} F$ is the natural topology in $\mathbb{R}$;

5) (M, F) is not a prefoliation since for each $U \in \operatorname{top} F = \operatorname{top} M$ we have $M_U = M|U \subsetneq F|U = F_U$ by Corollary 1.

6. Proper leaves

Using the definition of a prefoliation one can prove

LEMMA 3. *If* (M, F) *is a prefoliation of a* $\mathbb{K}$*-differential space then for each* $\beta \in F$ *and for each* $p \in D_\beta$*, there exist* $G \in \operatorname{top} F$ *and* $\alpha \in M$ *such that* $p \in G \subset D_\beta$ *and* $\beta|G = \alpha|G$.

Proof. Let $\beta \in F$ and $p \in D_\beta$. By the definition of a prefoliation there exists $U \in \operatorname{top} F$ such that $p \in U$ and $M_U = F_U$. Remark that, since $D_\beta \in \operatorname{top} F$, the set $V := U \cap D_\beta$ is an open (respective to $\operatorname{top} F$) neighbourhood of p contained in U. By Proposition 1 and Corollary 1 we have $M_V = (M_U)_V = (F_U)_V = F_V = F|V$. Therefore, $\beta|V \in M_V$ which means that there exist $W \in \operatorname{top} M$ and $\alpha \in M$ such that $p \in W \cap V \subset D_{\beta|V} = V$ and $W \subset D_\alpha$ and $\alpha|W \cap V = \beta|W \cap V$. By Corollary 2, $\operatorname{top} M \subset \operatorname{top} F$, so if we set $G := W \cap V$ then $p \in G \in \operatorname{top} F$ and $G \subset D_\beta$ and $\beta|G = \alpha|G$. ∎

Let (M, F) be a prefoliation of a $\mathbb{K}$-differential space M and let L be a leaf of (M, F).

DEFINITION 7. *The leaf* L *is said to be proper if* $(\operatorname{top} F)\,|L = (\operatorname{top} M)\,|L$.

We have the following

THEOREM 3. *Let* (M, F) *be a prefoliation of a* $\mathbb{K}$*-differential space* M *and* L *be a proper leaf of* (M, F) *then*

$$M_L = F|L.$$

Proof. The inclusion $M_L \subset F|L$ holds for each L by Corollary 4.

We show that $F|L \subset M_L$. Let $\gamma \in F|L$. By a local connectedness of $\operatorname{top} F$ we have $L \in \operatorname{top} F$ and consequently $\gamma \in F|L \subset F$ by Proposition 3. By Lemma 3, for each $p \in D_\gamma$ there exist $V \in \operatorname{top} F$ and $\beta \in M$ such that $p \in V \subset D_\gamma \subset L$ and $\gamma|V = \beta|V$. Since L is proper, there exists $W \in \operatorname{top} M$ such that $V = W \cap L$. Let $p \in D_\gamma$ and define $U := W \cap D_\beta \in \operatorname{top} M$ and $\alpha := \beta|U \in M$ by Proposition 3. Then we have

1) $p \in U \cap L \subset D_\gamma$ since $U \cap L \subset W \cap L = V \subset D_\gamma$;

2) $U \subset D_\alpha$ (in fact the equality holds);

3) $\alpha|U \cap L = \beta|U \cap L = \beta|W \cap L \cap D_\alpha = \gamma|W \cap L \cap D_\alpha = \gamma|U \cap L$ since $W \cap L \cap D_\alpha \subset V$.

By 1)-3) and the definition of the localization we have $\gamma \in M_L$. ∎

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