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# ON UNIVERSAL ELEMENTS FOR SOME FAMILIES OF FUNCTIONS

**Abstract.** A point $x \in X$ is called universal element for a family $\Phi$ of functions from $X$ to $Y$ if the set $\{f(x); f \in \Phi\}$ is dense in $Y$. In this article we show that every residual $G_\delta$-set in a completely regular space $X$ (every residual set in $\mathbb{R}^k$) is the set of all universal elements for some family of continuous functions from $X$ to $\mathbb{R}$ (for some family of quasicontinuous functions from $\mathbb{R}^k$ to $\mathbb{R}$). Moreover we investigate the sets of all universal elements for some families of monotone functions and for some families of functions having the property of Denjoy-Clarkson.

Let $(X, T_X)$ and $(Y, T_Y)$ be topological spaces with topologies $T_X$ and $T_Y$ respectively. Let

$$\mathcal{A} = \{T_j : X \to Y; j \in J\}, \quad \text{where } J \text{ is an index set,}$$

be a family of mappings from $X$ to $Y$.

A point $x \in X$ is called a universal element for the family $\mathcal{A}$ if the set $\{T_j(x); j \in J\}$ is dense in $Y$. Let $U$ denote the set of all universal elements for $\mathcal{A}$.

REMARK. If $(X, T_X)$ is a topological space, if $(Y, T_Y)$ is second countable topological space and if $T_j (j \in J)$ is continuous then $U$ is $G_\delta$-set.

Proof. Let $W_k (k \geq 1)$ be the nonempty members of a countable base of the topology of $Y$. Then $U = \bigcap_{k=1}^{\infty} \bigcup_{j \in J} T_j^{-1}(W_k)$ is $G_\delta$-set.

In [6, 7, 9] the following theorem is proved:

THEOREM 1. *Let $X$ be a Baire space, let $Y$ be second countable and let $T_j$ be continuous for each $j \in J$. Then the following assertions are equivalent:*

(1) *The set $U$ is residual in $X$;*

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(2) *The set $U$ is a dense $G_\delta$-subset of $X$;*
(3) *The set $U$ is dense in $X$;*
(4) *For each pair of nonempty sets $V \in T_X$ and $W \in T_Y$ there is an index $j \in J$ such that*

$$T_j(V) \cap W \neq \emptyset.$$

In [8] G. Herzog and R. Lemmert extend this result for families of quasicontinuous functions. Remind that a function $f : X \to Y$ is called quasicontinuous ([11, 12]) at a point $x \in X$ if for all sets $V \in T_X$ and $W \in T_Y$ with $x \in V$ and $f(x) \in W$ there is a nonempty set $S \in T_X$ such that $S \subset V$ and $f(S) \subset W$.

In [8] the authors proved the following theorem:

THEOREM 2. *Let $X$ be a Baire space, let $Y$ be second countable, and let $T_j$ be quasicontinuous for each $j \in J$. Then the following assertions are equivalent:*

(1) *The set $U$ is residual;*
(2) *The set $U$ is dense in $X$;*
(3) *For each pair of nonempty sets $V \in T_X$ and $W \in T_Y$ there is an index $j \in J$ such that*

$$T_j(V) \cap W \neq \emptyset.$$

Moreover it is obvious that if the family $\mathcal{A}$ contains only one function having the graph dense in the product space $X \times Y$ and if there exists no singleton dense in $Y$ then $\mathcal{A}$ satisfies the condition (3) from Theorem 2 and the set $U$ for this family is empty. The functions with dense graphs may have some important properties, for example in the case $X = Y = \mathbb{R}$, where $\mathbb{R}$ denotes the set of all reals, such functions may have the Darboux property or may be additive ([10]).

In connection with Theorem 1 observe that if the space $X$ contains an isolated point $x \in X$ then for each family $\mathcal{A}$ of functions from $X$ to $Y$ satisfying the condition (4) from Theorem 1 the point $x$ is a universal element for $\mathcal{A}$.

It is well now that there are regular topological spaces $(X, T_X)$ such that each function $f : X \to \mathbb{R}$ is continuous ([4], p.55).

For each a family of continuous functions from $X$ to $\mathbb{R}$ satisfying condition (4) from Theorem 1 we have $U = X$. Therefore, if $A \subset X$ is a proper $G_\delta$-set dense in $X$ then $A \neq U$ for every family $\mathcal{A}$ of continuous fonctions from $X$ to $\mathbb{R}$ satisfying condition (4) from Theorem 1.

But the following theorem is true:

THEOREM 3. *Let $X$ be a completely regular topological space. Then for each $G_\delta$-set $B \subset X$ there is a family $\mathcal{A}$ of continuous functions from $X$ to $\mathbb{R}$ for which $U = B$.*

Proof. The set $A = X \setminus B$ is an $F_\sigma$-set. So there are closed sets $A_n$, $n \geq 1$, such that

$$A = \bigcup_{n \geq 1} A_n \text{ and } A_n \subset A_{n+1} \text{ for } n \geq 1.$$

Let $(w_n)$ be a sequence of all rationals such that $w_n \neq w_m$ for $n \neq m$. Since $X$ is a completely regular space, it satisfies the axioms of separation $T_1$ and $T_{\frac{7}{2}}$. So for each point $t \in B$ the set $\{t\}$ is closed and there are continuous functions

$f_{n,t} : X \to [\min(0, w_n), \max(0, w_n)]$ such that $f_{n,t}(t) = w_n$ and $f_{n,t}(A_n) = \{0\}$. Let

$$\mathcal{A} = \{f_{n,t}; t \in B \text{ and } n \geq 1\}.$$

If a point $x \in B$ then

$$\{f(x); f \in \mathcal{A}\} \supset Q = \{w_n; n \geq 1\},$$

and consequently $x \in U$.

In the contrary case, if $x \in A = X \setminus B$ there is an index $k$ such that $x \in A_n$ for $n \geq k$. Then

$$f_{n,t}(x) = 0 \text{ for } t \in B \text{ and } n \geq k,$$

and

$$f_{n,t}(x) \subset [-\max_{n<k} |w_n|, \max_{n<k} |w_n|] \text{ for } t \in B \text{ and } n < k.$$

So the set

$$\{f(x); f \in \mathcal{A}\}$$

is not dense in $\mathbb{R}$ and consequently $x \in X \setminus U$. This completes the proof.

Let $(Z, \rho)$ be a metric space. Remind that the functions $f : X \to Z$ of a family $\Phi$ are equicontinuous at a point $x \in X$ if for every positive real $r$ there is a set $V \in T_X$ containing $x$ such that

$$f(V) \subset K(f(x), r) = \{u \in Z; \rho(u, f(x)) < r\} \text{ for each } f \in \Phi.$$

THEOREM 4. *Let $Y$ be a metric space. If $\mathcal{A} = \{T_j : X \to Y; j \in J\}$ is a family of functions equicontinuous at each point $x \in X$ satisfying condition (4) from Theorem 1 then $U = X$.*

Proof. Assume, by a contradiction, that there is a point $x \in X \setminus U$. Then there is a ball $K(y, r)$ with center $y$ and radius $r > 0$ such that

$$(*) \qquad \{f(x); f \in \mathcal{A}\} \cap K(y, r) = \emptyset.$$

Since the functions $f \in \mathcal{A}$ are equicontinuous at the point $x$, there is a set $V \in T_X$ containing $x$ and such that

$$(**) \qquad f(V) \subset K(f(x), \frac{r}{2}) \text{ for each } f \in \mathcal{A}.$$

But the family $\mathcal{A}$ satisfies condition (4) from Theorem 1, so there is an index $j \in J$ such that

$$T_j(V) \cap K(y, \frac{r}{2}) \neq \emptyset.$$

Let $v \in V$ be a point such that $T_j(v) \in K(y, \frac{r}{2})$. By $(**)$ we obtain

$$\rho(T_j(x), y) \leq \rho(T_j(x), T_j(v)) + \rho(T_j(v), y) < \frac{r}{2} + \frac{r}{2} = r,$$

which contradicts with $(*)$. This contradiction finishes the proof.

In connection with Theorem 2 observe that there is a topological space $(X, T_X)$ such that all quasicontinuous functions $f : X \to \mathbb{R}$ are constant. For example, such is each uncountable space $X$ with the topology

$$T_* = \{\emptyset\} \cup \{A \subset X; X \setminus A \text{ is countable}\}.$$

If $\mathcal{A}$ is a family of quasicontinuous functions from a such space $X$ to $\mathbb{R}$ satisfying condition (3) from Theorem 2 then $U = X$. Such spaces may contain some dense residual subsets $A \neq X$. So, for such $A$ we have $A \neq U$ for each family of quasicontinuous functions from $X$ to $\mathbb{R}$ satisfying condition (3) from Theorem 2.

However the following theorem is true:

THEOREM 5. *Let $X = \mathbb{R}^k$ and let $T_X$ be the Euclidean topology $T_e$ in $\mathbb{R}^k$. If $B \subset X$ is a residual set then there is a family $\mathcal{A}$ of quasicontinuous functions from $X$ to $\mathbb{R}$ such that $U = B$.*

Proof. Since the set $A = X \setminus B$ is of the first category, there is an $F_\sigma$-set $E \supset A$ of the first category. Let

$$E = \bigcup_{n \geq 1} E_n, \text{ where } E_{n+1} \supset E_n \text{ for } n \geq 1,$$

and $E_n$ are closed and nowhere dense.

As in the proof of Theorem 3 we define a family $\mathcal{A}_1$ of continuous functions from $X$ to $\mathbb{R}$ such that all sets

$$\{f(x); f \in \mathcal{A}_1\} \text{ are dense for } x \in X \setminus E,$$

and all sets

$$\{f(x); f \in \mathcal{A}_1\} \text{ are bounded for } x \in E_n, \quad n \geq 1.$$

For $n \geq 1$ put $H_n = E_n \cap B$. Moreover let $(w_n)$ be a sequence of all rationals such that $w_n \neq w_m$ for $n \neq m$. Without loss of the generality we can suppose that $H_n \neq \emptyset$ for $n \geq 1$. For each index $n \geq 1$ there is a family $K_{n,i}$, $i \geq 1$, of pairwise disjoint closed balls such that

- $K_{n,i} \cap E_n = \emptyset$ for $i \geq 1$;
- $\forall_{x \in E_n} \forall i \, \forall \varepsilon > 0 \, \exists_{j>i} : K_{n,j} \subset B(x, \varepsilon) = \{t; |x - t| < \varepsilon\}$.

Now for indices $n, i \geq 1$ define a continuous function $f_{n,i} : K_{n,i} \to [-i, i]$ such that $f_{n,i}(K_{n,i}) = [-i, i]$ and $f_{n,i}(x) = 0$ on the boundary $S_{n,i}$ of $K_{n,i}$. Let

$$g_{n,m}(x) = f_{n,i}(x) \text{ for } x \in K_{n,i}, \quad i \geq 1,$$

$$g_{n,m}(x) = w_m \text{ for } x \in H_n,$$

and

$$g_{n,m}(x) = 0 \text{ otherwise on } \mathbb{R}^k.$$

Every function $g_{n,m}$, $n, m \geq 1$, is quasicontinuous at each point. Let

$$\mathcal{A} = \mathcal{A}_1 \cup \{g_{n,m}; n, m \geq 1\}.$$

Then all functions of the family $\mathcal{A}$ are quasicontinuous at each point. If $x \in X \setminus E$ then the set

$$\{f(x); f \in \mathcal{A}\} \supset \{f(x); f \in \mathcal{A}_1\}$$

is dense.

If $x \in H_n = E_n \cap B$ for some index $n$ then the set

$$\{f(x); f \in \mathcal{A}\} \supset \{g_{n,m}(x); m \geq 1\} = \{w_m; m \geq 1\}$$

is also dense. So

$$B = (X \setminus E) \cup (E \cap B) = (X \setminus E) \cup \bigcup_{n \geq 1} H_n \subset U.$$

If

$$x \in X \setminus B = A \subset E = \bigcup_{n \geq 1} E_n$$

then there is an index $n$ such that $x \in E_n \setminus B$. So,

$$x \in X \setminus \bigcup_{i \geq 1} K_{m,i} \text{ for } m \geq n,$$

and consequently the difference

$$\{f(x); f \in \mathcal{A}\} \setminus \{f(x); f \in \mathcal{A}_1\}$$

is finite. Since $x \in E_n$, the set

$$\{f(x); f \in \mathcal{A}_1\}$$

is bounded and $x \in X \setminus U$. So $U = B$ and the proof is completed.

The notion of quasicontinuity is a particular case of cliquishness.

Remind that a function $f: X \to Y$, where $(Y, \rho)$ is a metric space, is cliquish at a point $x \in X$ if for each positive real $\varepsilon$ and each set $V \in T_X$ containing $x$ there is a nonempty set $W \in T_X$ contained in $V$ such that $osc_W f < \varepsilon$ ([12]).

In the next example we show that there is a countable family of monotone functions from $\mathbb{R}$ to $\mathbb{R}$ which are discontinuous only at one point (so they are cliquish) satisfying condition (3) from Theorem 2 whose set of all universal points is empty.

EXAMPLE 1. Let $A = \{(x_n, y_n); n \geq 1\} \subset \mathbb{R}^2$ be a countable set dense in $\mathbb{R}^2$ and such that each vertical section

$$A_x = \{y \in \mathbb{R}; (x, y) \in A\}$$

is of the cardinality $card(A_x) \leq 1$.

For $n \geq 1$ let

$$g_n(x) = \begin{cases} E(y_n) - 1 & \text{for } x < x_n \\ y_n & \text{for } x = x_n \\ E(y_n) + 1 & \text{for } x > x_n, \end{cases}$$

where $E(x)$ denotes the greatest integer which is $\leq x$.

Then the functions $g_n$ are monotone and continuous at all points $x \neq x_n$. Moreover the values of $g_n$ are integers for $x \neq x_n$. So

$$\text{if } \mathcal{A} = \{g_n; n \geq 1\} \text{ then } U = \emptyset.$$

We will show that the family $\mathcal{A}$ satisfies condition (3) of Theorem 2.

Let $V, W \in T_e$ be nonempty sets. Since the set $A$ is dense in $\mathbb{R}^2$, there is an index $k$ such that $(x_k, y_k) \in V \times W$. Consequently,

$$y_k \in g_k(V) \cap W \neq \emptyset.$$

So the family $\mathcal{A}$ satisfies condition (3) from Theorem 2.

THEOREM 6. *Let $X$ be a Baire space, let $Y$ be second countable and let $\mathcal{A} = \{T_j; j \in J\}$ be a family of functions from $X$ to $Y$. If there is a set $A \subset X$ of the first category such that for each $f \in \mathcal{A}$ the restricted functions $f|_{(X \setminus A)}$ are quasicontinuous (on $X \setminus A$) and for each pair of nonempty sets $V \in T_X$ and $W \in T_Y$ there is an index $j \in J$ such that $T_j(V \setminus A) \cap W \neq \emptyset$, then the set $U$ of all universal elements for $\mathcal{A}$ is residual in $X$.*

Proof. This theorem is an immediate corollary from Theorem 2. Since the space $X \setminus A$ is a Baire space and for each $f \in \mathcal{A}$ the restricted function $f|_{(X \setminus A)}$ is quasicontinuous, so the set $U$ for the family $\mathcal{A} = \{T_j|_{(X \setminus A)}; j \in J\}$ is residual in $(X \setminus A)$. So it is also residual in $X$. This finishes the proof.

In connection with Example 1 we have:

THEOREM 7. *Let $X = Y = \mathbb{R}$ and $T_X = T_Y = T_e$ be the Euclidean topology in $\mathbb{R}$. If $\mathcal{A} = \{g_n : X \to Y; n \geq 1\}$ is a countable family of monotone functions satisfying the condition* (3) *from Theorem 2 such that for each positive real $\varepsilon$ there is an index $k$ such that $osc g_n(x) < \varepsilon$ for each point $x \in X$ and for each index $n \geq k$, then the set $U$ for the family $\mathcal{A}$ is residual.*

Proof. For each index $n \geq 1$ let $D(g_n)$ denote the set of all discontinuity points of the function $g_n$. Since the functions $g_n$ are monotone, the sets $D(g_n)$ are countable. So the set

$$A = \bigcup_{n \geq 1} D(g_n)$$

is also countable. As the countable set $A$ is of the first category and the restricted functions $g_n|_{(X \setminus A)}$, $n \geq 1$, are continuous. It suffices to prove that the family $\mathcal{A}$ satisfies the remaining hypotheses of Theorem 6. For this, fix bounded open intervals $V, W \subset \mathbb{R}$ and denote by $(x, y)$ the center of the rectangle $V \times W$. Find a positive real

$$\varepsilon < \frac{\min(|V|, |W|)}{4},$$

where $|V|$ denotes the length of the interval $V$. Let $k$ be an index such that

$$osc g_n(x) < \varepsilon \text{ for all } x \in X \text{ and } n \geq k$$

and let

$$S = (x - \varepsilon, x + \varepsilon) \times (y - \varepsilon, y + \varepsilon).$$

Since the graphs of monotone functions are nowhere dense in $\mathbb{R}^2$, the set

$$S_1 = S \setminus \bigcup_{n<k} cl(Gr(g_n)),$$

where $cl$ denotes the closure operation, is open and nonempty. There are open intervals $V_1 \subset (x - \varepsilon, x + \varepsilon)$ and $W_1 \subset (y - \varepsilon, y + \varepsilon)$ such that

$$V_1 \times W_1 \subset S_1.$$

Since the family $\mathcal{A}$ satisfies the condition (3) from Theorem 2, there is an index $m \geq k$ such that

$$g_m(V_1) \cap W_1 \neq \emptyset.$$

Let $(x_1, g_m(x_1))$ be a point belonging to $V_1 \times W_1$. If $x_1 \in \mathbb{R} \setminus A$, then

$$\emptyset \neq g_m(V_1 \setminus A) \cap W_1 \subset g_m(V \setminus A) \cap W.$$

So suppose that $x_1 \in A$. Since $osc g_m(x_1) < \varepsilon$, there is an open interval $V_2 \subset V_1$ containing $x_1$ and such that $osc_{V_2} g_m < \varepsilon$. Fix a point $x_2 \in V_2 \setminus A$. Then

$$|g_m(x_2) - g_m(x_1)| < \varepsilon \text{ and } g_m(x_2) \in W.$$

Consequently, $(x_2, g_m(x_2)) \in V \times W$ and

$$g_m(V \setminus A) \cap W \neq \emptyset.$$

So the family $\mathcal{A}$ satisfies all hypotheses of Theorem 6 and the set $U$ for the family $\mathcal{A}$ is residual in $\mathbb{R}$. This finishes the proof.

Finishing we will consider the families of functions $f : \mathbb{R} \to \mathbb{R}$ connected with the density topology.

Remember that $x$ is a density point of a set $A \subset \mathbb{R}$ if there is a Lebesgue-measurable set $B \subset A$ such that

$$\lim_{h \to 0^+} \frac{\mu(B \cap (x-h, x+h))}{2h} = 1,$$

where $\mu$ denotes the Lebesgue measure in $\mathbb{R}$, and that the family

$$T_d = \{A \subset \mathbb{R}; \forall_{x \in A} x \text{ is a density point of } A\}$$

is a topology called the density topology ([1, 14]). It is well now that all set belonging to the density topology are measurable (Bruckner [1] p.19 (the proof of Theorem 5.2), Tall [13], Wilczyński [12], pp. 675–685).

From the Lebesgue density theorem it follows that a set $A \subset \mathbb{R}$ is of the first category with respect to $T_d$ if and only if $\mu(A) = 0$.

If $T_e$ denotes the Euclidean topology in $\mathbb{R}$ then the continuity of functions from $(\mathbb{R}, T_d)$ to $(\mathbb{R}, T_e)$ is called the approximate continuity.

All Lebesgue-measurable functions $f : \mathbb{R} \to \mathbb{R}$ are almost everywhere approximately continuous ([1]).

The quasicontinuity of mappings $f : (\mathbb{R}, T_d) \to (\mathbb{R}, T_e)$ is called approximate quasicontinuity ([5]).

From Theorem 2 it follows that if $X = Y = \mathbb{R}$, $T_X = T_d$, $T_Y = T_e$ and $\mathcal{A}$ is a family of approximately quasicontinuous functions from $X$ to $Y$ satisfying the condition (3) of Theorem 2 then $\mu(\mathbb{R} \setminus U) = 0$, **i.e.** almost all points $x \in \mathbb{R}$ are universal elements for the family $\mathcal{A}$.

Observe that if for a function $f : \mathbb{R} \to \mathbb{R}$ and a point $x$ there is a measurable set $A \ni x$ such that the restricted function $f|_A$ is continuous at $x$ and

$$(i) \qquad \limsup_{h \to 0^+} \frac{\mu(A \cap [x-h, x+h])}{2h} > 0,$$

then $f$ is approximately quasicontinuous at $x$.

We will say that a function $f : \mathbb{R} \to \mathbb{R}$ has the property $(p_i)$ if $f$ has at each point a path satisfying the condition $(i)$ (**i.e.** for each point $x$ there is a measurable set $A \ni x$ satisfying the condition $(i)$ such that the restricted function $f|_A$ is continuous at $x$.

A natural generalization of the property $(p_i)$ is the property of Denjoy-Clarkson ([2, 3]).

Remind that a nonempty measurable set $A \subset \mathbb{R}$ has the property of Denjoy-Clarkson if for every open interval $I$ such that $A \cap I \neq \emptyset$ the measure $\mu(A \cap I) > 0$. Moreover we suppose that $\emptyset$ has the property of Denjoy-Clarkson.

We say that a function $f : \mathbb{R} \to \mathbb{R}$ has the property of Denjoy-Clarkson if for each open set $U \in T_e$ the preimage $f^{-1}(U)$ has the property of Denjoy-Clarkson.

EXAMPLE 2. Let $C \subset [0,1]$ be a Cantor set of positive measure such that $0, 1 \in C$. Enumerate in a sequence $(I_n)$ all components of the complement $[0,1] \setminus C$ and in each open interval $I_n$ find a closed interval $K_n = [a_n, b_n] \subset I_n$. For $n \geq 1$ define a continuous function $f_n : K_n \to [-n, n]$ such that $f_n(a_n) = f_n(b_n) = 0$ and $f_n(K_n) = [-n, n]$. Let

$$f(x) = f_n(x) \text{ for } x \in K_n \quad \text{and} \quad f(x) = 0 \text{ otherwise on } \mathbb{R}.$$

There are nonmeasurable sets $B, E$ such that

$$C = B \cup E, \quad B \cap E = \emptyset$$

and for each measurable set $G \subset C$ of positive measure the intersections $G \cap B$ and $G \cap E$ are nonempty. Let $(w_n)$ be an enumeration of all rationals. For each pair $(x, n) \in B \times N$, where $N$ denotes the set of positive integers, define

$$g_{x,n}(u) = f(x) \text{ for } u \neq x \quad \text{and} \quad g_{x,n}(u) = w_n \text{ for } u = x.$$

For $n \geq 1$ let

$$h_n(x) = f(x) + w_n \text{ for } x \in \mathbb{R} \setminus C \text{ and } h_n(x) = f(x) = 0 \text{ for } x \in C.$$

Let

$$\mathcal{A} = \{h_n; n \geq 1\} \cup \{g_{x,n}; x \in B \text{ and } n \in N\}.$$

Evidently all functions from the family $\mathcal{A}$ have the property of Denjoy-Clarkson. Moreover the family $\mathcal{A}$ satisfies condition (3) from Theorem 2 for $X = Y = \mathbb{R}$, $T_X = T_d$ and $T_Y = T_e$, but $U = \mathbb{R} \setminus E$ is not residual in $(\mathbb{R}, T_d)$.

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