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QUANTUM GROUPOIDS OF THE FINITE TYPE AND QUANTIZATION ON ORBIT SPACES

Abstract. We show that the Hopf algebra on a transformation groupoid $\Gamma = E \times G$, where G is a finite group acting on the total space of a principal fibre boundle over $M = E/G$, is the cross product of the algebras $C^{\infty}(E)$ and $\mathbf{C}G$. We study duality properties of this algebra, and consider quantization on orbit spaces program in this context.

1. Introduction

The quantization on homogeneous spaces program began with Mackey's fundamental work [6] who studied particle motion on spaces G'/G where G is a closed subgroup of G'. This work gave rise to various modifications and generalizations, for instance [1, 4, 5]. Together with the advent of the quantum group theory the Mackey's quantization program acquired a new momentum. It was Shahn Majid who noticed that if the quantum algebra of observables is a Hopf algebra, then the noncocommutative coproduct corresponds to a non-Abelian group structure on the phase space which, in turn, means that the underlying Riemannian manifold has curvature. This could put "quantum mechanics and gravity on an equal but mutually dual footing" (Majid elaborated his previous ideas in [8]). In this context quantization on orbit spaces provides heuristic models allowing one to elaborate new techniques and better physical intuitions. The main results obtained by Majid in this field refer to finite quantum groups [7, Chapter 6.1].

In the present work, we generalize this kind of research by changing from finite quantum groups to quantum groupoids of the final type (by the latter we mean the situation when a finite quantum group is acting on a non-necessarily finite space, see below). In fact, we show that even in the previous research a certain finite quantum groupoid was implicitly involved.

To define a quantum groupoid is not a trivial thing (see [2, 9, 10]), and even for finite quantum groupoids there exist several definitions some of which are equivalent with each other [11, 12]. We start with constructing a transformation groupoid $\Gamma = E \times G$, where G is a finite group and E a principal fibre bundle over $M = E/G$, and then we show that the groupoid algebra is isomorphic with the cross product $C^\infty(E) \rtimes \mathbf{C}G$. This material is presented in Section 2. In Section 3, we demonstrate that our cross product algebra exhibits nice self-duality property and, in Section 4, we generalize Majid's version of quantization on homogeneous spaces to the context of quantum groupoids of the final type. This generalization could be important from the physical point of view since space-time M appears in it naturally when E is interpreted as the total space of the frame bundle.

2. Groupoids of the finite type

Let $\tilde{E}$ be a differential manifold with a group $\tilde{G}$ acting on it smoothly and freely to the right, $\tilde{E} \times \tilde{G} \to \tilde{E}$. We have the bundle $(\tilde{E}, \pi_M, M = \tilde{E}/\tilde{G})$. The frame bundle over M with the Lorentz group $\tilde{G}$ as its structural group is a special case of this construction. Let G be a finite subgroup of $\tilde{G}$, and $s : M \to \tilde{E}$ a cross section of the bundle $(\tilde{E}, \pi_M, M)$. We do not assume that it is continuous, we simply chose one element of E from each fibre (it can be easily seen that if the cross section $s : M \to \tilde{E}$ is smooth, the bundle $(\tilde{E}, \pi_M, M)$ is a trivial $\tilde{G}$-bundle).

We define $E = \bigcup_{x \in M} s(x)G$. Since G acts freely (to the right) on E, $E \times G \to E$, the Cartesian product $\Gamma = E \times G$ has the transformation groupoid structure. Let $\gamma_1 = (p_1, g_1)$ and $\gamma_2 = (p_2, g_2)$ be elements of Γ. They are composed in the following way: $(p_1, g_1) \circ (p_2, g_2) = (p_1, g_1 g_2)$, if $p_2 = p_1 g_1$. The source and range mappings for $\gamma = (p, g)$ can now be written as

$$d(\gamma) = p = s(x) \cdot g_1,$$

$$r(\gamma) = pg = s(x) \cdot g_2,$$

$x \in M$, for $g_1, g_2 \in G$, respectively; with $g_2 = g_1 g$.

The above groupoid $\Gamma = E \times G$, for both E and G finite, will be called the *finite groupoid*; if only G is finite it will be called the *groupoid of the finite type*.

Let us consider the following algebras:

1. The groupoid algebra $\mathcal{A} = C^\infty(\Gamma, \mathbf{C})$ with the convolution as multiplication

$$(a * b)(\gamma) = \sum_{\gamma_1 \in \Gamma_{d(\gamma)}} a(\gamma_1) b(\gamma_1^{-1} \gamma).$$

2. The algebra $A = C^\infty(E, \mathbf{C})$ with the usual pointwise multiplication.

3. The group algebra $H = \mathbf{C}G$ (linear combinations of elements of G with coefficients from $\mathbf{C}$ and linear extension of group multiplication as product) together with its Hopf algebra structure [7, Examples 1.5.3 and 1.5.4].

Let $A \bowtie H$ be the left cross product of the algebras A and H with the multiplication

$$(a \otimes h)(b \otimes g) = \sum a(h \triangleright b)hg,$$

where

$$(h \triangleright b)(p) = b(ph),$$

and the multiplication extends by linearity.

PROPOSITION 1. *The algebras $A \bowtie H$ and $\mathcal{A}$ are isomorphic. The isomorphism $J : A \bowtie H \to \mathcal{A}$ is given by*

$$J(a \otimes g) = a \otimes \delta_g$$

on primitive elements and extended by linearity to other elements. One has

$$J(a \otimes g)(p, g_1) = a(p)\delta_g(g_1)$$

where δ_g is the Kronecker δ function.

Proof. First, let us notice that $A \bowtie H$ and $\mathcal{A}$ are isomorphic as $C^\infty(M)$-modules. Indeed, the isomorphism $J^{-1} : \mathcal{A} \to A \bowtie H$ is given by

$$J^{-1}(a \otimes \delta_g) = a \otimes g.$$

Then we check that J is the homomorphism of algebras, i.e., we check by direct computation that

$$J((a \otimes h)(b \otimes g))(p, g_1) = (J(a \otimes h) * J(b \otimes g))(p, g_1).$$

It remains to show that $J(1 \otimes e) = 1 \otimes \delta_e$, where e is the unity of G, is the convolution unit. Indeed, for any $f \in \mathcal{A}$ we have

$$\begin{aligned}(f * (1 \otimes \delta_e))(p, g) &= \sum_{\bar{g} \in G} f(p, \bar{g})(1 \otimes \delta_e)(p\bar{g}, \bar{g}^{-1}g) \\ &= \sum_{\bar{g} \in G} f(p, \bar{g})\delta_e(\bar{g}^{-1}g) = f(p, g).\end{aligned}$$ □

Let E_x be the fiber over $x \in M$, and $\Gamma_x = E_x \times G$. Let us consider the algebras: $A_x = C^\infty(E_x, \mathbf{C})$ which is isomorphic with the algebra $\mathbf{C}(E_x)$ of all functions on E_x with the pointwise multiplication, and $\mathcal{A}_x = C^\infty(\Gamma_x, \mathbf{C})$ which is isomorphic with the algebra $\mathbf{C}\Gamma_x$ of all linear combinations of elements from Γ_x with convolution as multiplication.

PROPOSITION 2. *The algebras $A_x \bowtie H$ and $\mathcal{A}_x$, for every $x \in M$ are isomorphic. Moreover, the algebras A_x and H are strictly paired, and therefore $A_x^* = H$ and $H^* = A_x$ as vector spaces.*

Proof. The first part of the proof is analogous to that of Proposition 1. To prove the second part let us notice that $E = \bigcup_{x \in M} s(x)G$. Then we have $E_x = \{s(x)g : g \in G\}$, and the pairing form is

$$\langle \varphi, h \rangle = \sum_{g \in G} h(g)\phi(s(x)g),$$

$\varphi \in \mathbf{C}(E_x)$, $h = \sum_{g \in G} h(g)g \in G$. This form is nondegenerate, i.e.,

$$\forall h \in H, \langle \varphi, h \rangle = 0 \Rightarrow \varphi = 0,$$

$$\forall \varphi \in A_x, \langle \varphi, h \rangle = 0 \Rightarrow h = 0.$$

The first of these equalities we obtain by substituting for h the subsequent elements of the group G that generate the algebra H; the second of these equalities by putting $\varphi = \delta_{s(x)g}$. □

The above proof depends on the choice of the cross section $s(x)$, but we should remember that this cross section enters into the very construction of the groupoid Γ (through the definition of E). The same cross section ensures the isomorphism of the algebras A_x and $\mathbf{C}(G)$ [7, Example 1.5.2] which allows us to equip A_x with the Hopf algebra structure. For $f \in A_x$ we define the coproduct

$$\Delta f(p_1, p_2) = \Delta f(s(x)g_1, s(x)g_2) = f(s(x)g_1 g_2),$$

$p_1, p_2 \in E_x$, the counit

$$\epsilon f = f(s(x)e) = f(s(x)),$$

and the antipode

$$(Sf)(s(x)g) = f(s(x)g^{-1}).$$

COROLLARY 3. *The algebras A_x and H are strictly dual as Hopf algebras.* □

It should be noticed that the structure of the algebra A_x depends on the choice of the section $s : M \to E$. Therefore, we in fact have a "bundle of coalgebras" over M.

We are now able to define a *quantum grouppoid of the finite type.* The algebra $\mathcal{A}$ will be called its *total algebra* and the algebra $\mathcal{Z} := C^\infty(M)$ its *base algebra.* The *source* and *target* maps, $\alpha : \mathcal{Z} \to \mathcal{A}$, are equal and are defined to be

$$\alpha(f) = pr^* f$$

for $f \in \mathcal{Z}$, where pr is the composition of the natural projection of Γ onto E with the bundle projection π_M. The total algebra $\mathcal{A}$ has the natural $(\mathcal{Z}, \mathcal{Z})$-bimodular structure given by multiplication of elements of the type $pr^* f$. We have the coproduct $\Delta\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

$$\Delta(\delta_p \otimes \delta_g) = \delta_p \otimes \delta_g \otimes \delta_p \otimes \delta_g$$

on simple elements, to be extended by linearity. Δ is a mapping of $(\mathcal{Z}, \mathcal{Z})$-bimodules, but it does not preserve unit. The counit $\epsilon : \mathcal{A} \to \mathcal{A}$ is

$$\epsilon(a \otimes \delta_g)_x = a(s(x))$$

for every $a \in \mathcal{A}$. It can be easily seen that $\epsilon(\mathbf{1}_\mathcal{A}) = 1$. And the antipode $S : \mathcal{A} \to \mathcal{A}$ is

$$S(a)(\gamma) = a(\gamma^{-1}),$$

for every $a \in \mathcal{A}$ and $\gamma \in \Gamma$.

Here we also have a "bundle of quantum groupoids of the finite type" dpending on the section $s : M \to E$.

It can be readily checked that our groupoid of the final type satisfies all conditions of the Hopf bi-algebroid definition given by Lu [2] with the exception that the coproduct Δ does not preserve unit. For this reeason we shall also call it the *weak Hopf algebroid*. It is worthwhile to notice that in some groupoid definitions preservation of unit is not required (see for instance [12]).

3. Pontriagin duality

Let G be a finite Abelian group, and $\hat{G} = \{\chi : G \to \mathbf{C} \setminus \{0\}\}$ the set of its characters. Pontriagin theorem asserts that $\hat{G}$ itself has the Abelian group structure, and that there are the following isomorphisms

(1) $\mathbf{C}\hat{G} \simeq \mathbf{C}(G)^* = \mathbf{C}(G)$,

(2) $\mathbf{C}G \simeq \mathbf{C}(\hat{G})$.

These two isomorphisms are connected by the Fourier transforms in the following way

$$\tilde{h}(u) = \sum_{\chi \in \hat{G}} h(\chi)\chi(u),$$

$$\tilde{\varphi}(\chi) = \frac{1}{|G|} \sum_{u \in G} \chi(u^{-1})\varphi(u)$$

for $h, \tilde{\varphi} \in \mathbf{C}\hat{G}, \varphi, \tilde{h} \in \mathbf{C}(G)$ (see [3]).

In our case, because of the isomorphism between $A_x = \mathbf{C}(E_x)$ and $\mathbf{C}(G)$ we have

$$\mathbf{C}\hat{G} = (\mathbf{C}G)^* = \mathbf{C}(E_x)$$

for every $x \in M$ (equality (2) follows from equality (1) by adding another hat), and for the Fourier transform

$$\tilde{f}(s(x)u) = \sum_{\chi \in \hat{G}} f_\chi \chi(u)$$

where $\tilde{f} \in \mathbf{C}(E_x)$, $s(x) \in E_x$, $u \in G$ and $f \in \mathbf{C}\hat{G}$ is a linear combination of characters, i.e., $f = \sum_{\chi\in\hat{G}} f_\chi \chi$.

On the strength of Proposition 2, for any finite group, we immediately have

$$\mathcal{A}_x \simeq \mathbf{C}(E_x) \rtimes\!\!\!\lhd \mathbf{C}G$$

or

$$\mathcal{A}_x \simeq \mathbf{C}\hat{G} \rtimes\!\!\!\lhd \mathbf{C}G.$$

This cross product is also called *quantum double* of $\mathbf{C}G$. For a finite Abelian group, $\mathcal{A}_x$ exhibits the self-duality property

$$\hat{\mathcal{A}}_x \simeq \mathbf{C}\hat{\hat{G}} \rtimes\!\!\!\lhd \mathbf{C}\hat{G} = \mathbf{C}G \rhd\!\!\!\ltimes \mathbf{C}\hat{G}.$$

For the algebra $\mathcal{A} \simeq A \rtimes\!\!\!\lhd H$ everything above is true "along fibres".

4. Quantization on orbit spaces

Let us consider a *classical system* (G, X, α) where G is a Lie group acting on a space X to the right, $\alpha : X \times G \to X$. In the following we shall assume that X is a smooth manifold and α transitive. We regard $C^\infty(X)$ as the algebra of its observables. The right action α of G on X induces its left action on $C^\infty(X)$. In this case, G is called the momentum group.

The kinematic part of the quantization program of the above classical system consists in finding quantum algebra of observables in which classical observables are suitably contained. In Majid's approach [7, chapter 6.1], in which G and X are finite, this means finding an algebra B and maps

$$\mathbf{C}(X) \stackrel{\frown}{\to} B \stackrel{\frown}{\leftarrow} \mathbf{C}G,$$

such that

$$\hat{u}\hat{f}\hat{u}^{-1} = \widehat{\alpha_u(f)},$$

for all $u \in G$ and $f \in \mathbf{C}(X)$. Majid shows that the cross product algebra $\mathbf{C}(X) \rtimes\!\!\!\lhd \mathbf{C}G$, together with its canonical inclusions ($1 \otimes \mathbf{C}G$ and $\mathbf{C}(G) \otimes 1$), is a universal solution of the above algebraic quantization problem (Proposition 6.1.1).

On the strength of Proposition 1 we have the isomorphism

$$J^{-1} : \mathcal{A} \to C^\infty(E) \rtimes\!\!\!\lhd \mathbf{C}G$$

(with the correct inclusions) where E is the bundle introduced in Section 1. Indeed, for $a, b \in C^\infty(E)$ we have

$$a \mapsto J(a \otimes e) = a \otimes \delta_e \in \mathcal{A},$$

and the convolution becomes the usual multiplication

$$(J(a\otimes e) * J(b\otimes e))(x,g) = \sum_{\bar g\in G}(a\otimes\delta_e)(x,\bar g)(b\otimes\delta_e)(x\bar g,\bar g^{-1}g)$$
$$= \sum_{\bar g\in G} a(x)\delta_e(\bar g)b(x\bar g)\delta_e(\bar g^{-1}g) = (J(ab\otimes e))(x,g).$$

For $g,h\in \mathbf{C}G$ we have

$$g\mapsto J(1\otimes g) = 1\otimes\delta_g\in\mathcal{A},$$

and the multiplication is preserved

$$(J(1\otimes g)\otimes J(1\otimes h))(x,g_1) = \sum_{\bar g\in G} 1(x)\delta_e(\bar g)1(x\bar y)\delta_h(\bar g^{-1}g_1)$$
$$= \delta_{gh}(g_1) = (J(1\otimes gh))(x,g_1).$$

The repetition of Majid's proof (of Proposition 6.1.1 in [7]) shows that the algebra $\mathcal{A}$, together with its two natural inclusions is also a universal solution of the algebraic quantization problem of the classical system (G,E,α).

Moreover, there exists a representation of the algebra $\mathcal{A} = C^\infty(E)\bowtie\mathbf{C}G$ in $C^\infty(E)$ called the *Schrödinger representation*. It is defined to be a mapping $\mathcal{S}:\mathcal{A}\to \mathrm{Lin}(C^\infty(E))$ given by

$$(\mathcal{S}(f)(b))(p) = a(p)b(pg)$$

for every $f = a\otimes\delta_g\in\mathcal{A}$, $a,b\in C^\infty(E)$. The algebras $C^\infty(E)$ and $\mathbf{C}$G are involutive with the following involutions: for $a\in C^\infty(E)$, $a^* = \bar a$, and for $h=\sum_{g\in G}h_g g\in \mathbf{C}$G, $h^* = \sum_{g\in G}\bar h_g g$. Now $f^*(\gamma) = \overline{f(\gamma^{-1})}$, for every $f\in\mathcal{A}$ and $\gamma\in\Gamma$. In this case, the mapping $J : C^\infty(E)\bowtie\mathbf{C}G\to\mathcal{A}$ is an *-isomorphism. Indeed, we readily check that

$$f^*(\gamma) = (a\otimes\delta_q)^*(p,h)$$

and

$$\overline{f(\gamma^{-1})} = \overline{f(ph,h^{-1})}$$

are equal to each other.

To prove that the algebra $\mathcal{A} = C^\infty(X)\bowtie\mathbf{C}G$ and its two canonical inclusions are involutive algebra maps it is enough to check the equality (see [7, Proposition 6.1.5])

$$(h\triangleright a)^* = (Sh)^*\triangleright a^*$$

for every $a\in C^*(X)$ and $h\in\mathbf{C}G$, but this is immediate

$$(h\triangleright a)^*(p) = \overline{a(ph)},$$

and

$$((Sh)^*\triangleright a^*)(p) = a^*(ph) = \overline{a(ph)}.$$

In this case the action is said to be *unitary*, and the situation is analogous to what is usually done in quantum mechanics.

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