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Leszek Pysiak

GROUPOIDS, THEIR REPRESENTATIONS AND IMPRIMITIVITY SYSTEMS

1. Introduction

Groupoids play a crucial role in the study of symmetry in differential geometry and physics; also in algebraic topology the very useful notion is that of the fundamental groupoid (see [10] for a survey).

The importance of the theory of representations of groupoids has its origin in models unifying general relativity and quantum mechanics (see [2, 3, 4]) based on noncommutative convolution algebras. We have used the technique of representations of groupoids to study the theory of spacetime singularities (see [1]).

The representations of goupoids were defined and studied in different manners by various authors (see [11] for example).

The aim of this paper is to define, clarify and explain some basic concepts of the goupoids representation theory related to similar concepts for groups.

To simplify the exposition we assume that groupoids are finite sets, although most of results holds for more general case.

In section 2 we describe groupoids, transformation groupoids and define transitive action groupoids. In Lemma 2.2 we prove the isomorphism of transitive free action groupoid with a pair groupoid.

In section 3 we define a representation of groupoid, describe basic properties of the representation, recall the concepts of a system of impimitivity and of an induced system of imprimitivity.

In section 4 we investigate a correspondence between the set of representations of a transformation groupoid $\Gamma = X \times G$ and the set of systems of imprimitivity of the group G. We prove in Theorem 4.1 that this correspondence is one–to–one. The proof in the finite case is simple and transparent. A version for C *–algebras of that correspondence one can find in [5], chapter III–3 (see also [8]). In the paper [1] we present a partial construction of the correspondence in the case of $\Gamma = E \times G$ where E is the frame bundle over the spacetime M and G is the structural group of the bundle.

Finally, we conclude, using the imprimitivity theorem of Mackey [7], that every representation of groupoid is unitarily equivalent to induced one.

2. Preliminaries

2.1. Basic concepts. A groupoid and a transformation groupoid

A groupoid is a certain generalization of a group, in which multiplication is only partially defined. When it is defined, it is associative, and each element has an inverse in a suitable sense. Let us recall (see, for instance the book of Paterson [8], chapter 1) that a groupoid Γ consists of set Γ, a distinguished subset $\Gamma_2 \subset \Gamma \times \Gamma$, called the set of composable elements, a mapping of multiplication: $\cdot : \Gamma_2 \to \Gamma$, defined by $(x, y) \mapsto x \cdot y$ and an inversion map $x \mapsto x^{-1}$ from Γ to Γ, such that $(x^{-1})^{-1} = x$. These two mappings satisfy:

(i) if $(x, y), (y, z) \in \Gamma_2$ then $(xy, z), (x, yz) \in \Gamma_2$ and $(xy)z = x(yz)$,
(ii) $(y, y^{-1}) \in \Gamma_2$ for every $y \in \Gamma$ and if $(x, y) \in \Gamma_2$ then $x^{-1}(xy) = y$ and $(xy)y^{-1} = x$.

We also define the set of units $\Gamma^0 = \{x^{-1}x : x \in \Gamma\}$ ($\Gamma^0 \subset \Gamma$). Let us define two mappings $d, r : \Gamma \to \Gamma^0$ by $d(x) = x^{-1}x$ and $r(x) = xx^{-1}$ called the source mapping and range mapping respectively. It is easily seen that $(x, y) \in \Gamma_2$ if and only if $d(x) = r(y)$.

For each $u \in \Gamma^0$ let denote $\Gamma_u = d^{-1}(u)$ and $\Gamma^u = r^{-1}(u)$. Now the set $\Gamma_u^u := \Gamma^u \cap \Gamma_u$ has a structure of group. It is called the isotropy group at u.

A groupoid Γ will be called a transitive groupoid if for every pair $u, v \in \Gamma$ there exists $x \in \Gamma$ such that $d(x) = u$ and $r(x) = v$.

In natural way we define a subgroupoid Γ_1 of Γ as a subset $\Gamma_1 \subset \Gamma$ closed with respect to the multiplication and inversion map. Clearly the unit space of subgroupoid Γ_1 will be subset of the groupoid Γ.

We can also consider additional structures as topology or differential structure in the set Γ. Then the composition and inversion mapping have to be respectively continuous or smooth. But in this paper we study algebraic properties of some groupoids.

An important class of groupoids is that of transformation groupoids (or action groupoid). Let X be a set with a group G acting on it to the right, $X \times G \to X$.

A transformation groupoid will be the set $\Gamma = X \times G$ equipped with the following structure: Two elements γ_1, γ_2 of Γ, i.e. two pairs $\gamma_1 = (x_1, g_1)$, $\gamma_2 = (x_2, g_2)$, where $x_1, x_2 \in X$ and $g_1, g_2 \in G$, will be composable if and only if $x_2 = x_1 g_1$ and then the product will be defined by the following formula:

$$\gamma_1 \circ \gamma_2 = (x_1 g_1, g_2) \circ (x_1, g_1) = (x_1, g_1 g_2).$$

The inverse of $\gamma = (x, g) \in \Gamma$ will be $\gamma^{-1} = (xg, g^{-1})$. If we represent $\gamma = (x, g)$ as an arrow beginning at x and ending at xg, then two arrows γ_1, γ_2 can be composed if the beginning of γ_2 coincides with the end of γ_1. Now, the set of units of the groupoid Γ is $\Gamma^0 = \{\gamma^{-1}\gamma : \gamma \in \Gamma\} = \{(x, e) : x \in X\}$, where e is the unity of the group G. We can write $\Gamma^0 = X$, identifying an element (x, e) with $x \in X$. Next

$$\Gamma_x = \{(x, g) : g \in G\},$$
$$\Gamma^x = \{(xg^{-1}, g) : g \in G\}.$$

It is clear that the set Γ_x can be represented as the set of arrows which begin in x (or in $(x, e) \in \Gamma$) and the set Γ^x as the set of arrows which end at x.

2.2. A transitive action groupoid

The transformation groupoid $\Gamma = X \times G$ which is a transitive groupoid will be called a transitive action groupoid. It is clear that Γ is a transitive action groupoid if and only if G acts transitively on the set X. In this case the set X forms a single orbit of the group G and can be identifield with the set $K \setminus G$ of right cosets of a subgroup K of G. Denote by $o \in X$ the origin of X, i.e. the point of X for which we have

$$\{g \in G : og = o\} = K.$$

It is clear the isotropy group $\Gamma_o^o = \{(o, k) : k \in K\} \simeq K$.

LEMMA 2.1. *Let $x \in X$. Choose an element $g_x \in G$ such that $og_x = x$ and denote by K_x the subgroup of G of the form $K_x = \{g_x^{-1} k g_x : k \in K\}$. Then the isotropy group at x $\Gamma_x^x = \{(x, \overline{k}) : \overline{k} \in K_x\}$. Hence all the isotropy groups Γ_x^x $(x \in X)$ are mutually isomorphic and isomorphic to K.*

Proof. We observe that $\gamma \in \Gamma_x^x$ if and only if it is of the form $\gamma = (x, g)$ where $g \in G$ satisfies $x = xg^{-1}$ or equivalently $og_x = og_x g^{-1}$ or finally $og_x g g_x^{-1} = x$. But this means that $g_x g g_x^{-1} \in K$ and we have $g = g_x^{-1} k g_x$ for some $k \in K$. □

Now, let observe that if for a transitive action groupoid $\Gamma = X \times G$, G acts freely on X, then $K = \{e\}$ (and all the isotropy groups Γ_x^x are trivial). In that case we can identify X with G.

Now, return to a general transformation groupoid $\Gamma = X \times G$. The set X is the union of disjoint G–orbits: $X = \bigcup X_i$ and the group G acts transitively on every set X_i. This leads to decomposition of Γ in the union of disjoint subgroupids $\Gamma_i = X_i \times G$:

$$\Gamma = \bigcup \Gamma_i$$

and every Γ_i is a transitive action groupoid. It may occur that the isotropy group for different orbits X_i are not isomorphic mutually. This implies that the different orbits X_i would not be bijective or in the case of differential transformation groupoid (see [3]) would not be of the same differential dimension. We have studied this fact in [1] in the case of $\Gamma = E \times G$ where E is the frame bundle over spacetime M and G is the structural group of E. The group G acts transitively on each fiber E_m of the bundle E over a point $m \in M$. It follows that the groupoid Γ splits into the union of subgroupoids $\Gamma_m = E_m \times G : \Gamma = \bigcup_{m \in M} \Gamma_m$. Now, every subgroupoid Γ_m is a transitive action groupoid. But only if a point $m \in M$ is a regular point of spacetime M, G acts freely on E_m and then E_m is diffeomorphic to G. In singular points $m \in M$ we have $E_m \simeq K \setminus G$ for some subgroup K of G.

Another important class of groupoids is formed by the pair groupoids. For a set X we consider $\Gamma = X \times X$ and define the multiplication as follows:

$$(x,y) \circ (y,z) = (x,z) \quad \text{for all } x,y,z \in X.$$

Now, recall the definition of an isomorphism of two groupoids Γ, Γ'. Let Γ_2, Γ'_2 denote their respective sets of composable elements. The mapping $\Phi : \Gamma \to \Gamma'$ is called an isomorphism of groupoids if it is a bijection such that

(i) $(x,y) \in \Gamma_2$ if and only if $(\Phi(x), \Phi(y)) \in \Gamma'_2$,
(ii) for $(x,y) \in \Gamma_2$ we have $\Phi(x \circ y) = \Phi(x) \circ \Phi(y)$,
(iii) for $x \in \Gamma \quad \Phi(x^{-1}) = [\Phi(x)]^{-1}$.

In the paper [2] we have investigated the groupoid $\Gamma = E \times G$, where E is a subbundle of the frame bundle mentioned above and G is a finite group acting freely and transitive on each fiber E_m of the bundle E. We have introduced the groupoid $M \times G \times G$ with the multiplication defined by $(m, g', \overline{g}) \circ (m, g, g') = (m, g, \overline{g})$. We have proved that these two groupoids (Γ and $M \times G \times G$) are isomorphic. For $m \in M$ fixed, the subgroupoid $\Gamma_m = E_m \times G$ of the first groupoid is also isomorphic to the respective subgroupoid $\{m\} \times G \times G$ of the second one. This fact can be reformulated in a more simple manner using more general notation:

Let X be a finite set with a (finite) group G acting on X transitively and freely. Let $\Gamma = X \times G$ be the corresponding transitive action groupoid. Then

LEMMA 2.2. *The groupoid Γ is isomorphic to the pair groupoid $X \times X$.*

Proof. The isomorphism $\Phi : \Gamma \to X \times X$ will be given by for $\gamma = (x,g)$, $x \in X$, $g \in G$ by

$$\Phi(x,g) = (xg, x).$$

It is clear that Φ is a bijection of Γ and $X \times X$ which leads to bijection of composable sets. Now

$$\Phi[(xg,h) \circ (x,g)] = \Phi[(x,gh)] = (xgh,x)$$

and

$$\Phi(xg,h) \circ \Phi(x,g) = (xgh,xg) \circ (xg,x) = (xgh,x).$$

Similarly $\Phi[(x,g)^{-1}] = \Phi(xg,g^{-1}) = (x,xg) = [\Phi(x,g)]^{-1}$. □

3. Representations of groupoids

Let us recall the definition of a groupoid representation in the finite case.

Let Γ be a finite set equipped with a groupoid structure. Let $X = \Gamma^o$ be its set of units. Let be given a collection $\mathcal{H}$ of finite dimensional complex Hilbert spaces $\mathcal{H} = \{H_x\}_{x \in X}$ with x ranging over X.

DEFINITION 3.1. A unitary representation (shortly u.r.) $\mathcal{U}$ of the groupoid Γ is the pair $(\mathcal{U},\mathcal{H})$ where $\mathcal{H}$ is the collection of Hilbert spaces and $\mathcal{U}$ is the mapping

$$\Gamma \ni \gamma \mapsto \mathcal{U}(\gamma) \in U(H_{d(\gamma)}, H_{r(\gamma)})$$

where d and r are the source and the range mappings. (Here we denote by $U(H_x, H_y)$ the space of Hilbert isomorphisms from H_x to H_y, i.e. linear maps leaving invariant inner products). The following conditions are imposed on the mapping $\mathcal{U}$:

(i) $\mathcal{U}(x) = \mathrm{id}|_{H_x}$ for $x \in \Gamma^o$,
(ii) $\mathcal{U}(\gamma_1 \circ \gamma_2) = \mathcal{U}(\gamma_1) \circ \mathcal{U}(\gamma_2)$ for all $(\gamma_1, \gamma_2) \in \Gamma_2$,
(iii) $\mathcal{U}(\gamma^{-1}) = \mathcal{U}(\gamma)^{-1}$.

Let us observe that the composition in (ii) is well defined since $r(\gamma_2) = d(\gamma_1)$.

Further, note that if the groupoid Γ is transitive then all the spaces of the collection $\{H_x\}_{x \in X}$ are of the same dimension. It is clear because there exists an isomorphism between the spaces H_{x_1} and H_{x_2} for each pair of points $x_1, x_2 \in X$. But, in general, the spaces H_x need not be of the same dimension.

DEFINITION 3.2. Let $(\mathcal{U}_1, \mathcal{H}_1)$ and $(\mathcal{U},\mathcal{H})$ be two u.r. of the groupoid Γ. Suppose that the collections of Hilbert spaces $\mathcal{H}_1 = \{H_{1x}\}_{x \in X}$ and $\mathcal{H} = \{H_x\}_{x \in X}$ satisfy $H_{1x} \subset H_x$ for every $x \in X$. Moreover: $\mathcal{U}_1(\gamma) = \mathcal{U}(\gamma)|_{H_{1d(\gamma)}}$ for all $\gamma \in \Gamma$. Then $(\mathcal{U}_1, \mathcal{H}_1)$ is called a subrepresetation of the representation $(\mathcal{U},\mathcal{H})$.

DEFINITION 3.3. A representation $(\mathcal{U},\mathcal{H})$ of the groupoid Γ is called irreducible if there exists no proper subrepresentation, i.e. if one has the collection $\mathcal{H}_1 = \{H_{1x}\}_{x \in X}$ and $\mathcal{U}(\gamma)$ satisfying the condition $\mathcal{U}(\gamma)H_{1d(\gamma)} = H_{1r(\gamma)}$ for all $\gamma \in \Gamma$ then it implies $H_{1x} = H_x$ or $H_{1x} = \{0\}$ for every $x \in X$.

DEFINITION 3.4. Let $(\mathcal{U}_1, \mathcal{H}_1)$ and $(\mathcal{U}_2, \mathcal{H}_2)$ be two u.r. of the groupoid Γ. Assume that there exists a family of Hilbert space isomorphisms

$$A_x : H_{1x} \to H_{2x}, \quad x \in X$$

such that

$$A_{r(\gamma)}\mathcal{U}_1(\gamma) = \mathcal{U}_2(\gamma)A_{d(\gamma)}$$

for every $\gamma \in \Gamma$. Then we say that the representations $(\mathcal{U}_1, \mathcal{H}_1)$, $(\mathcal{U}_2, \mathcal{H}_2)$ are unitarily equivalent.

The main object of the paper is to relate the representations of transformation groupoids to the induced representations of groups and to the systems of imprimitivity (see [1, 5, 6]).

To fix notation, we recall the following definitions, for simplicity assuming the finiteness of groups.

Let G be a finite group and K its subgroup. Let (L, V) be a finite dimensional unitary representation (u.r.) of K in a Hilbert space V. Let H^L denote the space of all functions f from G to V satisfying the following condition:

$$f(kg) = L(k)f(g) \quad \text{for all } k \in K \text{ and } g \in G. \tag{1}$$

Then we define the operators in H_L:

$$(U^L(g_0)f)(g) = f(gg_0) \qquad \text{for } g_0 \in G. \tag{2}$$

In the space H^L we introduce the inner product as follows:

$$(f_1, f_2) = \sum_{K\backslash G} (f_1(g), f_2(g))_L$$

where $K \setminus G$ indicates that we take just one element g for each right coset Kg and $(\ ,\ )_L$ denotes the inner product od the space L. The unitarity of L implies that $(f_1(g), f_2(g))_L$ does not depend of the choosen element g of a coset. Now it is easy to see that the operators $U^L(g_0)$ form a u.r. of the group G.

DEFINITION 3.5. The u.r. (U^L, H^L) of G is called the representation of G induced by L.

Now we return to the systems of imprimitivity. Let $(\mathcal{U}, \mathcal{H})$ be a u.r. of finite group G in a finite dimensional Hilbert space $\mathcal{H}$. Let G acts on a finite set X to the right: $X \times G \to X$. Suppose that there exists a decomposition of Hilbert space $\mathcal{H}$ into direct sum of Hilbert spaces H_x ($H_x \subset H$), $x \in X$: $\mathcal{H} = \bigoplus_{x\in X} H_x$. It is clear that the decomposition defines a family of orthogonal projection operators P_x of the space $\mathcal{H}$ onto space H_x, for all $x \in X$:

$$P_x : \mathcal{H} \to H_x$$

which satisfy:

$$(3) \qquad P_x \circ P_y = \begin{cases} P_x & \text{for } x = y \\ 0 & \text{for } x \neq y, \end{cases}$$

$$(4) \qquad \sum_{x \in X} P_x = \mathrm{id}|_{\mathcal{H}}.$$

The family $\{P_x\}_{x \in X}$ leads to a projection operators valued (probability) measure on X. Denote the family shortly by P.

DEFINITION 3.6 (see [9, 6]). By a system of imprimitivity (S.I. for short) of the group G for the representation $(\mathcal{U}, \mathcal{H})$ with the base X we shall mean the quadruple $(G, \mathcal{U}, X, P)$, where P is the family of projection operators satisfying the condition (3) and (4) together with the following one

$$(5) \qquad \mathcal{U}(g) P_x \mathcal{U}(g^{-1}) = P_{xg^{-1}} \quad \text{for every } g \in G \text{ and } x \in X.$$

The condition (5) expresses the covariance of P with respect to $\mathcal{U}$.

S.I. is said to be transitive if G acts transitively on X. (In this case $X = K \setminus G$ for a subgroup K of G).

LEMMA 3.1. *Let $(G, \mathcal{U}, X, P)$ be a transitive S.I. Denote $H_x = P_x \mathcal{H}$ for $x \in X$. Then all the Hilbert spaces H_x $(x \in X)$ are mutually isomorphic.*

Proof. Let $X = K \setminus G$ and $o \in X$ be the origin of X. Then $ok = o$ for every $k \in K$. Now choose $x_1 \in X$, $x_1 \neq o$. Then there exists $g_1 \in G$ such that $og^{-1} = x_d 1$. Let $h \in H_o$. Then $h = P_o h$ and by the condition (5) we have:

$$\mathcal{U}(g_1) h = \mathcal{U}(g_1) P_o h = P_{og_1^{-1}} \mathcal{U}(g_1) h = P_{x_1} \mathcal{U}(g_1) h.$$

This means that $\mathcal{U}(g_1) h \in H_{x_1}$ and that the operator $\mathcal{U}(g_1)$ realizes the unitary isomorphism of the space H_o onto H_{x_1}. □

Now we shall recall that for a induced representation of $(\mathcal{U}^L, \mathcal{H}^L)$ (see Definition 3.5) there exists a canonical transitive S.I. (see [9] for example), which is given by the following family of projections

$$P^L_{x_i} : \mathcal{H}^L \to H^L_{x_i}, \quad x_i \in X \qquad \text{and}$$

$$(6) \qquad P^L_{x_i}(f) = \begin{cases} 0 & \text{for } g \notin x_i \\ f(g) & \text{for } g \in x_i, \end{cases}$$

where $f \in \mathcal{H}^L$, $g \in G$, and $x_i = Kg_i$ for a fixed representative g_i of the right coset x_i. $\mathcal{H}$ is easy to see that

$$\mathcal{U}^L(g_0) P^L_{x_i} = P_{x_i g_0^{-1}} \mathcal{U}^L(g_o)$$

for every $g_0 \in G$.

DEFINITION 3.7. The S.I. $(G, \mathcal{U}^L, X, P^L)$, where $P^L = \{P^L_{x_i}\}_{x_i \in X}$ and $P^L_{x_i}$ are defined by the formula (6), is called the induced S.I. of G for the representation $\mathcal{U}^L$.

We shall recall also the Imprimitivity Theorem of Mackey (see [6, 7, 9]).

THEOREM 3.1 (Mackey). *Every transitive S.I. $(G,\mathcal{U},X,P)$ is unitarily equivalent to an induced S.I. $(G,\mathcal{U}^L,X,P^L)$ of G. More precisely, if $\mathcal{H}$ is the space of the representation $\mathcal{U}$ and $\mathcal{H}^L$ is the space of $\mathcal{U}^L$ then there exists a unitary isomorphism $A:\mathcal{H}\to\mathcal{H}^L$ such that*

$$\begin{cases} A\mathcal{U}(g)h = \mathcal{U}^L(g)Ah & \text{for every } g\in G, h\in H \\ AP_x h = P_x^L Ah & \text{for every } x\in X, h\in H \end{cases} \tag{7}$$

4. Representations of transformation groupoids

We continue with the finite case. Let $\Gamma = X\times G$ be a transformation groupoid (Γ being a finite set).

THEOREM 4.1. *There exists a one–to–one correspondence J between unitary representations of the transformation groupoid Γ and the systems of imprimitivity of the group G:*

$$J:\{(\overline{\mathcal{U}},\overline{\mathcal{H}})\}\to\{(G,\mathcal{U},X,P)\}.$$

Proof. 1°. Let $(\overline{\mathcal{U}},\overline{\mathcal{H}})$ be a u.r. of Γ. By Definition 3.1 we have that $\overline{\mathcal{H}} = \{H_x\}_{x\in X}$, where H_x are finite dimensional complex Hilbert spaces. Let us form the Hilbert direct sum of the space H_x: $\mathcal{H} = \bigoplus_{x\in X} H_x$. Then we can identify the space H_x with a subspace of $\mathcal{H}$ and define P_x as the projection operator of $\mathcal{H}$ onto H_x, for every $x\in X$. Let define operators $\mathcal{U}(g_0):\mathcal{H}\to\mathcal{H}$ by the formula:

$$\mathcal{U}(g_0)\Big(\sum_{x\in X} h_x\Big) = \sum_{x\in X}\overline{\mathcal{U}}(x,g_0^{-1})h_x, \tag{8}$$

where $h_x\in H_x$ and $g_0\in G$. First, observe that $\mathcal{U}(g_0)$ is well defined, because $\overline{\mathcal{U}}(x,g_0^{-1}): H_x\to H_{xg_0^{-1}}$. Now check that $\mathcal{U}$ is a representation of G:

$$\begin{aligned}[\mathcal{U}(g_1)\mathcal{U}(g_2)]\Big(\sum_{x\in X} h_x\Big) &= \overline{\mathcal{U}}(g_1)\Big(\sum_{x\in X}\mathcal{U}(x,g_2^{-1})h_x\Big)\\ &= \sum_{x\in X}\overline{\mathcal{U}}(xg_2^{-1},g_1^{-1})\overline{\mathcal{U}}(x,g_2^{-1})h_x,\end{aligned}$$

(because $\overline{\mathcal{U}}(x,g_2^{-1})h_x\in H_{xg_2^{-1}}$).

Now, by the condition (ii) of Definition 3.1 and by the law of multiplication of the groupoid it follows that the last sum is equal to:

$$\sum_{x\in X}\overline{\mathcal{U}}(x,g_2^{-1}g_1^{-1})h_x = \sum_{x\in X}\overline{\mathcal{U}}(x,(g_1g_2)^{-1})h_x = \mathcal{U}(g_1g_2)\Big(\sum_{x\in X}h_x\Big).$$

Finally the unitarity of the operators $\overline{\mathcal{U}}(x,g_0^{-1})$ implies that $(\mathcal{U},\mathcal{H})$ is unitary representation of G.

Now $\mathcal{U}P_x h = \overline{\mathcal{U}}(x, g_{-1})h = P_{xg^{-1}}\overline{\mathcal{U}}(x, g^{-1})h = P_{xg^{-1}}\mathcal{U}(g)h$ for each $h \in H$ and we obtain a S.I. $(G, \mathcal{U}, X, P)$. Thus, we have constructed the mapping J: $J(\overline{\mathcal{U}}, \overline{\mathcal{H}}) = (G, \mathcal{U}, X, P)$.

2°. Choose a S.I. $(G, \mathcal{U}, X, P)$. Denote $H_x = P_x\mathcal{H}$ and define: $\overline{\mathcal{U}}(x, g) : H_x \to H_{gx}$ by the formula:

$$\overline{\mathcal{U}}(x, g)h = \mathcal{U}(g^{-1})|_{H_x}h \qquad \text{for } h \in H_x. \tag{9}$$

Observe that $\overline{\mathcal{U}}(x, g) = P_{xg}\mathcal{U}(g^{-1})h$, by the condition (5) of Definition 3.6. But it means that $\overline{\mathcal{U}}(x, g)h \in H_{gx}$. Let us check the conditions (i), (ii), (iii) of Definition 3.1. Indeed, one has $\overline{\mathcal{U}}(x, e)h = \mathcal{U}(e)|_{H_x}h = h$, for $h \in H_x$. Further $\overline{\mathcal{U}}(xg_2, g_1) \circ \mathcal{U}(x, g_2) = \mathcal{U}(g_1^{-1})|_{xg_2} \circ \mathcal{U}(g_2^{-1})|_{H_x} = \mathcal{U}(g_1 g_2)^{-1}|_{H_x} = \overline{\mathcal{U}}(x, g_1 g_2)$. And finally $\overline{\mathcal{U}}(xg, g_{-1}) = \mathcal{U}(g)|_{H_{xg}} = \overline{\mathcal{U}}(x, g)^{-1}$. Thus we have constructed the representation $(\overline{\mathcal{U}}, \overline{\mathcal{H}})$ of Γ, corresponding to the S.I. given.

3°. It is easily seen that the mapping constructed in part 2° is inverse the mapping J constructed in part 1° □

In the section 3 we have seen that for a transitive action groupoid Γ if $(\mathcal{U}, \mathcal{H})$ is a u.r. of Γ then all the members H_x, $x \in X$ of the collection $\mathcal{H}$ are mutually isomorphic Hilbert spaces. This fact, by Theorem 4.1 coincides with Lemma 3.1.

Now if a transformation groupoid Γ is not transitive, $\Gamma = \bigcup \Gamma_i$, as we have seen in Section 2, where Γ_i is transitive action subgroupoid of Γ based on a G–orbit X_i. Then, if $(\mathcal{U}, \mathcal{H})$ is a u.r. of such groupoid, where $\mathcal{H} = \{H_x\}_{x \in X}$, one can not expect that the spaces H_{x_1}, H_{x_2} still will be isomorphic if x_1 and x_2 belong to different orbits X_1 and X_2. One can consider the restrictions $(\mathcal{U}_i, \mathcal{H}_i)$ of $(\mathcal{U}, \mathcal{H})$ to subgroupoids $\Gamma_i = X_i \times G$. These restrictions will be different as the representations of different, not isomorphic (in general) groupoids. But still Theorem 3.1 holds in this case.

Now we shall associate the representations of a transitive action groupoid with induced representations. Put $\Gamma = X \times G$, with $X = K \setminus G$. Let (L, V) be u.r. of the subgroup K and consider the induced S.I. of G $(G, \mathcal{U}^L, X, P^L)$ for the representation $\mathcal{U}^L$. (See Definition 3.7).

Definition 4.1. The u.r. of Γ $(\overline{\mathcal{U}}^L, \overline{\mathcal{H}}^L)$ given by formula

$$(\overline{\mathcal{U}}^L, \overline{\mathcal{H}}^L) = J^{-1}(G, \mathcal{U}^L, X, P^L)$$

is called the representation of Γ induced by L. (Here J denotes the bijection described in Theorem 4.1).

Theorem 4.2. *Every unitary representation $(\mathcal{U}, \mathcal{H})$ of the transitive action groupoid Γ is unitarily equivalent to an u.r. $(\overline{\mathcal{U}}^L, \overline{\mathcal{H}}^L)$ i.e. to representation of Γ induced by an representation L of K.*

Proof. This result is a simple consequence of Theorem 4.1 and of the Imprimitivity Theorem of Mackey (see Theorem 3.1). □

In a forthcoming paper which is in preparation we will present a more general concept of induced representations of groupoids and state a generalized version of Theorem 4.2.

Now we conclude this section with the following important and simple theorem.

THEOREM 4.3. *Let* $(\mathcal{U}^L, \mathcal{H}^L)$ *be the representation of* Γ *induced by* L. *Then* $(\mathcal{U}^L, \mathcal{H}^L)$ *is irreducible if and only if the representation* (L, V) *of the group* K *is irreducible.*

Proof. Now the statement of the theorem is clear by the construction of the induced S.I. (Definition 3.7) and by form of the bijection J of Theorem 4.1. □

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WARSAW SCHOOL OF INFORMATION AND MANAGEMENT
FACULTY OF COMPUTER SCIENCE
ul. Newelska 6
01-447 WARSAW, POLAND

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