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U. C. De, Anup Kumar Sengupta

NOTES ON THREE-DIMENSIONAL QUASI-SASAKIAN MANIFOLDS

Abstract. A non-cosympletic quasi-Sasakian manifold of dimension 3 is Ricci-semi-symmetric if and only if it is Einstein.

1. Introduction

On a 3-dimensional quasi-Sasakian manifold, the structure function β has been defined by Z. Olszak [5] and with the help of this function he has obtained necessary and sufficient conditions for the manifold to be conformally flat [6]. Next he has proved that if the manifold is additionaly conformally flat with β =constant, then (a) the manifold is locally a product of $\mathbb{R}$ and a 2-dimensional Kahlerian space of constant Gauss curvature (the cosympletic case), or (b) the manifold is of constant positive curvature (the non-cosympletic case, here the quasi-Sasakian structure is homothetic to a Sasakian structure). An example of a 3-dimensional quasi-Sasakian structure being conformally flat with non-constant structure function is also described in [6].

The object of the present paper is to study 3-dimensional quasi-Sasakian manifolds. We prove that a parallel symmetric $(0,2)$ tensor field in a 3-dimensional non-cosympletic quasi-Sasakian manifold is a constant multiple of the associated metric tensor and a parallel 2-form is the zero form on such manifolds. A Riemannian manifold is called semisymmetric (respectively, Ricci-semisymmetric) if $R(X,Y).R = 0$ (respectively, $R(X,Y).S = 0$) [4], [7] where $R(X,Y)$ is treated as a derivation of the tensor algebra for any tangent vectors X, Y.

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2. Preliminaries

Let M be a (2n+1)-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ,ξ,η,g), where ϕ is a tensor field of type (1,1), ξ is a vector field, η is a 1-form and g is the Riemannian metric on M such that [2], [3]

$$\phi^2 = -I + \xi\otimes\eta, \qquad \eta(\xi) = 1$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad X,\ Y\in T(M).$$

Then also

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X,\xi).$$

Let Φ be fundamental 2-form of M defined by $\Phi(X,Y) = g(X,\phi Y)$, $X, Y \in T(M)$. Then $\Phi(X,\xi) = 0$, $X\in T(M)$. M is said to be quasi-Sasakian if the almost contact structure (ϕ, ξ, η) is normal and the fundamental 2-form Φ is closed ($d\Phi = 0$), which was first introduced by Blair [1]. The normality condition gives that the induced almost complex structure of $M\times\mathbb{R}$ is integrable or equivalently, the torsion tensor field $N = [\phi,\ \phi] + 2\xi\otimes d\eta$ vanishes identically on M. The rank of a quasi-Sasakian structure is always odd [1], it is equal to 1 if the structure is cosympletic and it is equal to (2n+1) if the structure is Sasakian.

3. Quasi-Sasakian structure of dimension three

An almost contact metric manifold M of dimension 3 is quasi-Sasakian if and only if [5]

$$\nabla_X\xi = \beta\phi X, \qquad X\in T(M) \tag{3.1}$$

for a certain function β on M such that $\xi\beta = 0$, ∇ being the operator of the covariant differentiation with respect to the Levi-Civita connection of M. Clearly, such a quasi-Sasakian manifold is cosympletic if and only if $\beta = 0$. As a consequence of (3.1), we have [5]

$$(\nabla_X\phi)(Y) = \beta(g(X,Y)\xi - \eta(Y)X), \quad X,\ Y\in T(M). \tag{3.2}$$

In a 3-dimensional Riemannian manifold, we always have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &\quad -S(X,Z)Y - \tfrac{r}{2}(g(Y,Z)X - g(X,Z)Y) \end{aligned} \tag{3.3}$$

where Q is the Ricci operator i.e., $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold.

Let M be a 3-dimensional quasi-Sasakian manifold. The Ricci tensor S of M is given by [6]

$$S(X,Y) = (\tfrac{r}{2} - \beta^2)g(Y,Z) + (3\beta^2 - \tfrac{r}{2})\eta(Y)\eta(Z) - \eta(Y)d\beta(\phi Z) - \eta(Z)d\beta(\phi Y) \tag{3.4}$$

where r is the scalar curvature of M.

As a consequence of (3.4), we get for the Ricci operator Q

$$QX = (\tfrac{r}{2} - \beta^2)X + (3\beta^2 - \tfrac{r}{2})\eta(X)\xi + \eta(X)(\phi \;\; grad\, \beta) - d\beta(\phi X)\xi, \tag{3.5}$$

where the gradient of a function f is related to the exterior derivative df by the formula $df(X) = g(\;\; grad\, f, X)$. From (3.4) we have

$$S(X,\xi) = 2\beta^2\eta(X) - d\beta(\phi X). \tag{3.6}$$

Moreover, as a consequence of (3.3)–(3.5), we find

$$R(X,Y)\xi = \beta^2(\eta(Y)X - \eta(X)Y) + d\beta(Y)\phi X - d\beta(X)\phi Y, \quad X, Y \in T(M). \tag{3.7}$$

4. Parallel $(0,2)$-tensor fields

Let us consider a parallel symmetric $(0,2)$-tensor field α on a 3-dimensional quasi-Sasakian manifold M. Then, by $\nabla\alpha = 0$, we have

$$\alpha(R(W,X)Y,Z) + \alpha(Y,R(W,X)Z) = 0. \tag{4.1}$$

In the above and in the sequel we assume that W,X,Y,Z are arbitrary vector fields on M if it is not otherwise stated. As α is symmetric, putting $W = Y = Z = \xi$ in (4.1), we obtain

$$\alpha(\xi, R(\xi,X)\xi) = 0. \tag{4.2}$$

Let us assume that M is non-cosympletic. Take a non-empty connected open subset U of M on which $\beta\neq0$ and restrict our considerations to this set. Since $\beta\neq0$, by applying (3.7) and $\xi\beta = 0$ into (4.2) we get

$$\alpha(X,\xi) = \alpha(\xi,\xi)g(X,\xi). \tag{4.3}$$

Differentiating (4.3) covariantly along Y and applying again (4.3) and (3.1), we find

$$g(X,\phi Y)\alpha(\xi,\xi) - \alpha(X,\phi Y) = 0. \tag{4.4}$$

Putting ϕY instead of Y in (4.4) and using (4.3), we get

$$\alpha(X,Y) = \alpha(\xi,\xi)g(X,Y).$$

Hence, since α and g are parallel tensor fields, $\lambda = \alpha(\xi,\xi)$ is constant on U. Thus, $\alpha = \lambda g$ on the subset U. By the parallelity of α and g it must be $\alpha = \lambda g$ on the whole of M.

Thus we have the following:

LEMMA 4.1. *A parallel symmetric* $(0,2)$ *tensor field in a 3-dimensional non-cosympletic quasi-Sasakian manifold is a constant multiple of the associated metric tensor.*

Let us now assume that α is a parallel 2-form on M, that is $\alpha(X,Y) = -\alpha(Y,X)$ and $\nabla\alpha = 0$. Then

$$\alpha(\xi,\xi) = 0. \tag{4.5}$$

Covariant differentiation of (4.5) implies

$$\alpha(\nabla_X\xi,\xi) = 0. \tag{4.6}$$

By (3.1) we obtain from (4.6)

$$-\beta\alpha(\phi X,\xi) = 0.$$

Assume $\beta\neq 0$ on an open connected subset $U \neq \emptyset$. Then on U we have

$$\alpha(\phi X,\xi) = 0. \tag{4.7}$$

Putting ϕX instead of X in (4.7) and using (4.5) we obtain

$$\alpha(X,\xi) = 0. \tag{4.8}$$

Covariant differentiation of the above gives

$$\alpha(X,\phi Y) = 0. \tag{4.9}$$

Putting ϕY instead of Y in (4.9) and using (4.8) we get

$$\alpha(X,Y) = 0.$$

Hence $\alpha = 0$ on U. Since α is parallel on M, $\alpha = 0$ on M.

Thus we have the following:

LEMMA 4.2. *On a* 3*-dimensional non-cosympletic quasi-Sasakian manifold there does not exist a non-zero parallel* 2*-form.*

5. Ricci-semisymmetric quasi-Sasakian manifolds

LEMMA 5.1. *Let M be a Ricci-semisymmetric* 3*-dimensional non-cosympletic quasi-Sasakian manifold. Then the structure function β is constant.*

Proof. Let $R(X,Y).S = 0$, for any X,Y $\in$ T(M). Then we have

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0. \tag{5.1}$$

Putting $U = V = \xi$ in (5.1), we have

$$S(R(X,Y)\xi,\xi) = 0.$$

Hence, by applying (3.6) and (3.7), we get after certain calculations

$$\eta(Y)d\beta(\phi X) - \eta(X)d\beta(\phi Y) = 0 \tag{5.2}$$

on the set on which $\beta \neq 0$. Taking $Y = \xi$ in (5.2) we obtain

$$d\beta(\phi X) = 0.$$

Now taking ϕX instead of X in the above and using $\xi\beta = 0$, we get $d\beta = 0$ on the set where $\beta \neq 0$. Therefore β is a constant function on M.

THEOREM 5.1. *Let M be a 3-dimensional non-cosympletic quasi-Sasakian manifold. Then the following conditions are equivalent:*
(i) *M is an Einstein manifold;*
(ii) *the Ricci tensor S of M is parallel i.e., $\nabla S = 0$;*
(iii) *M is Ricci-semisymmetric.*

Proof. (i)$\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are clear. We will prove (iii) $\Rightarrow$ (i).

Let us assume that condition (iii) holds. Then equation (5.1) holds good. Also by Lemma 5.1 we have β is a constant function. Since the manifold M is non-cosympletic, $\beta \neq 0$.

Now putting $X = U = \xi$ in (5.1) and then using (3.7), we have

$$-S(Y,V) + \eta(Y)S(\xi,V) + g(V,Y)S(\xi,\xi) - \eta(V)S(\xi,Y) = 0,$$

which implies, by using (3.6),

$$S(Y,V) = 2\beta^2 g(Y,V).$$

This completes the proof.

REMARKS. It is obvious that by the formula (3.3) the conditions (i)-(iii) in Theorem 5.1 can be replaced by the following conditions:
(i′) M is of constant curvature;
(ii′) M is locally symmetric ($\nabla R = 0$);
(iii′) M is semisymmetric.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI
KALYANI-741235, INDIA
E-mail: ucde@klyuniv.ernet.in

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