DEMONSTRATIO MATHEMATICA
Vol. XXXVII No 3 2004

Mariusz Łapiński, Krzysztof Prażmowski

# ON SET-THEORETIC AND CYCLIC REPRESENTATION OF THE STRUCTURE OF BARYCENTRES

**Abstract.** In the paper certain abstract combinatorial structures are studied as representations of the structure of barycentres of all the subsimplices of a given simplex in an arbitrary Desarguesian affine space. General properties of the configuration of barycentres are characterized in terms of those combinatorial structures. Most essential parameters of these structures are established and relevant automorphism groups are characterized.

## 1. Introduction

The barycentre of a segment is just the midpoint of this segment. It is known (cf. [1], [6], [2]) that barycentre of a triangle can be obtained as the intersection of its medians, i.e. of lines which join vertices of the triangle with barycentres of its opposite sides. Continuing this procedure we can find the barycentre of a given simplex and barycentres of all its subsimplices. Many lines are drawn to achieve this, and it may turn out that some of them are parallel. It may be interesting to figure out what is the abstract schema of the configuration which arises in this way, i.e. of the configuration which is formed by the family of all barycentres of all the subsimplices of a given simplex in an arbitrary Desarguesian affine space.

In the paper we define several combinatorial structures which characterize such configurations. The first of them, denoted by $\mathfrak{B}(N)$ ($N$ is any set of cardinality $n < \infty$) determines all the necessarily collinear triples of points in the family $\mathcal{B}(\Delta)$ of all barycentres of all subsimplices of an $n$-simplex $\Delta$ (cf. 2.1). The next two configurations, $\mathfrak{D}(N)$ and $\mathfrak{C}(N)$ are obtained by adding to every family of necessarily parallel lines formed from points in $\mathcal{B}(\Delta)$ (cf. 2.3) their common direction as a new point. The last configuration $\mathfrak{H}(N)$ characterizes the projective structure of added directions. All the basic numerical parameters of the configurations defined in the paper (the number of points, the number of lines, and ranks of points and lines,

*Key words and phrases*: barycentre, partial linear space.

cf. [4]) are determined in section 3. Finally, for configurations $\mathfrak{B}(N)$ and $\mathfrak{D}(N)$ we find their automorphism groups (propositions 4.4 and 4.5). It is evident that a permutation of vertices of a simplex $\Delta$ determines a bijection of $\mathcal{B}(\Delta)$ which preserves both collinearity and parallelity. From our abstract point of view this means that any permutation of the set $N$ determines an automorphism of every structure defined in the paper. However, it turned out that in two cases presented in the paper corresponding automorphism groups of configurations are greater and contain also automorphisms which cannot be associated with any affine automorphism of a given simplex. We close the paper with some comments and (open) problems, mainly concerning projective embeddings and representations of the structures defined in this paper.

## 2. Basic constructions

Let $\mathcal{A} = \{a_1, \dots, a_n\}$ be a set of points of an affine space $\mathfrak{A}$, represented in a vector space $\mathbb{V}$ with the coordinate field $\mathfrak{F}$. Recall that if $char(\mathfrak{F})\not| k$ then for every $k$-set $A = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\} =: N$ the barycentre $\mathbf{B}_{\mathfrak{A}}(a_{i_1}, \dots, a_{i_k}) = \mathbf{B}(a_{i_1}, \dots, a_{i_k})$ of the set $\{a_{i_1}, \dots, a_{i_k}\}$ is defined by the formula $\mathbf{B}(a_{i_1}, \dots, a_{i_k}) := \frac{a_{i_1} + \ldots + a_{i_k}}{k}$. We simply write $\mathbf{B}(A) = \mathbf{B}_{\mathfrak{A}}(A) = \mathbf{B}(\{a_i \colon i \in A\})$ for $A \subseteq N$. Therefore, if $n < char(\mathfrak{F})$ or $char(\mathfrak{F}) = 0$, then the barycentre $\mathbf{B}_{\mathfrak{A}}(A)$ exists for every system $a_1, \dots, a_n$ of points of $\mathfrak{A}$ and every $A \subset N$. Note that for some set $\mathcal{A}$ of points of an affine space $\mathfrak{A}$ and $A_1, A_2 \subseteq N$ it may happen that $A_1 \neq A_2$ and $\mathbf{B}_{\mathfrak{A}}(A_1) = \mathbf{B}_{\mathfrak{A}}(A_2)$. But

> if $\mathbf{B}_{\mathfrak{A}}(A_1) = \mathbf{B}_{\mathfrak{A}}(A_2)$ for *every* affine space $\mathfrak{A}$ and for *every* system $a_1, \dots, a_n$ of points of $\mathfrak{A}$ ($n < char(\mathfrak{F})$ or $char(\mathfrak{F}) = 0$) then, necessarily, $A_1 = A_2$.

That's why we consider $A \subseteq N$ as an "abstract" barycentre $\mathbf{B}(A)$. In this section we shall demonstrate how to analogously define an "abstract" collinearity and an "abstract" parallelity on the set of "abstract" barycentres.

Let us define on the set $X = \wp(N) \setminus \{\emptyset\}$ of nonempty subsets of $N$ an incidence structure $\mathfrak{B}(N) := \langle X, \mathcal{L}_0 \rangle$, with the family of lines $\mathcal{L}_0$ defined by

$$\mathcal{L}_0 = \{\{B_1, B_2, B_3\} \colon B_1, B_2, B_3 \in X, B_1 = B_2 \cup B_3, B_2 \cap B_3 = \emptyset\}.$$

Clearly, $\mathfrak{B}(N)$ is a partial linear space. Further properties of $\mathfrak{B}(N)$ are given in the sequence of the following lemmas.

LEMMA 2.1. *Let $B_1, B_2, B_3$ be three pairwise distinct nonempty subsets of $N$. The following conditions are equivalent:*

(i) *For* any *family $\{a_1, \dots, a_n\}$ of points of an affine space $\mathfrak{A}$, the points $\mathbf{B}(B_1)$, $\mathbf{B}(B_2)$, and $\mathbf{B}(B_3)$ are collinear in $\mathfrak{A}$;*

(ii) *$B_1, B_2, B_3$ are collinear in $\mathfrak{B}(N)$.*

Proof. Set $k_i = |B_i|$ for $i = 1, 2, 3$; clearly, $k_i > 0$. Assume that (ii) holds true. Without loss of generality we can write $B_1 = B_2 \cup B_3$, $B_2 \cap B_3 = \emptyset$. Then $k_1 = k_2 + k_3$ and $\mathbf{B}(B_i) = \frac{1}{k_i}\sum_{j\in B_i} a_j$ for $i = 1, 2, 3$. Thus $\mathbf{B}(B_1) = \frac{k_2}{k_1}\frac{1}{k_2}\sum_{j\in B_2} a_j + \frac{k_3}{k_1}\frac{1}{k_3}\sum_{j\in B_3} a_j = \frac{k_2}{k_1}\mathbf{B}(B_2) + (1-\frac{k_2}{k_1})\mathbf{B}(B_3)$, which proves (i).

Now, assume that (i) holds and $B_1, B_2, B_3$ are pairwise distinct. Let $\mathbf{B}(B_1) = \alpha \cdot \mathbf{B}(B_2) + (1-\alpha) \cdot \mathbf{B}(B_3)$ i.e.

$$\sum_{j\in B_1} \frac{1}{k_1} a_j = \sum_{j\in B_2} \frac{\alpha}{k_2} a_j + \sum_{j\in B_3} \frac{1-\alpha}{k_3} a_j.$$

One can take points $a_j$ such that they are linearly independent. If there was $j_0$ in exactly one of the sets $B_1, B_2, B_3$ then we obtain either $\alpha a_{j_0} = 0$ or $(1-\alpha)a_{j_0} = 0$, so $\alpha = 0$ or $\alpha = 1$, which contradicts assumptions.

Now, suppose that there is $j_0 \in B_1 \cap B_2 \cap B_3$. This yields $\frac{1}{k_1}a_{j_0} = \frac{\alpha}{k_2}a_{j_0} + \frac{1-\alpha}{k_3}a_{j_0}$, so $\frac{1}{k_1} = \frac{\alpha}{k_2} + \frac{1-\alpha}{k_3}$, which gives $\alpha k_1(k_3 - k_2) = k_2(k_3 - k_1)$. If there is $j_1 \in B_2 \cap B_3 \setminus B_1$ then $\frac{\alpha}{k_2} + \frac{1-\alpha}{k_3} = 0$, which yields, contradictory, $\frac{1}{k_1} = 0$. If $j_3 \in B_1 \cap B_2 \setminus B_3$ then $\frac{1}{k_1} = \frac{\alpha}{k_2}$, so $\frac{1-\alpha}{k_3} = 0$, which is impossible.

Finally, if $j_2 \in B_1 \cap B_3 \setminus B_2$ then $\frac{1}{k_1} = \frac{1-\alpha}{k_3}$, which leads to $\frac{\alpha}{k_2} = 0$. Therefore, if there is $j_0$ as above then $B_1 = B_2 = B_3$. Hence we conclude with $B_1 \cap B_2 \cap B_3 = \emptyset$.

Thus every $j \in B_1 \cup B_2 \cup B_3$ belongs to exactly two of the sets $B_1, B_2, B_3$. Set $A_{i_1} = B_{i_2} \cap B_{i_3}$ for all $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. Suppose that $A_1, A_2, A_3 \neq \emptyset$. As above, considering suitable $j_i \in A_i$, we obtain the following system of equations

$$\begin{aligned}
&(1) \qquad & \alpha(k_3 - k_2) &= -k_2 && \text{with } j_1 \in A_1,\\
&(2) \qquad & \alpha k_1 &= k_1 - k_3 \neq 0 && \text{with } j_2 \in A_2,\\
&(3) \qquad & \alpha k_1 &= k_2 && \text{with } j_3 \in A_3.
\end{aligned}$$

Then we calculate $\alpha(k_3 - k_2) = -\alpha k_1$ by (1) and (3), so $k_3 - k_2 = -k_1$, and $k_2 = k_1 - k_3$ by (2) and (3). This gives $k_3 = 0$, which cannot happen. Therefore, there is $i_1$ with $A_{i_1} = \emptyset$, and then $B_{i_1} = B_{i_2} \cup B_{i_3}$, where $\{i_1, i_2, i_3\} = \{1, 2, 3\}$, as required. □

For given $D'_1, D'_2, D''_1, D''_2 \in X$ we define:
$D'_1D'_2 \simeq D''_1D''_2$ iff there are $D_1, D_2 \in X$ and $D', D'' \subset N$ with $D_1 \cap D_2 = \emptyset$, $|D_1| = |D_2|$ such that $D'_i = D_i \cup D'$, $D''_i = D_i \cup D''$, and $D', D''$ are disjoint with $D_i$.

Lemma 2.2. *Let $D'_1, D'_2, D''_1, D''_2 \in X$.*

(i) *$D'_1, D'_2 \simeq D''_1D''_2$ iff $D'_1, D'_2 \simeq D''_2D''_1$, iff $D''_1, D''_2 \simeq D'_1D'_2$.*
(ii) *$D'_1, D'_2 \simeq D''_1D''_2$ iff $D'_1 \setminus D'_2 = D''_1 \setminus D''_2$ and $D'_2 \setminus D'_1 = D''_2 \setminus D''_1$.*

(iii) *If* $D'_1, D'_2 \simeq D''_1 D''_2$ *then for any family* $\{a_1, \ldots, a_n\}$ *of points of an affine space* $\mathfrak{A}$ *the vectors* $\overrightarrow{\mathbf{B}(D'_1)\mathbf{B}(D'_2)}$ *and* $\overrightarrow{\mathbf{B}(D''_1)\mathbf{B}(D''_2)}$ *are parallel in* $\mathfrak{A}$.

Proof. The statements (i) and (ii) are evident. To prove (iii) assume that $D'_1, D'_2 \simeq D''_1 D''_2$; let $D_i, D', D''$ be taken in accordance with definition. Set $k = |D_1| = |D_2|$, $k' = |D'|$, and $k'' = |D''|$. Then $|D'_i| = k' + k$ and $\overrightarrow{\mathbf{B}(D'_1)\mathbf{B}(D'_2)} = \frac{1}{k'+k}\sum_{j\in D'_2} a_j - \frac{1}{k'+k}\sum_{j\in D'_1} a_j = \frac{1}{k'+k}(\sum_{j\in D_2} a_j - \sum_{j\in D_1} a_j)$ and, analogously, $\overrightarrow{\mathbf{B}(D''_2)\mathbf{B}(D''_1)} = \frac{1}{k''+k}(\sum_{j\in D_2} a_j - \sum_{j\in D_1} a_j)$. This yields our claim. □

LEMMA 2.3. *Let* $B'_1, B'_2, B''_1, B''_2 \in X$. *Assume that* $B'_1 \neq B'_2$ *and* $B''_1 \neq B''_2$. *The following conditions are equivalent:*

(i) *For any family* $\{a_1, \ldots, a_n\}$ *of points of an affine space* $\mathfrak{A}$ *the vectors* $\overrightarrow{\mathbf{B}(B'_1)\mathbf{B}(B'_2)}$ *and* $\overrightarrow{\mathbf{B}(B''_1)\mathbf{B}(B''_2)}$ *are parallel in* $\mathfrak{A}$.

(ii) *There are* $D'_1, D'_2, D''_1, D''_2 \in X$ *such that* $D'_1, D'_2 \simeq D''_1 D''_2$, $D'_1 \neq D'_2$, $D''_1 \neq D''_2$. $B'_1, B'_2, D'_1, D'_2$ *are on a line in* $\mathcal{L}_0$, *and* $B''_1, B''_2, D''_1, D''_2$ *are on a line in* $\mathcal{L}_0$.

Proof. The implication (ii)⇒(i) follows immediately from 2.2(iii) and 2.1.

Now we assume that (i) holds; let $\alpha \cdot \overrightarrow{\mathbf{B}(B'_1)\mathbf{B}(B'_2)} = \overrightarrow{\mathbf{B}(B''_2)\mathbf{B}(B''_1)}$ with $\alpha \neq 0$. Let us set $k'_i = |B'_i|$, $k''_i = |B''_i|$, $B' = B'_1 \cap B'_2$, $B'' = B''_1 \cap B''_2$, $A'_i = B'_i \setminus B'$, and $A''_i = B''_i \setminus B''$. Note that

$$\overrightarrow{\mathbf{B}(B'_1)\mathbf{B}(B'_2)} = \sum_{j\in A'_2}\frac{1}{k'_2}a_j - \sum_{j\in A'_1}\frac{1}{k'_1}a_j + \sum_{j\in B'}(\frac{1}{k'_2} - \frac{1}{k'_1})a_j \quad \text{and}$$

$$\overrightarrow{\mathbf{B}(B''_1)\mathbf{B}(B''_2)} = \sum_{j\in A''_2}\frac{1}{k''_2}a_j - \sum_{j\in A''_1}\frac{1}{k''_1}a_j + \sum_{j\in B''}(\frac{1}{k''_2} - \frac{1}{k''_1})a_j.$$

We have $\sum_{j\in B'_2}\frac{\alpha}{k'_2}a_j - \sum_{j\in B'_1}\frac{\alpha}{k'_1}a_j = \sum_{j\in B''_2}\frac{1}{k''_2}a_j - \sum_{j\in B''_1}\frac{1}{k''_1}a_j$, i.e.

$$(4)\qquad \sum_{j\in A'_2}\frac{\alpha}{k'_2}a_j - \sum_{j\in A'_1}\frac{\alpha}{k'_1}a_j + \sum_{j\in B'}(\frac{\alpha}{k'_2} - \frac{\alpha}{k'_1})a_j = $$

$$= \sum_{j\in A''_2}\frac{1}{k''_2}a_j - \sum_{j\in A''_1}\frac{1}{k''_1}a_j + \sum_{j\in B''}(\frac{1}{k''_2} - \frac{1}{k''_1})a_j.$$

As in the proof of 2.1, we can assume that the points $a_i$ are linearly independent vectors of a real vector space. Without loss of generality we can take $\alpha > 0$. Note that $A^t_i = \emptyset = A^t_{3-i}$ yields $B^t_1 = B^t_2$ for $t = {}'\,,{}''$ so, under our assumptions, $A^t_j \neq \emptyset$ for some $j = 1, 2$.

For simplicity we write $t$ for $'\,,''$, and we use $t$ as a number with $3 - {}' = {}''$, $3 - {}'' = {}'$. We set $\alpha' = \alpha$ and $\alpha'' = 1$. Then coefficients of $a_j$ in the equation (4) are

$$\begin{cases} (-1)^s \frac{\alpha^t}{k^t_s} & \text{for } j \in A^t_s \\ \frac{\alpha^t}{k^t_2} - \frac{\alpha^t}{k^t_1} & \text{for } j \in B^t. \end{cases}$$

Let us begin with some simple but useful facts (the proof of each of them is ended with $\triangle$). Note that every $a_j$ which occurs with a non zero coefficient on one side of the equation (4), must occur on the second side of this equation.

(i) Comparing signs of coefficients in the equation (4) we see that $A^t_s \cap A^{3-t}_{3-s} = \emptyset$ for $t = {}', {}''$ and $s = 1, 2$.

(ii) There are no $i_1, i_2 \in N$ such that $i_s \in A^t_s \cap B^{3-t}$ for $s = 1, 2$ and some $t = {}', {}''$.

Proof. If there were $i_1, i_2$ as above, we would have both $\frac{\alpha^{3-t}}{k^{3-t}_2} - \frac{\alpha^{3-t}}{k^{3-t}_1} = \frac{\alpha^t}{k^t_2} > 0$ and $\frac{\alpha^{3-t}}{k^{3-t}_2} - \frac{\alpha^{3-t}}{k^{3-t}_1} = -\frac{\alpha^t}{k^t_1} < 0$. $\triangle$

(iii) There are no $i, j \in N$ such that $i \in A^t_s \cap A^{3-t}_s$ and $j \in A^t_s \cap B^{3-t}$ for some $t = {}', {}''$ and $s = 1, 2$.

Proof. Assume contrary, then $\frac{\alpha^t}{k^t_s} = \frac{\alpha^{3-t}}{k^{3-t}_s}$ and $(-1)^s \cdot \frac{\alpha^t}{k^t_s} = \frac{\alpha^{3-t}}{k^{3-t}_2} - \frac{\alpha^{3-t}}{k^{3-t}_1}$, which gives contradictory $\frac{\alpha^{3-t}}{k^{3-t}_{3-s}} = 0$. $\triangle$

(iv) There are no $i', i'' \in N$ such that $i' \in A'_s \cap B''$ and $i'' \in A''_s \cap B'$ for some $s = 1, 2$.

Proof. Assume the contrary for e.g. $s = 2$. Then $\frac{\alpha}{k'_2} = \frac{1}{k''_2} - \frac{1}{k''_1}$ and $\frac{1}{k''_2} = \frac{\alpha}{k'_2} - \frac{\alpha}{k'_1}$. This gives $\frac{\alpha}{k'_1} + \frac{1}{k''_1} = 0$, which is impossible. The reasoning for $s = 1$ is analogous. $\triangle$

(v) There are no $i, j \in N$ with $i \in A^t_s \cap B^{3-t}$ and $j \in B' \cap B''$ for some $t = {}', {}''$ and $s = 1, 2$.

Proof. Suppose that the contrary holds. Then we obtain $\frac{\alpha'}{k'_2} - \frac{\alpha'}{k'_1} = \frac{\alpha''}{k''_2} - \frac{\alpha''}{k''_1}$ and $(-1)^s \frac{\alpha^t}{k^t_s} = \frac{\alpha^{3-t}}{k^{3-t}_2} - \frac{\alpha^{3-t}}{k^{3-t}_1}$. And then $\frac{\alpha^t}{k^t_{3-s}} = 0$, which is impossible. $\triangle$

(vi) If $A^t_s \cap A^{3-t}_s \neq \emptyset$ then $A^t_s = A^{3-t}_s$.

Proof. Let $j \in A^t_s \cap A^{3-t}_s$. Take any $i \in A^t_s$. Then $i \notin A^{3-t}_{3-s}$ by (i), and $i \notin B^{3-t}$ by (iii). Thus $i \in A^{3-t}_s$; this gives $A^t_s \subseteq A^{3-t}_s$. Analogously, $A^{3-t}_s \subseteq A^t_s$. $\triangle$

(vii) If $A^t_s \cap B^{3-t} \neq \emptyset$ then $A^{3-t}_s = \emptyset$.

Proof. Let $j \in A^t_s \cap B^{3-t}$. Suppose that there is $i \in A^{3-t}_s$. By (i), $i \notin A^t_{3-s}$, and by (iv), $i \notin B^t$. This yields $i \in A^t_s$, which contradicts (iii). $\triangle$

Then, we pass to the proof of our lemma. There are several cases to consider:

1. We have $A'_s \neq \emptyset$ for some $s$; assume that $s = 2$ and take any $i'_2 \in A'_2$. From (ii), $i'_2 \in A''_2$ or $i'_2 \in B''$.

1.1. Assume that $i'_2 \in A''_2$. From (vi) we obtain $A'_2 = A''_2$.

1.1.1. Suppose that $A'_1 \neq \emptyset$, let $i'_1 \in A'_1$. By (ii), $i'_1 \in A''_1$ or $i'_1 \in B''$.

1.1.1.1. Assume that $i'_1 \in A''_1$. Again, with (vi) we get $A'_1 = A''_1$. Suppose that $B^t \neq \emptyset$ and $\frac{1}{k_2^t} - \frac{1}{k_1^t} \neq 0$. Then by (iii), $B' = B''$, so $B'_s = B''_s$ for $s = 1, 2$.

If $B' = \emptyset = B''$ then, again, $B'_s = B''_s$ for $s = 1, 2$. Finally, if $B' \neq \emptyset$ and $\frac{1}{k_2^t} = \frac{1}{k_1^t}$ then $k'_1 = k'_2$, so $|A'_1| = |A'_2|$ and thus $B'_1 B'_2 \simeq B''_1 B''_2$.

1.1.1.2. Assume that $i'_1 \in B''$. From (vii), $A''_1 = \emptyset$, so $A'_1 = B'' = B''_1$ and $B''_1, B''_2, A''_2$ are collinear. Moreover, from (v), $B' \cap B'' = \emptyset$. This yields two possible solutions.
If $B' = \emptyset$, then $A'_2 A'_1 = A''_2 B''_1$. If $B' \neq \emptyset$ but $\frac{1}{k'_1} = \frac{1}{k'_2}$, then $|A'_1| = |A'_2|$ and thus $B'_1 B'_2 \simeq A'_1 A'_2 \simeq A''_2 B''_1$.

1.1.2. Analogously, the claim of the lemma holds if we assume that $A''_1 \neq \emptyset$.

1.1.3. Assume that $A'_1 = \emptyset = A''_1$. Then $B_1^t = B^t$ and $k_1^t \neq k_2^t$ for $t = ','' $. Thus $B' = B''$ and $B'_1 B'_2 = B''_1 B''_2$.

1.2. Assume that $i'_2 \in B''$. From (vii) we infer $A''_2 = \emptyset$, and from (v), $B' \cap B'' = \emptyset$; thus $A'_2 = B''$. Moreover, $A''_1 \neq \emptyset$. If $A''_1 \cap A'_1 \neq \emptyset$ we get the case analogous to 1.1.1.2. Assume that there is $j \in A''_1 \cap B'$. From (vii) we get $A'_1 = \emptyset$, and $A''_1 = B'$. In this case the points $B'_1 = B' = A''_1, B'_2, B''_1, B''_2 = B'' = A'_2$ are collinear. □

For $B_1, B_2 \in X$ we write $B_1 \sim B_2$ if $B_1 \cap B_2 = \emptyset$ and $|B_1| = |B_2|$. To every set $\{B_1, B_2\}$ such that $B_1 \sim B_2$ we assign a new abstract point $(B_1, B_2)^\infty$ and we define a new class of lines

$$\mathcal{L}_1 = \{\{B_1 \cup B, B_2 \cup B, (B_1, B_2)^\infty\} : B_1, B_2 \in X, B_1 \sim B_2, B \subset (N \setminus (B_1 \cup B_2))\}.$$

The set of all those new abstract points will be denoted by $\mathcal{D}$, formally

$$\mathcal{D} = \{(B_1, B_2)^\infty : B_1, B_2 \in X,\ B_1 \sim B_2\}.$$

Let $\mathfrak{D}(N) := \langle X \cup \mathcal{D}, \mathcal{L}_1 \rangle$. Clearly, $\mathfrak{D}(N)$ is a partial linear space.

LEMMA 2.4. *Two distinct points $B_1, B_2$ of $X$ belong to a line $\overline{B_1, B_2}$ in $\mathcal{L}_1$ iff $|B_1| = |B_2|$. If $|B_1| = |B_2|$ and $B_1 \neq B_2$ then the improper point $\overline{B_1, B_2}^\infty$ of $\overline{B_1, B_2}$ is $(B_1 \setminus B_2, B_2 \setminus B_1)^\infty$.*

Proof. Evidently, if $B_i = B'_i \cup B$, $B'_1 \sim B'_2$, and $B \cap (B'_1 \cup B'_2) = \emptyset$ then $|B_1| = |B_2|$.

Conversely, if $|B_1| = |B_2|$ we set $B := B_1 \cap B_2$, $B'_i := B_i \setminus B$ to get $B'_1 \sim B'_2$, and then the points $B_1$, $B_2$, and $(B'_1, B'_2)^\infty$ are on a line of $\mathcal{L}_1$. □

## 3. Combinatorial representation of the structure of barycentres and its numerical parameters

In this section we shall establish the most essential parameters of the structures $\mathfrak{B}(N)$, and $\mathfrak{D}(N)$, as well as some others derived from these.

Recall that the structures $\mathfrak{B}(N)$ and $\mathfrak{D}(N)$ are partial linear spaces. As usual we use the symbol $\overline{B_1, B_2}$ ($B_1 \neq B_2$) to denote the line which joins the points $B_1$ and $B_2$ in currently considered structure. Recall also that, if $\mathfrak{M} = \langle Z, \mathcal{L} \rangle$ is a partial linear space, then

$$\sum_{z \in Z} r(z) = \sum_{L \in \mathcal{L}} l(L), \tag{5}$$

where $r(z)$ is the rank of $z$ (the number of lines which pass through $z$), and $l(L)$ is the rank of $L$, i.e the number of points on $L$.

Evidently,

LEMMA 3.1. *The number $\nu_0(n)$ of points of the structure $\mathfrak{B}(N)$ is*

$$\nu_0(n) = 2^n - 1.$$

LEMMA 3.2. *The number $\lambda_0(n)$ of lines of the structure $\mathfrak{B}(N)$ is*

$$\lambda_0(n) = \frac{3^n}{2} - 2^n + \frac{1}{2}.$$

Proof. By definition, each line of $\mathfrak{B}(N)$ is of the form $\{B_1, B_2, B_3\}$ for some $B_1, B_2, B_3 \in X$ such that $B_1 = B_2 \cup B_3$ and $B_2 \cap B_3 = \emptyset$. Then $i = |B_1| \geq 2$ and the set $B_1$ can be chosen in $\binom{n}{i}$ ways. For a given $B_1$, a set $B_2$ is an arbitrary nonempty proper subset of $B_1$, so it can be chosen in $2^i - 2$ ways, and $B_3 = B_1 \setminus B_2$. Following this way, the pair $(B_2, B_3)$ is counted twice, so for a given $B_1$ we have $\frac{2^i - 2}{2} = 2^{i-1} - 1$ possibilities to get a line of the above form.

Finally, we get $\lambda_0(n) = \sum_{i=2}^{n} \binom{n}{i} \cdot (2^{i-1} - 1) = \frac{3^n}{2} - 2^n + \frac{1}{2}$, as required. □

Clearly, each line of $\mathfrak{B}(N)$ is of the same rank $l_0(n) = l_0 = 3$.

LEMMA 3.3. *If $B \in X$ and $|B| = k$, then the rank of the point $B$ in $\mathfrak{B}(N)$ is*

$$r_0(B) = r_0(n, k) = 2^{k-1} + 2^{n-k} - 2.$$

*If $B_1, B_2 \in X$ have the same rank, then $|B_1| + |B_2| = n + 1$ or $|B_1| = |B_2|$.*

Proof. Let $B \in L \in \mathcal{L}_0$. Then two possibilities occur.

(i) $L = \{B, B_1, B_2\}$ with $B = B_1 \cup B_2$. A set $B_1$ is an arbitrary but nonempty and proper subset of $B$, so – as in 3.2 – we have $\frac{2^k - 2}{2} = 2^{k-1} - 1$ lines of this form which contain $B$.

(ii) $L = \{B, B_1, B_2\}$, where $B_2 = B \cup B_1$ and $B_1 \subset N \setminus B$. A set $B_1$ can be chosen in $2^{n-k} - 1$ ways, since it is arbitrary, but nonempty.

Thus we obtain the required formula for $r_0(B)$.

Now, if $r_0(n, k) = r_0(n, i)$ from the formula proved above we get

$$(2^k x - 2^{n+1})(x - 2^k) = 0 \text{ with } x = 2^i.$$

The only solutions of this equation are $x = 2^k$ and $x = 2^{n+1-k}$, which proves our claim. □

LEMMA 3.4. *The number $\lambda_1(n)$ of lines of $\mathfrak{D}(N)$ is*

$$\lambda_1(n) = \frac{1}{2}\binom{2n}{n} - 2^{n-1}.$$

Proof. By 2.4, each line in $\mathcal{L}_1$ is uniquely determined by a pair of distinct subsets of $N$ with the same cardinality. Thus

$$\lambda_1(n) = \frac{1}{2}\sum_{i=1}^{n-1}\left[\binom{n}{i}^2 - \binom{n}{i}\right] = \frac{1}{2}\left[\left(\binom{2n}{n} - 2\right) - (2^n - 2)\right],$$

as required. □

LEMMA 3.5. *The number $\delta_1(n)$ of "directions" (the cardinality of the set $\mathcal{D}$) is*

$$\delta_1(n) = \frac{1}{2}\sum_{i=1}^{E(\frac{n}{2})}\left[\binom{n}{i}\cdot\binom{n-i}{i}\right].$$

Proof. Each direction $(B_1, B_2)^\infty$ is uniquely determined by two disjoint subsets $B_1, B_2$ of $X$ with the same cardinality. Let $B_1 = i$; then $2i \leq n$, so $i \leq E(\frac{n}{2})$. A set $B_1$ can be chosen in $\binom{n}{i}$ ways, and then $B_2$, as a subset of $N \setminus B_1$, can be chosen in $\binom{n-i}{i}$ ways. A pair $B_1, B_2$ is counted twice, which yields our formula. □

As an immediate consequence of 3.1 and 3.5 we obtain a formula for the number $\nu_1(n)$ of points of $\mathfrak{D}(N)$:

$$\nu_1(n) = \nu_0(n) + \delta_1(n) = 2^n - 1 + \frac{1}{2}\sum_{i=1}^{E(\frac{n}{2})}\left[\binom{n}{i}\cdot\binom{n-i}{i}\right].$$

By definition, every line of $\mathfrak{D}(N)$ has $l_1(n) = 3$ points.

LEMMA 3.6. *Let $|B_1| = k$. The number of lines of $\mathfrak{D}(N)$ which pass through $B_1$ is*

$$r_1(n, k) = \binom{n}{k} - 1.$$

*If* $B_1 \sim B_2$ *then the number of lines of* $\mathfrak{D}(N)$ *which pass through* $q = (B_1, B_2)^\infty$ *is*

$$r_q = r_1^\infty(n,k) = 2^{n-2k}.$$

Proof. Each line $L$ through $q$ is of the form $L = \{B_1', B_2', q\}$, where $B_i' = B_i \cup B$ for some $B \subset (N \setminus (B_1 \cup B_2))$. For given $B_1, B_2$, we obtain distinct lines $L$ with distinct sets $B$ chosen as above. This yields the formula for $r_q$. The formula for $r_1(n,k)$ follows by 2.4. □

The straightforward consequence of the above is

COROLLARY 3.7. *From* (5) *and* 3.3 *we obtain*

$$3 \cdot \lambda_0(n) = \sum_{i=1}^{n} \binom{n}{i} r_0(n,i) = \sum_{i=1}^{n} \binom{n}{i} (2^{k-1} + 2^{n-k} - 2).$$

*Since every line of* $\mathfrak{D}(N)$ *has exactly one point in* $\mathcal{D}$ *and exactly two points in* $X$, *from* 3.6 *and* (5) *we obtain*

$$\lambda_1(n) = \frac{1}{2} \sum_{i=1}^{E(\frac{n}{2})} \binom{n}{i} \binom{n-i}{i} r_1^\infty(n,i) = \frac{1}{2} \sum_{i=1}^{E(\frac{n}{2})} \binom{n}{i} \binom{n-i}{i} (2^{n-2i}),$$

$$2 \cdot \lambda_1(n) = \sum_{i=1}^{n} \binom{n}{i} r_1(n,i) = \sum_{i=1}^{n-1} \binom{n}{i} \left( \binom{n}{i} - 1 \right).$$

Now, we are going to deal with the most interesting structure investigated in this section. Note that, formally, we cannot use the family $\mathcal{L}_0 \cup \mathcal{L}_1$ as a family of lines. Thus the lines of the structure $\mathfrak{C}(N)$, which we define now, are of three types. First, we distinguish the set $\mathcal{S}$ of all pairs $B_1, B_2$ such that they are collinear in both $\mathfrak{B}(N)$ and $\mathfrak{D}(N)$.

LEMMA 3.8. *Let* $B_1, B_2$ *be two distinct points in* $X$. *The following conditions are equivalent:*

(i) *There are lines* $K' \in \mathcal{L}_0$, $K'' \in \mathcal{L}_1$ *such that* $B_1, B_2 \in K', K''$.
(ii) $B_1 \sim B_2$.

*If the condition* (i) *is satisfied and* $K' = \{B_1, B_2, B_3\}$ *then neither* $B_1, B_3$ *nor* $B_2, B_3$ *are collinear in* $\mathfrak{D}(N)$.

Proof. The implication (ii) $\Longrightarrow$ (i) is evident, just take $K' = \{B_1, B_2, B_1 \cup B_2\}$ and $K'' = \{B_1, B_2, (B_1, B_2)^\infty\}$.

Let (i) hold. By 2.4, $|B_1| = |B_2|$, so neither $B_1 \subset B_2$ nor $B_2 \subset B_1$. Thus $K' = \{B_1, B_2, B_3\}$ with $B_3 = B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$. This yields $B_1 \sim B_2$. Finally, by the above, $|B_3| \neq |B_1| = |B_2|$, which, by 2.4, finishes the proof. □

Let us set

$$\mathcal{L}_0' = \mathcal{L}_0 \setminus \{\{B_1, B_2, B_1 \cup B_2\}\colon\ B_1, B_2 \in X \ \&\ B_1 \sim B_2\}$$
$$\mathcal{L}_1' = \mathcal{L}_1 \setminus \{\{B_1, B_2, (B_1, B_2)^\infty\}\colon\ B_1, B_2 \in X \ \&\ B_1 \sim B_2\}$$
$$\mathcal{L}_2 = \{\{B_1, B_2, B_1 \cup B_2, (B_1, B_2)^\infty\}\colon\ B_1, B_2 \in X \ \&\ B_1 \sim B_2\}.$$

Finally, we define

$$\mathfrak{C}(N) := \langle X \cup \mathcal{D}, \mathcal{L}_0' \cup \mathcal{L}_1' \cup \mathcal{L}_2 \rangle.$$

Clearly, $\mathfrak{C}(N)$ is a partial linear space.

LEMMA 3.9. *The number $\nu_2(n)$ of points of $\mathfrak{C}(N)$ is $\nu_2(n) = \nu_1(n)$. The cardinality of the family $\mathcal{L}_2$ is $\delta_1(n)$. Consequently, the number $\lambda_2(n)$ of lines of $\mathfrak{C}(N)$ is*

$$\lambda_2(n) = \lambda_0(n) + \lambda_1(n) - \delta_1(n)$$
$$= \frac{3^n}{2} - 2^n - 2^{n-1} + \frac{1}{2} + \frac{1}{2}\binom{2n}{n} - \frac{1}{2}\sum_{i=1}^{\mathrm{E}(\frac{n}{2})}\left[\binom{n}{i}\cdot\binom{n-i}{i}\right].$$

Evidently, lines in $\mathcal{L}_0' \cup \mathcal{L}_1'$ have rank 3, and lines in $\mathcal{L}_2$ have rank 4. It is more difficult to determine the rank of a point in $\mathfrak{C}(N)$.

LEMMA 3.10. *Let $B \in X$, $|B| = k$ and let $r_2'(n,k)$ be the number of lines in $\mathcal{L}_2$ through $B$. The rank of $B$ in $\mathfrak{C}(N)$ is*

$$r_2(n,k) = r_0(n,k) + r_1(n,k) - r_2'(n,k).$$

*If $k > \mathrm{E}(\frac{n}{2})$ then $r_2'(n,k) = 0$, and if $k \le \mathrm{E}(\frac{n}{2})$ then $r_2'(n,k) = \binom{n-k}{k}$.*

Proof. To find lines in $\mathcal{L}_2$ through $B$ we must find all the points $B'$ with $B \sim B'$, that is all the $k$-subsets of the set $N \setminus B$. This justifies the formula for $r_2'(n,k)$.

Originally, each such a pair $B, B'$ was used twice: to produce a line in $\mathcal{L}_0$ and a line in $\mathcal{L}_1$. Thus we obtain the required formula for $r_2(n,k)$. □

Finally, we shall build a structure of collinearity on the universe of directions. For this purpose we should first determine triangles of $\mathfrak{B}(N)$, since in any case, elements of a triangle must belong to a plane "spanned" by this triangle.

PROPOSITION 3.11.

(i) *Any three pairwise disjoint nonempty subsets $B_1, B_2, B_3$ of $N$ yield a triangle in $\mathfrak{B}(N)$, denoted by $\triangle\ (B_1, B_2, B_3)$. Set $B_i' := B_j \cup B_s$ for $\{i,j,s\} = \{1,2,3\}$ – these are points on lines which join vertices of this triangle, and $B_0 = B_1 \cup B_2 \cup B_3$ – a diagonal point. Let $\Pi = \Pi(B_1, B_2, B_3) = \{B_1, B_2, B_3, B_1', B_2', B_3', B_0\}$. Then we obtain 15 new triangles in $\mathfrak{B}(N)$ with vertices in $\Pi(B_1, B_2, B_3)$ of the following forms:*

*1:* $\Delta\ (B_i, B_j, B_0)$ *for* $i, j = 1, 2, 3,\ i \neq j$ *(3 triangles);*

*2:* $\Delta\ (B_i, B_j, B_i')$ *for* $i, j = 1, 2, 3,\ i \neq j$ *(6 triangles);*

*3:* $\Delta\ (B_i, B_j', B_0)$ *for* $i, j = 1, 2, 3,\ i \neq j$ *(6 triangles).*

*We write* $\mathcal{T}(B_1, B_2, B_3)$ *for the family of the triangles determined as above.*

(ii) *If* $A_1, A_2, A_3$ *is a triangle in* $\mathfrak{B}(N)$ *then there are pairwise disjoint nonempty subsets* $B_1, B_2, B_3$ *of* $N$ *such that* $\Delta\ (A_1, A_2, A_3) \in \mathcal{T}(B_1, B_2, B_3)$.

Proof. The statement (i) is evident and follows just from the definition of the family $\mathcal{L}_0$. Let $A_1, A_2, A_3$ be a triangle $\Delta$. Thus $A_1, A_2, A_3$ are pairwise collinear in $\mathfrak{B}(N)$. One can see that this can occur only in the following cases. Either

1. $A_i \cap A_j = \emptyset$ for all $1 \leq i < j \leq 3$ – then $\Delta \in \mathcal{T}(A_1, A_2, A_3)$;

or there are $i, j, s$ with $\{i, j, s\} = \{1, 2, 3\}$ such that

2. $A_i \subseteq A_j \subseteq A_s$ – then $\Delta \in \mathcal{T}(A_i, A_j \setminus A_i, A_s \setminus A_j)$; or
3. $A_i, A_j \subseteq A_s$ and $A_i \cap A_j = \emptyset$ – then $\Delta \in \mathcal{T}(A_i, A_j, A_s \setminus (A_i \cup A_j))$; or
4. $A_i \subseteq A_j$ and $A_j \cap A_s = \emptyset$ – then $\Delta \in \mathcal{T}(A_s, A_j \setminus A_i, A_i)$.

This proves the claim. □

In view of the above a *plane* of $\mathfrak{B}(N)$ is a set $\Pi(B_1, B_2, B_3)$ (cf. 3.11(i)), determined by a triple $B_1, B_2, B_3$ of pairwise disjoint elements of $X$. For an arbitrary plane $P$ we set

$$P^{\infty} = \{(D_1, D_2)^{\infty} \colon D_1, D_2 \simeq A_1, A_2,\ A_1, A_2 \in P\}.$$

PROPOSITION 3.12. *Let* $B_1, B_2, B_3$ *be pairwise disjoint nonempty subsets of* $N$*, let* $P = \Pi(B_1, B_2, B_3)$*, and let* $\mathcal{L}(P)$ *be the set of lines of* $\mathfrak{C}(N)$ *which join points of* $P$*. The plane* $P$ *is one of the following:*

**3-affine:** $|P^{\infty}| = 3$ *and there are six lines* $K_i, L_i$ $(i = 1, 2, 3)$ *in* $\mathcal{L}(P) \setminus \mathcal{L}_0'$*. They can be paired such that* ${K_i}^{\infty} = {L_i}^{\infty}$*,* $K_i \in \mathcal{L}_1'$*, and* $L_i \in \mathcal{L}_2$*.*

*These are all pairs* $\overline{B_i, B_j}, \overline{B_j', B_i'}$ *with* $1 \leq i < j \leq 3$.

**2-affine:** $|P^{\infty}| = 2$ *and there are exactly three lines* $K_1, L_1, L_2$ *in* $\mathcal{L}(P) \setminus \mathcal{L}_0'$*. We have* $K_1 \in \mathcal{L}_1'$*,* $L_1, L_2 \in \mathcal{L}_2$*, and* ${K_1}^{\infty} = {L_1}^{\infty}$*.*

*Typical form:* $\overline{B_i', B_j'}$, $\overline{B_i, B_j}$, *and* $\overline{B_s, B_s'}$ *for some* $i, j, s$ *with* $\{i, j, s\} = \{1, 2, 3\}$.

**1-affine:** $|P^{\infty}| = 1$ *and there are exactly two lines* $K, L$ *in* $\mathcal{L}(P) \setminus \mathcal{L}_0'$*. We have* $K \in \mathcal{L}_1'$*,* $L \in \mathcal{L}_2$*, and* $K^{\infty} = L^{\infty}$*.*

*Typical form:* $\overline{B_j', B_i'}$ *and* $\overline{B_i, B_j}$ *for some* $i, j$.

**0-affine:** $|P^{\infty}| = 1$ *and there is exactly one line* $K$ *in* $\mathcal{L}(P) \setminus \mathcal{L}_0'$*. Then* $K \in \mathcal{L}_2$*.*

*Typical form:* $\overline{B_i, B_i'}$ *for some* $i$.

**projective:** $P^{\infty} = \emptyset$*, so all the lines from* $\mathcal{L}(P)$ *are in* $\mathcal{L}_0'$*.*

*Consequently,* $|P^\infty| \leq 3$. *Moreover, in notation of* 3.11(i): $\overline{B'_i, B'_j} \in \mathcal{L}'_0 \cup \mathcal{L}'_1$, *and* $\overline{B_i, B_j}, \overline{B_i, B'_i} \in \mathcal{L}'_0 \cup \mathcal{L}_2$,

Proof. By 2.4, a pair $A_1, A_2$ yields a direction iff $|A_1| = |A_2|$. Set $n_i = |B_i|$, then, in notation of 3.11(i), $|B'_i| = n_j + n_s$, $|B_0| = n_1 + n_2 + n_3$. Clearly, $|B_j| = n_j < |B'_i|$ for $i \neq j$. If $|B_i| = |B_j|$ then $B_i, B_j \simeq B'_i, B'_j$. But then $|B_i| \neq |B'_i|$. If $|B'_i| = |B'_j|$ then $|B_i| = |B_j|$, so we have the situation as before. If $|B_i| = |B'_i|$ then (note: $B_i \cap B'_i = \emptyset$) there is no pair $A_1, A_2 \in \Pi(B_1, B_2, B_3)$ with $B_i, B'_i \simeq A_1, A_2$. Finally, we see that only the following can occur:

1. $n_i = n_j \neq n_s \neq n_i + n_j$ – then $B_i, B_j$ yield a sole direction in $P$, which has lines $\overline{B_i, B_j}$ and $\overline{B'_i, B'_j}$ of $\mathfrak{C}(N)$. Then $P$ is 1-affine.
2. $n_1 = n_2 = n_3$ – then $P$ contains three pairs of lines of $\mathfrak{C}(N)$ with three distinct directions: $(B_i, B_j)^\infty$ with $1 \leq i < j \leq 3$. Then $P$ is 3-affine.
3. $n_i = n_j \neq n_s = n_i + n_j$ – then $B_i, B_j$ yield a line $\overline{B'_i, B'_j}$ of $\mathfrak{C}(N)$ parallel to $\overline{B_i, B_j}$, and points $B_s, B'_s$ (collinear in $\mathfrak{B}(N)$) yield a line $\overline{B_s, B'_s}$ of $\mathfrak{C}(N)$, which is not parallel to any other line contained in $P$. Then $P$ is 2-affine.
4. $n_s = n_i + n_j$ and $n_i \neq n_j$ – then the only direction, which is contained in $P$ is $(B_s, B'_s)^\infty$, and it is a direction of just one pair of points contained in $P$. Now $P$ is 0-affine.
5. $\neq (n_1, n_2, n_3, n_1 + n_2, n_2 + n_3, n_3 + n_1)$. Then $P$ has no pair of points which yield a direction.

This justifies our claim. □

LEMMA 3.13. *If* $P', P''$ *are two distinct planes of* $\mathfrak{C}(N)$, *then* $|P'^\infty \cap P''^\infty| \leq 1$.

Proof. Let $P' = \Pi(B'_1, B'_2, B'_3)$, $P'' = \Pi(B''_1, B''_2, B''_3)$. Assume that $|P'^\infty \cap P''^\infty| \geq 2$; then without loss of generality, we can assume that $P^t$ is $i$-affine for $i \geq 2$ and $t = {}', {}''$.

A direction $\mathfrak{b}$ of a plane $P = \Pi(B_1, B_2, B_3)$ is called *side direction* if it is a direction of a side of the triangle $B_1, B_2, B_3$; we call $\mathfrak{b}$ a *medial direction* if it is a direction of a line of the form $\overline{B_i \cup B_j, B_s}$. Note that if two planes have a common side direction, they have in common the corresponding side.

Consequently, if both $P'$ and $P''$ are 3-affine then $P' = P''$.

Let $P', P''$ be 2-affine. In accordance with 3.12, $P^{t\infty}$ has one side direction $\mathfrak{s}^t$ and one medial direction $\mathfrak{m}^t$ for $t = {}', {}''$. Assume, first, that $\mathfrak{s}' = \mathfrak{s}''$, so $\mathfrak{m}' = \mathfrak{m}''$. Then the triangles which determine corresponding planes have a common side; say $B'_1 = B''_1$ and $B'_3 = B''_3$, and then $B'_1 \cup B'_3 = B''_1 \cup B''_3$. Therefore $\mathfrak{m}' = (B'_2, B'_1 \cup B'_3)^\infty = (B''_2, B''_1 \cup B''_3)^\infty = \mathfrak{m}''$, which gives $B'_2 = B''_2$ and thus $P' = P''$.

Now, suppose that $\mathfrak{s}' = \mathfrak{m}''$ and $\mathfrak{m}' = \mathfrak{s}''$. Say, $\mathfrak{m}'' = (B''_2, B''_1 \cup B''_3)^\infty = (B'_1, B'_2)^\infty = \mathfrak{s}'$; then $B'_2 = B''_2$ (or $B''_2 = B'_1$, which gives the same result) and $B_1' = B_1'' \cup B_3''$. Further, $\mathfrak{s}'' = (B_1'', B_3'')^\infty = (B_3', B_1' \cup B_2')^\infty = \mathfrak{m}'$ yields $B_3' = B_3''$ (or $B_3' = B_1''$), $B_1'' = B_1' \cup B_2'$ , and $\overline{B_1', B_3'} = \overline{B_1'', B_3''}$. This gives, contradictory, $\overline{B_1', B_2'} = \overline{B_1', B_3'}$.

Finally, suppose that $P'$ is 3-affine and $P''$ is 2-affine. Then, since $P''$ has one side direction, triangles which determine $P'$ and $P''$ have one side in common, say $B_1' = B_1''$ and $B_3' = B_3''$. Then $(B_2'', B_1'' \cup B_3'')^\infty$ is a side direction in $P'$, say $(B_2'', B_1'' \cup B_3'')^\infty = (B_2', B_3')^\infty$. Since $B_1'' \cup B_3'' = B_1' \cup B_3' \neq B_2', B_3'$ we obtain a contradiction. □

By 3.13, the structure $\mathfrak{H}(N) := \langle \mathcal{D}, \mathcal{L}_3 \rangle$, where

$$\mathcal{L}_3 = \{P^\infty \colon P \text{ an } i\text{-affine plane of } \mathfrak{C}(N),\ i = 3, 2\}$$

is a partial linear space, defined on the set $\mathcal{D}$ of directions. Note that the structure $\langle \mathcal{D}, \{P^\infty \colon \text{ a 3-affine plane of } \mathfrak{C}(N)\} \rangle$ is a partial linear space in which every line has rank 3, and it is just a disjoint union of horizons of Desarguesian closures (cf. [5]) of binomial graphs $\Psi_k^0$ (cf. [3]), i.e. graphs of the relation $\sim$ in $X$.

From definition, the rank of each line of $\mathfrak{H}(N)$ is 2 or 3.

LEMMA 3.14. *Let $k \leq \mathrm{E}(\frac{n}{2})$ and $B_1, B_2 \in X$ with $B_1 \sim B_2$, $|B_1| = k$. Then the rank $r_3(k)$ of the point $(B_1, B_2)^\infty$ in the structure $\mathfrak{H}(N)$ is*

$$r_3(n,k) = r_{3,3}(n,k) + r_{3,2}(n,k),$$

*where $r_{3,i}(n,k)$ is the number of i-affine planes of $\mathfrak{B}(N)$, which contain the line $\overline{B_1, B_2}$, and*

$$r_{3,2}(n,k) = \begin{cases} \binom{n-2k}{2k} & \text{if } 2\not| k \\ \binom{n-2k}{2k} + \binom{k}{\frac{k}{2}} & \text{if } 2|k \end{cases} \quad \text{and } r_{3,3}(n,k) = \binom{n-2k}{k}.$$

*Consequently, $r_{3,3}(k) \neq 0$ iff $k \leq \mathrm{E}(\frac{n}{3})$.*

Proof. Set $\mathfrak{b} = (B_1, B_2)^\infty$. Evidently, if $\mathfrak{b}$ is the direction of a line contained in a 3-affine plane $P$ then $B_1, B_2 \in P$ and $P = \Pi(B_1, B_2, B_3)$ for some $B_3$ such that $|B_3| = k$ and $B_3 \subseteq N \setminus (B_1 \cup B_2)$. This justifies the formula for $r_{3,3}(k)$.

Now, let $\mathfrak{b}$ be the direction of a line contained in a 2-affine plane $P$. Two possibilities appear. Either, $P = \Pi(B_1, B_2, B_3)$ for some $B_3 \subseteq N \setminus (B_1 \cup B_2)$ with $|B_3| = |B_1 \cup B_2|$ – there are $\binom{n-2k}{2k}$ sets $B_3$ with these properties. Or, $P = \Pi(B_i, D_2, D_3)$, where $B_{3-i} = D_2 \cup D_3$ and $|D_2| = |D_3|$ and $i = 1$ or $i = 2$. To find such $D_j$ we must require $2|k$, i.e. $k = 2 \cdot m$ for some $m$. A pair $\{D_1, D_2\}$ for $B_i$ can be chosen in $\frac{1}{2}\binom{2m}{m}$ ways, so there are $\binom{2m}{m}$ triangles of the second form. □

LEMMA 3.15. *The number $\lambda_3(n)$ of lines of $\mathfrak{H}(N)$ is the sum*

$$\lambda_3(n) = \lambda_{3,3}(n) + \lambda_{3,2}(n),$$

*where $\lambda_{3,i}(n)$ is the number of lines of rank $i$, and*

$$\lambda_{3,3}(n) = \frac{1}{6}\sum_{k=1}^{\mathrm{E}(\frac{n}{3})} \binom{n}{k}\binom{n-k}{k}\binom{n-2k}{k},$$

$$\lambda_{3,2}(n) = \frac{1}{2}\sum_{k=1}^{\mathrm{E}(\frac{n}{4})} \binom{n}{k}\binom{n-k}{k}\binom{n-2k}{2k}.$$

Proof. It suffices to note that we have to count all the triples $B_1, B_2, B_3$ such that $P = \Pi(B_1, B_2, B_3)$ exists, and it is a 3-affine or 2-affine plane. A 3-affine $P$ is obtained if $B_i$ are pairwise disjoint and of the same cardinality $k$, so $3k \leq n$. To get 2-affine $P$ we need $k = |B_i| = |B_j|$ and $|B_s| = 2k$; thus $4k \leq n$. □

## 4. Automorphisms of the structure of barycentres

Note that if $\varphi$ is a permutation of the set $N$ then $\varphi$ induces an automorphism $f = \varphi^*$ of $\mathfrak{B}(N)$ by the condition $f(B) = \{\varphi(i) : i \in B\}$ for $B \in X$. The aim of this section is to find all the automorphims of $\mathfrak{B}(N)$ and all the automorphisms of $\mathfrak{D}(N)$.

Let us fix $f \in \mathrm{Aut}(\mathfrak{B}(N))$.

As an immediate consequence of 3.3 we have the following

COROLLARY 4.1. *If $B \in X$ then either $|f(B)| = |B|$ or $|f(B)| = n+1-|B|$.*

LEMMA 4.2. *If an automorphism $f$ fixes $N$, then it is determined by a permutation $\varphi$ of $N$.*

Proof. By 4.1, $f$ preserves the family $\{B \in X : |B| = 1 \text{ or } B = N\}$. Therefore, there is a permutation $\varphi$ of $N$ such that

$$f(\{i\}) = \{\varphi(i)\}$$

for every $i \in N$. This means that

$$f(B) = \{\varphi(i) : i \in B\} = \varphi^*(B) \tag{6}$$

holds for every $B \in X$ with $|B| = 1$. Assume that (6) holds for all $B$ with $|B| < k \leq n$. Let $|B| = k$, we take any $i \in B$ and write $B = A \cup \{i\}$ with $A = B \setminus \{i\}$. By assumption, $f(A) = \varphi^*(A)$. Then $B$ is the third point of the line $K = \overline{A, \{i\}}$ of $\mathfrak{B}(N)$, so $f(B)$ must be the third point of the line $f(K) = \overline{\varphi(A), \{\varphi(i)\}}$. Then, either, $f(K) = \{\varphi(A), \{\varphi(i)\}, \varphi(A) \setminus \{\varphi(i)\}\}$, which is impossible, since $\varphi(i) \notin \varphi(A)$, or $f(K) = \{\varphi(A), \{\varphi(i)\}, \varphi(A) \cup \varphi(\{i\})\}$ and thus $f(B) = \{\varphi(A) \cup \{\varphi(i)\} = \varphi(A \cup \{i\}) = \varphi(B)$. Thus (6) holds for all $B \in X$. □

LEMMA 4.3. *Let $a, d \in N$ and $\varphi$ be an arbitrary bijection of the set $N \setminus \{a\}$ onto $N \setminus \{d\}$. Define*

$$f_0(B) = \begin{cases} \varphi(i) & \text{if } B = \{i\} \text{ for } i \in N, i \neq a, \\ N & \text{if } B = \{a\}, \\ \{d\} & \text{if } B = N. \end{cases}$$

*Then the permutation $f_0$ of the points of $\mathfrak{B}(N)$ of rank $2^{n-1} - 1$ can be extended, in a unique way, to an automorphism $f$ of $\mathfrak{B}(N)$.*

Proof. Set $A = \{a\}$ and $D = \{d\}$ and assume that $f$ is an automorphism of $\mathfrak{B}(N)$ extending the map $f_0$. As in the proof of 4.2, for arbitrary $i, j \in N \setminus \{a\}$ we obtain $f(\{i,j\}) = \{\varphi(i), \varphi(j)\}$ and, generally, $f(B) = \varphi(B)$ for all $B \in X$ with $a \notin B$. In particular, for $A' = N \setminus A$ we have $f(A') = \varphi(A')$.

Let $a \in B$; set $B_a := B \setminus A$. Then $B$ is the third point of the line $\overline{B_a, A}$, and thus $f(B)$ is the third point of the line $\overline{f(B_a), f(A)}$. We have $f(B_a) = \varphi(B_a)$ and $f(A) = N$, thus $f(B) = N \setminus \varphi(B_a)$. Note that $d \in f(B)$ in this case.

We have now an explicit definition of the map $f$:

$$f(B) = \begin{cases} \varphi(B) & \text{if } a \notin B \\ N \setminus \varphi(B \setminus \{a\}) & \text{if } a \in B \end{cases}$$

for all $B \in X$. The aim is to show that $f$ is an automorphism.

Let $K = \{B, B', B''\}$ be a line of $\mathfrak{B}(N)$ with $B = B' \veebar B''$ ($\veebar$ stands for the operation of disjoint union). If $a \notin B$ then $f$ and $\varphi$ coincide on $K$, so $f(B) = f(B') \veebar f(B'')$ and $f(K)$ is a line of $\mathfrak{B}(N)$.
Now, let $a \in B$; thus $a \in B' \setminus B''$ (or $a \in B'' \setminus B'$, resp.). Then $B_a = B'_a \veebar B''$, so $\varphi(B_a) = \varphi(B'_a) \veebar \varphi(B'')$. By elementary set theory, $N \setminus \varphi(B'_a) = \varphi(B'') \veebar (N \setminus \varphi(B'_a))$, which yields $f(B') = f(B'') \veebar f(B')$ and thus $f(K)$ is a line, as required. □

The automorphism $f$ constructed in 4.3 will be denoted by $f = F_{(a,\varphi)}$ (note that $d$ is determined by the condition $d \in N \setminus \varphi(N \setminus \{a\})$). Especially interested are maps $F_{(a)} = F_{(a,id)}$; note that $(F_{(a)})^2 = id$ for every $a \in N$.

Now, a characterization of the group of automorphisms of $\mathfrak{B}(N)$ will be given.

PROPOSITION 4.4. *Let $\mathcal{P} = \{B \in X : B = N \vee |B| = 1\}$.*

(i) *Every permutation $\psi$ of the set $\mathcal{P}$ can be uniquely extended to an automorphism of $\mathfrak{B}(N)$. Therefore,* $\mathrm{Aut}(\mathfrak{B}(N)) \cong S_{n+1}$.

(ii) *If $f \in \mathrm{Aut}(\mathfrak{B}(N))$ then there are permutations $\varphi_1, \varphi_2$ of $N$ and $a_1, a_2 \in N$ such that, either*

- *$f = \varphi_1^*$ is determined by $\varphi_1$, or*

- $f = \varphi_1^* \circ F_{(a_1)} = F_{(a_2)} \circ \varphi_2^*$.

*Thus we can write* $\mathrm{Aut}(\mathfrak{B}(N)) = S_n \circledcirc \mathfrak{S} = \mathfrak{S} \circledcirc S_n$, *where* $\mathfrak{S}$ *is the group generated by* $\{F_{(a)} : a \in N\}$ *and* $\circledcirc$ *is the complex product of two subgroups in* $S_N$.

Proof. Let $\psi$ be a permutation of $\mathcal{P}$. First, assume that $\psi(N) = N$ and set $\varphi := \psi \restriction (\mathcal{P} \setminus \{N\})$. Then $\varphi$ is uniquely extendable to an automorphism $f$ of $\mathfrak{B}(N)$, and $f(N) = N$.

Then, assume that $\psi(N) \neq N$; let $\psi(N) = \{d\}$ and $\psi(\{a\}) = N$, for some $a, d \in N$. Set $\varphi := \psi \restriction (\mathcal{P} \setminus \{a\})$. By 4.3, $\psi$ is uniquely extendable to an automorphism $f$ of $\mathfrak{B}(N)$ with $f(N) = \{d\} = \psi(N)$. This proves (i).

Let $f \in \mathrm{Aut}(\mathfrak{B}(N))$. If $f(N) = N$ then by 4.2, $f$ is determined by a permutation of $N$. Assume that $f(N) \neq N$, let $f(N) = \{a_2\}$ and $f(\{a_1\}) = N$. Then $f_1 = f \circ F_{(a_1)}$ and $f_2 = F_{(a_2)} \circ f$ are automorphism of $\mathfrak{B}(N)$ fixing $N$. By 4.2 we find permutations $\varphi_1, \varphi_2$ of $N$ with $f_i = \varphi_i$ and the proof of (ii) is finished. □

Clearly, every permutation $\varphi$ of $N$ determines an automorphism $\varphi^*$ of the structure $\mathfrak{D}(N)$ as well. Recall that if $L = \{B_1, B_2, \mathfrak{q}\}$ with $\mathfrak{q} = (B_1', B_2')^\infty \in \mathcal{D}$ is a line of $\mathfrak{D}(N)$, then $|B_1| = |B_2|$ and $B_1' = B_i \setminus B_j$, $B_2' = B_j \setminus B_i$ for some $\{i, j\} = \{1, 2\}$ (cf. 2.4). In this case the direction of $f(L)$ is $(f(B_1'), f(B_2'))^\infty$. Note that the boolean complement operation $\kappa \colon X \ni B \mapsto N \setminus B$ yields an (involutory) automorphism of $\mathfrak{D}(N)$ as well. Clearly, if $|B_1| = |B_2|$ then $|\kappa(B_1)| = |\kappa(B_2)|$. Moreover, if $B_1' = B_i \setminus B_j$ and $B_2' = B_j \setminus B_i$ then $B_2' = \kappa(B_i) \setminus \kappa(B_j)$, $B_1' = \kappa(B_j) \setminus \kappa(B_i)$ so, $\kappa$ maps $L$ onto $\{\kappa(B_1), \kappa(B_2), \mathfrak{q}\}$, and $\kappa$ preserves every direction. In the sequel we shall prove that, in some sense, these are all the automorphism of $\mathfrak{D}(N)$.

PROPOSITION 4.5. *Let $f$ be an automorphism of $\mathfrak{D}(N)$. Then $f$ preserves $\mathcal{D}$ and there is a permutation $\varphi$ of $N$ such that for every $k \leq \frac{n}{2}$, either*

*a) $f(B) = \varphi^*(B)$ for all $B \in X$ such that $|B| = k$ or $|B| = n - k$, or*
*b) $f(B) = \kappa\varphi^*(B)$ for all $B \in X$ such that $|B| = k$ or $|B| = n - k$.*

*Consequently,* $\mathrm{Aut}(\mathfrak{D}(N)) \cong (C_2)^{\mathrm{E}(\frac{n}{2})} \oplus S_n$.

Proof. Note that $N$ is the only point of $\mathfrak{D}(N)$ such that no line of $\mathfrak{D}(N)$ passes through it, thus $N$ remains invariant under every automorphism of $\mathfrak{D}(N)$.

A line $L = \{B_1, B_2, (B_1, B_2)^\infty\}$ with $B_1 \sim B_2$ will be called a *direction line.* A line $L$ is *dual-direction* if $\kappa(L)$ is a direction line. Note that through every $\mathfrak{q} \in \mathcal{D}$ there passes exactly one direction line and exactly one dual-direction line.

If $L = \{B_1, B_2, \mathfrak{q}\}$ with $\mathfrak{q} \in \mathcal{D}$ then we write $\beta(L) = |B_1|$ and $\omega(L) = |B_1 \setminus B_2|$. In view of 3.6, the rank of $B_1$ is $r_1(n, \beta(L)) = \binom{n}{\beta(L)} - 1$, and the rank of $\mathfrak{q}$ is $r_1^\infty(n,\omega(L)) = 2^{n-2\omega(L)}$. Note that $\omega(L) \le \beta(L) \le n - \omega(L)$, $\omega(\kappa(L)) = \omega(L)$, and $\beta(\kappa(L)) = n - \beta(L)$. The equality $\beta(L) = \omega(L)$ characterizes direction lines, and the equality $\beta(L) = n - \omega(L)$ characterizes dual-direction lines.

Let $f \in \mathrm{Aut}(\mathfrak{B}(N))$ and $L$ be a line of $\mathfrak{D}(N)$. There is at least one proper point of $L$ mapped by $f$ onto a proper point of $f(L)$. Their ranks must be equal, so $\binom{n}{\beta(L)} = \binom{n}{\beta(f(L))}$, and thus

$$(7) \qquad \text{(a)}: \ \beta(f(L)) = \beta(L) \quad \text{or} \quad \text{(b)}: \ \beta(f(L)) = n - \beta(L).$$

Moreover

$$(8) \qquad \omega(f(L)) = \omega(L).$$

Indeed, if $f$ maps $L^\infty$ onto $f(L)^\infty$ then $2^{n-2\omega(L)} = 2^{n-2\omega(f(L))}$, so $\omega(L) = \omega(f(L))$. If $f(L^\infty) \notin \mathcal{D}$ then $\binom{n}{\beta(L)} - 1 = 2^{n-2\omega(f(L))}$ and $\binom{n}{\beta(f(L))} - 1 = 2^{n-2\omega(L)}$. In both cases (7a) and (7b) we obtain $2^{n-2\omega(L)} = 2^{n-2\omega(f(L))}$, so $\omega(L) = \omega(f(L))$.

By (7a), (7b), and (8), if $L$ is a direction line or $L$ is a dual-direction line then $f(L)$ is a direction or a dual-direction line as well.

Assume that $2k \ne n$, let $L$ be a direction line with $\beta(L) = k$; then $L$ and $\kappa(L)$ have only the point $\mathfrak{q} = L^\infty$ in common and, moreover, $\beta(L) \ne \beta(\kappa(L))$. Thus there is no line in $\mathfrak{D}(N)$ which joins a proper point of $L$ with a proper point of $\kappa(L)$. Suppose that $f(L)$ and $f(\kappa(L))$ are two direction lines, then they have a common proper point $D$; say $f(L) = \{D, D_1, \mathfrak{p}_1\}$ and $f(\kappa(L)) = \{D, D_2, \mathfrak{p}_2\}$ with $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathcal{D}$. Then $|D_1| = |D_2|$, so $D_1, D_2$ are collinear in $\mathfrak{D}(N)$. Similarly, it is impossible to obtain two dual-direction lines $f(L)$ and $f(\kappa(L))$. Thus only one of the lines $f(L)$ and $f(\kappa(L))$ is a direction line, the other one is a dual-direction line, and then their common point $f(\mathfrak{q})$ is improper. Therefore $f(\mathfrak{q}) \in \mathcal{D}$ whenever $\mathfrak{q} = (D_1, D_2)^\infty \in \mathcal{D}$ and $2|D_1| \ne n$.

Let $B \in X$; take any $b \in B$ and $b' \in N \setminus B$, and set $B' = (B \setminus \{b\}) \cup \{b'\}$. Then $\mathfrak{q} = (\{b\}, \{b'\})^\infty$ is the improper point of the line $L = \overline{B, B'}$; since $f(\mathfrak{q}) \in \mathcal{D}$ we get $f(B) \in X$. Since $B$ was arbitrary, we infer that $f$ leaves families $\mathcal{D}$ and $X$ invariant. Set $f^\infty = f \restriction \mathcal{D}$. and $g = f \restriction X$.

Then $g$ is a permutation of $X$ such that

$$|B_1| = |B_2| \text{ iff } |g(B_1)| = |g(B_2)|, \text{ for every } B_1, B_2 \in X \text{ (cf. 2.4)}.$$

Set $X_k = \{B \in X \colon |B| = k\}$ and $g_k = g \restriction X_k$. Note that if $B \in X_k$ then $|g(B)| = k$ or $|g(B)| = n - k$. Indeed, it suffices to consider any line $L$ through $B$; since $|B| = \beta(L)$ and $|g(B)| = \beta(f(L))$, (7a),(7b) give the claim. Thus, either $g_k$ is a bijection of $X_k$, or $g_k$ maps bijectively $X_k$ onto $X_{n-k}$. Since $f$ preserves the class of direction and dual-direction lines, $g_k$

is an automorphism of the binomial graph $\langle X_k, \Psi_k^0 \rangle$, or an isomorphism between $\langle X_k, \Psi_k^0 \rangle$ and $\langle X_{n-k}, \Psi_{n-k}^{2k-n} \rangle$, where $B_1 \Psi_i^m B_2$ means $|B_1 \cap B_2| = m$ for $B_1, B_2 \in X_i$. Then, by [3], there is a permutation $\varphi_k$ of $N$ such that, either $g_k = \varphi^* \restriction X_k$, or $g_k = \kappa\varphi^* \restriction X_k$.

Let $\mathfrak{q} = (D_1, D_2)^\infty \in \mathcal{D}$ with $|D_1| = i$. For every $k \geq i$ there are $B_1, B_2 \in X_k$ with $\mathfrak{q} \in \overline{B_1, B_2}$. Therefore

$$f^\infty(\mathfrak{q}) = \overline{g_k(B_1), g_k(B_2)}^\infty = (\varphi_k(D_1), \varphi_k(D_2))^\infty.$$

Consequently, $\{\varphi_{k'}(D_1), \varphi_{k'}(D_2)\} = \{\varphi_{k''}(D_1), \varphi_{k''}(D_2)\}$ for all $k', k'' \geq |D_1|$ and $D_1, D_2$ with $D_1 \sim D_2$. With $i = 1$ we come to $\varphi_{k'}^* \restriction X_2 = \varphi_{k''}^* \restriction X_2$ and thus, by [3], $\varphi_{k'} = \varphi_{k''}$. Set $\varphi = \varphi_1$.

With every sequence $\sigma = (\varepsilon_1, \ldots, \varepsilon_m, \varphi)$, where $\varphi \in S_n$, $m = \mathrm{E}(\frac{n}{2})$, and $\varepsilon_i = \pm 1$ we correlate a bijection $g = \Gamma(\sigma)$ of $X$ defined by

$$g(B) = \begin{cases} \varphi(B) & \text{if } |B| = k,\ \varepsilon_k = 1 \text{ or } \varepsilon_{n-k} = 1 \\ \kappa\varphi(B) & \text{if } |B| = k,\ \varepsilon_k = -1 \text{ or } \varepsilon_{n-k} = -1 \end{cases}.$$

By the above, $\Gamma$ maps $(C_n)^m \times S_n$ onto $\mathrm{Aut}(\mathfrak{D}(N))$. The formulae $\kappa \circ \varphi^* = \varphi^* \circ \kappa$ and $(\varphi^*)^\infty = (\kappa \circ \varphi^*)^\infty$ for a permutation $\varphi$ of $N$ give that $\Gamma$ is a group homomorphism, $\Gamma\colon (C_2)^m \oplus S_n \longrightarrow \mathrm{Aut}(\mathfrak{D}(N))$. Finally, let $\sigma = (\varepsilon_1, \ldots, \varepsilon_m, \varphi) \in (C_2)^m \times S_n$, and $\Gamma(\sigma) = id$. Then $|(\Gamma(\sigma))(B)| = |B|$ for every $B \in X$ and thus $\varepsilon_k = 1$ for each $k$ with $2k \neq n$. Therefore $\varphi \restriction X_1 = id$, which yields $\varphi = id_N$. Now, suppose that $n = 2m$ and $\varepsilon_m = -1$. Then we have $\kappa(A) = A$ for every $A \in X_m$, which is impossible. Thus $\Gamma$ has trivial kernel, so it is an isomorphism. □

The automorphism groups $\mathrm{Aut}(\mathfrak{C}(N))$ and $\mathrm{Aut}(\mathfrak{H}(N))$ will not be studied in more details. We would like to only note that every $f \in \mathrm{Aut}(\mathfrak{C}(N))$ must preserve the family $\mathcal{L}_2$ (as it consists of lines of rank 4). With rather tedious calculations it comes out that $f \in \mathrm{Aut}(\mathfrak{B}(N))$ and $f \in \mathrm{Aut}(\mathfrak{D}(N))$, which yields that $f$ is determined by a permutation of $N$.

## 5. Final conclusions, remarks, and comments

The structures considered in the paper, $\mathfrak{M} = \mathfrak{B}(N)$ and $\mathfrak{M} = \mathfrak{D}(N)$, are some particular partial Steiner triple systems. They are rather regular – any two points of $\mathfrak{M}$ with the same rank can be interchanged by an automorphism of $\mathfrak{M}$. On the other hand these groups of automorphisms are relatively small and quite elementary:

(i) $\mathrm{Aut}(\mathfrak{M}) \cong S_{n+1}$ has $(n+1)!$ elements and acts on $2^n - 1$ points for $\mathfrak{M} = \mathfrak{B}(N)$ with $|N| = n \geq 2$, and

(ii) $\mathrm{Aut}(\mathfrak{M}) \cong (C_2)^m \oplus S_n$ has $2^m \cdot n!$ elements and acts on $2^n - 1 + \sum_{i=1}^m \binom{n}{i}\binom{n-i}{i}$ points for $\mathfrak{M} = \mathfrak{D}(N)$ with $|N| = n = 2m$ or $n = 2m + 1$.

It is worth to note which partial Steiner triple systems defined in the paper can be embedded into Fano projective spaces, and then how the results of the paper can be interpreted in terms of the projective Fano geometry. The boolean representation $\chi\colon X \longrightarrow \underline{2}^n$, which associates with a subset $A$ of $N$ its characteristic function $\chi_A\colon N \longrightarrow \underline{2} = \{0,1\}$ yields a projective representation of $\mathfrak{B}(N)$ in the $(n-1)$-dimensional projective space $PG(n-1,2)$ over $GF(2)$. Indeed,

if $B_1, B_2, B_3 \in N$ are collinear in $\mathfrak{B}(N)$, then $\chi_{B_1}, \chi_{B_2}, \chi_{B_3}$ are collinear in $PG(n-1,2)$.

Since the proportionality relation in the vector space $\underline{2}^n$ coincides with the relation of equality, the set $\mathcal{P} = \underline{2}^n \setminus \{\theta\}$, where $\theta = \chi_\emptyset$, can be considered as the set of points of $PG(n-1,2)$. Lines of this space are sets of the form $\overline{f_1, f_2} = \{f_1, f_2, f_1 + f_2\}$, where $f_1, f_2 \in \mathcal{P}$ are arbitrary but distinct. Then $\mathfrak{B}(N)$ is obtained by removing some lines of $PG(n-1,2)$:

a line $L$ of $PG(n-1,2)$ is a line of $\mathfrak{B}(N)$ if there are points $f_1, f_2 \in L$ such that $f_1 \cdot f_2 = \theta$

(we use the natural ring structure of $\underline{2}^n$ here).
Since the set $\mathcal{F} = \{\chi_{\{a\}}\colon a \in N\} \cup \{\chi_N\}$ of points of $PG(n-1,2)$ is (projectively) independent, from 4.4 we infer that every automorphisms of $\mathfrak{D}(N)$ is an automorphism of $PG(n-1,2)$ as well. It is seen that (under our embedding) the group $\mathrm{Aut}(\mathfrak{B}(N))$ consists of all the projective collineations of $PG(n-1,2)$ which leave invariant the projective frame $\mathcal{F}$.

Unhappily, the above representation cannot be (fully) extended to a representation of $\mathfrak{D}(N)$ in a Fano space. Let $f_i = \chi_{A_i} \in \underline{2}^n$ for $A_i \in X$. By 2.4, the pair $(A_1, A_2)$ determines a direction in $\mathfrak{D}(N)$ iff $|\chi^{-1}(f_1)| = |\chi^{-1}(f_2)|$. By 2.2, $A_1, A_2 \simeq A_3, A_4$ yields $f_1 + f_2 = f_3 + f_4$, and thus

if $A_1, A_2 \simeq A_3, A_4$, then the vectors $\overrightarrow{\chi_{A_1}, \chi_{A_2}}$ and $\overrightarrow{\chi_{A_3}, \chi_{A_4}}$ are parallel in the $n$-dimensional affine space over the vector space $\underline{2}^n$.

However, one cannot uniquely correlate a direction $(A_1, A_2)^\infty \in \mathcal{D}$ with the (projective) direction of the affine line $\overline{\chi_{A_1}, \chi_{A_2}}$ over $\underline{2}^n$. Indeed, let $n \geq 4$; set $A_1 = \{1,2\}$, $A_2 = \{3,4\}$, $A_3 = \{1,3\}$, $A_4 = \{2,4\}$. The lines $\overline{\chi_{A_1}, \chi_{A_2}}$ and $\overline{\chi_{A_3}, \chi_{A_4}}$ are parallel, but $(A_1, A_2)^\infty \neq (A_3, A_4)^\infty$. We do not know whether the structure $\mathfrak{D}(N)$ can be embedded into a Fano space.

Clearly, $\underline{2} \subset \underline{3} = \{1,2,3\}$ and thus the characteristic function $\chi_A$ for $A \subset N$ can be considered as an element of $\underline{3}^n$ as well. In particular, we obtain an injection of $X$ into the affine space $\underline{3}^n$ over $GF(3)$. Let $f_i = \chi_{A_i} \in \underline{3}^n$ for $i = 1, \ldots, 4$ and let $A_1, A_2$ and $A_3, A_4$ be collinear in $\mathfrak{D}(N)$. Then

$A_1, A_2 \simeq A_3, A_4$ iff the affine lines $\overline{\chi_{A_1}, \chi_{A_2}}$ and $\overline{\chi_{A_3}, \chi_{A_4}}$ are parallel over $\underline{3}^n$.

Therefore, the map $\chi$ can be extended to an injection of $\mathfrak{D}(N)$ into the projective completion $PG(n,3)$ of the above affine space. However, this representation does not map lines *onto* lines. Moreover, not every automorphism of $\mathfrak{D}(N)$ can be extended to an automorphism of $PG(n,3)$. On the other hand, it seems that this injection can be extended to an injection of $\mathfrak{H}(N)$ into $PG(n,3)$, but so far it is only a conjecture.

Up to now, no (natural and regular) minimal representation of $\mathfrak{C}(N)$ and of $\mathfrak{H}(N)$ in a finite projective space is known. In particular, we do not know what is a minimal $m$ such that $\mathfrak{H}(N)$ can be embedded into a Fano projective space $PG(m,2)$. As noted on the page 631, if we remove from $\mathfrak{H}(N)$ the lines with rank 2, the resulting structure will be the disjoint union of the horizons of Desarguesian closures of some (binomial) graphs; it is known that every such a closure can be represented in a Fano space. More precisely, the horizon of the Desarguesian closure of the binomial graph $\Psi_k^0$ of $k$-element subsets of $N$ can be embedded into the space $PG(\binom{n}{k}-2,2)$ and this horizon contains a line only for $3k \le n$. Therefore, one can embed $\mathfrak{H}(N)$ into $PG(m,2)$ for $m = \sum_{k=1}^{\mathrm{E}(\frac{n}{3})} \binom{n}{k} - \mathrm{E}(\frac{n}{3}) - 1$. This embedding is constructed, in accordance with the general theory, in a somewhat "free" way – one should expect that there exists an embedding with smaller $m$, but no one is known yet.

## References

[1] M. Berger, *Geometrié*, CEDIC, Paris, 1977(8).
[2] D. Hilbert, P. Cohn-Vossen, *Anschauliche Geometrie*, Berlin, 1932.
[3] M. Ch. Klin, R. Pöschel, K. Rosenbaum, *Angewandte Algebra für Mathematiker und Informatiker*, VEB Deutcher Verlag der Wissenschaften, Berlin 1988.
[4] W. Lipski, W. Marek, *Analiza kombinatoryczna* (in Polish), PWN, Warszawa, 1986.
[5] K. Prażmowski, *Extensions of complete graphs to regular partial linear spaces*, ZN Geometria wykreślna i grafika inżynierska, Zesz.5(1999), Wrocław (Politechnika Wrocławska), 63–72.
[6] A. B. Romanowska, J, D. H. Smith, *Modal Theory, An Algebraic Approach to Order, Geometry, and Convexity*, Heldermann Verlag, Berlin 1985.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF BIAŁYSTOK
Akademicka 2
15-267 BIAŁYSTOK, POLAND
E-mail: krzypraz@math.uwb.edu.pl

*Received January 17, 2003.*