DEMONSTRATIO MATHEMATICA
Vol. XXXVII No 3 2004

Liaqat Ali Khan

TRANS-SEPARABILITY IN SPACES OF CONTINUOUS VECTOR-VALUED FUNCTIONS

Abstract. Let X be a completely regular Hausdorff space and E a Hausdorff topological vector space (TVS). In this note, we study the notion of trans-separability for certain subspaces of $C_b(X, E)$ endowed with the σ-compact-open and some related topologies.

Let X be a completely regular Hausdorff space and E a Hausdorff topological vector space (TVS) with a base $\mathcal{W}$ of balanced neighbourhoods of 0, and let $C_b(X, E)$ (resp. $C_{rc}(X, E)$) denote the vector space of all continuous E-valued functions f on X such that $f(X)$ is bounded (resp. relatively compact). Let $C_0(X, E)$ be the subspace of $C_b(X, E)$ consisting of those functions which vanish at infinity. If X is compact, $C_b(X, E) = C_{rc}(X, E) = C_0(X, E)$. When E is the real or complex field, $C_b(X, E) = C_{rc}(X, E)$ and we denote this by $C_b(X)$ and $C_0(X, E)$ by $C_0(X)$. The *σ-compact-open topology* σ (resp. *compact-open topology* k, *countable-open topology* σ_0) [4, 5, 11] on $C_b(X, E)$ is defined as the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form

$$N(A, W) = \{f \in C_b(X, E) : f(A) \subseteq W\},$$

where A varies over all σ-compact (resp. compact, countable) subsets of X and $W \in \mathcal{W}$. The uniform topology u on $C_b(X, E)$ is the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form $N(X, W)$, where $W \in \mathcal{W}$. Clearly, $\sigma_0 \leq \sigma \leq u$ and $k \leq \sigma$. Further, by ([11], Theorem 3.3), $\sigma = u$ iff $X = \bar{A}$ for some σ-compact subset A; $k = \sigma$ iff every σ-compact subset of X is relatively compact; $\sigma_0 = u$ iff X is separable. For any $A \subseteq X$, let $\| \,.\, \|_A$ denote the seminorm on $C_b(X)$ given by $\| f \|_A = \sup_{x \in A} |f(x)|$, $f \in C_b(X)$. We shall denote by $C_b(X) \otimes E$ the vector subspace of $C_b(X, E)$ spaned by the set of all functions of the form

1991 *Mathematics Subject Classification*: Primary 46E10; Secondary 46E40, 46A10.
Key words and phrases: topological vector spaces, vector-valued continuous functions, σ-compact-open topology, uniform topology, trans-separable spaces.

$\varphi \otimes a$, where $\varphi \in C_b(X)$, $a \in E$, and $(\varphi \otimes a)(x) = \varphi(x)a$, $x \in X$. Let βX denote the Stone-Čech compactification of X. Each g in $C_b(X)$ or $C_{rc}(X, E)$ has a continuous extension $\hat{g}$ to βX and the spaces $C_b(X) \otimes E$ and $C_{rc}(X, E)$ are linearly isomorphic to $C_b(\beta X) \otimes E$ and $C_b(\beta X, E)$, respectively. Recall that a TVS E is called *admissible* if the identity operator on E can be uniformly approximated on compact subsets by continuous operators with finite dimensional range [12]. Every F-space with a basis, in particular l_p $(0 < p < \infty)$, is admissible ([18], p. 101). We shall require the following known result. We sketch its proof for the sake of completeness (cf. [18], [11]).

THEOREM 1. *$C_b(X) \otimes E$ is u-dense (hence also σ-, σ_0-dense) in $C_{rc}(X, E)$ in each of the following cases:*

(i) *X is a normal Hausdorff space of finite covering dimension and E any TVS.*

(ii) *X is any Hausdorff space and E an admissible TVS.*

(iii) *X is any completely regular Hausdorff space and E a locally convex TVS.*

Proof. (i) Since X is a normal Hausdorff space of finite covering dimension, βX is also of finite coveing dimension [3]. Thus, by ([18], Theorem 1), $C_b(\beta X) \otimes E$ *is u-dense* $C_b(\beta X, E)$, and so the results follows from the above observation.

(ii) Let $f \in C_{rc}(X, E)$ and W a neighbouhood of 0 in E. Since $\overline{f(X)}$ is compact and E is admissible, there exists a continuous map $\psi : \overline{f(X)} \to E$ with finite dimensional range such that $\psi(f(x)) - f(x) \in W$ for all $x \in X$. By using the same argument as that of ([18], Proposition 1), we see that $\psi \circ f \in C_b(X) \otimes E$, and so f is in the u-closure of $C_b(X) \otimes E$.

(iii) If E is localy convex, then ([18], Theorem 1) remains valid without the assumption of finite covering dimension on X, and so the result follows as in (i). ∎

A locally convex space is L is called *seminorm-separable* if, for each continuous seminorm p on L, (L, p) is separable. The following two classical results are stated for reference purpose.

THEOREM 2. (i) [13, 20] $(C_b(X), \| . \|)$ *is separable iff X is a compact metric space.*

(ii) [16, 20] *If X is locally compact, then* $(C_0(X), \| . \|)$ *is separable iff X is a σ-compact metric space.*

THEOREM 3. [5] *The following statements are equivalent:*

(a) $(C_b(X), \sigma)$ *is seminorm-separable.*

(b) $(C_b(X), \sigma_0)$ *is seminorm-separable.*

(c) *The closure in* βX *of each* σ*-compact subset of* X *is metrizable in* βX.

A uniform space L is called *trans-separable* if every uniform cover of L admits a countable subcover ([7], [6]). This notion has occured to be useful in the work of ([14], [1], [15]) while studying the metrizability of precompact sets in locally convex spaces (see also [9]). Drewnowski [2] had actually coined the word "trans-separable" and it has been further used by Robertson [15]. Recently, in [10], the author has introduced a generalized notion of separability in the TVS setting and characterized it for vector-valued function spaces endowed with the strict and compact-open topologies. This notion may be defined equivalently, as follows. Let (L, τ) be a TVS whose topology is generated by a family $Q(\tau)$ of continuous F-seminorms [21, p. 2]. Then (L, τ) is called F*-seminorm-separable* if (L, q) is separable for each $q \in Q(\tau)$. Clearly, separability implies F-seminorm-separability; the converse holds in metric spaces.

The following result establishes the equivalence of the notions of trans-separability and F-seminorm-separability.

PROPOSITION 4. *A TVS* (L, τ) *is trans-separable iff* (L, q) *is separable for each* $q \in Q(\tau)$.

Proof. Suppose L is trans-separable, and let $q \in Q(\tau)$. For each $n \geq 1$, let $V_n = \{x \in L : q(x) < 1/n\}$, a balanced neighbourhood of 0 in L. Then, for each $n \geq 1$, $\mathbb{U}_n = \{x + V_n : x \in L\}$ is a uniform cover of L, and so it has a countable subcover $\mathbb{U}_n^* = \{x_k^{(n)} + V_n : k \in \mathbb{N}\}$. Let $D = \cup_{n=1}^{\infty}\{x_k^{(n)} : k \in \mathbb{N}\}$. To show that D is dense in (L, q), let $y \in L$ and $\varepsilon > 0$. Choose $N \geq 1$ such that $1/N < \varepsilon$. Since $\mathbb{U}_N^*$ is a cover of L then $y \in x_K^{(N)} + V_N$ for some $K \in \mathbb{N}$. Thus $q(y - x_K^{(N)}) < 1/N < \varepsilon$ and hence (L, q) is separable.

Conversely, suppose (L, q) is separable for each $q \in Q(\tau)$. Let $\{x + U : x \in L\}$ be any uniform cover of L, where U is a neighbourhood of 0 in L. Choose a balanced neighbourhood V of 0 in L with $V + V \subseteq U$. Pick $q \in Q(\tau)$ such that $W = \{x \in L : q(x) < 1\} \subseteq V$. Let $\{z_n\}$ be a countable dense subset in (L, q). Since $L = \cup_{x \in L}(x + V)$, then for each $z_n \in L$, there exists some $x_n \in L$ such that $z_n - x_n \in W$. Let $y \in L$. Choose z_k such that $q(y - z_k) < 1$. Then

$$y - z_k = (y - z_k) + (z_k - x_k) \in W + W \subseteq U,$$

and so $L = \cup_{n \geq 1}(x_n + U)$. ∎

THEOREM 5. *Let* E *be any non-trivial TVS. Then the following statements are equivalent:*

(a) $(C_b(X) \otimes E, \sigma)$ *is trans-separable.*

(b) $(C_b(X) \otimes E, \sigma_0)$ *is trans-separable.*

(c) *The closure in* βX *of each* σ*-compact subset of* X *is metrizable in* βX *and* E *is trans-separable.*

Proof. It is obvious that (a) implies (b).

That (b) implies (c) follows from Theorem 3 and the fact that both $(C_b(X), \sigma_0)$ and E are isomorphic to subspaces of $(C_b(X) \otimes E, \sigma_0)$ via the maps $\varphi \to \varphi \otimes a$ ($0 \neq a \in E$ fixed) and $z \to 1_X \otimes z$, respectively.

(c) implies (a): By Theorem 3, $(C_b(X), \sigma)$ is trans-separable. Fix a σ-compact subset A of X and a balanced $W \in \mathcal{W}$. We need to show that there is a countable set $H \subset C_b(X) \otimes E$ such that $C_b(X) \otimes E = H + N(A, W)$.

For every pair $m, n \in \mathbb{N}$ choose a balanced $U_{m,n} \in \mathcal{W}$ so that, for $V_{m,n} = U_{m,n} + mU_{m,n} + U_{m,n}$, one has

$$V_{m,n} + \cdots + V_{m,n} \ (n\text{-summands}) \subset W.$$

Also, choose a countable set $D_{m,n}$ in E so that $E = D_{m,n} + U_{m,n}$. Let D be the union of all these sets $D_{m,n}$ ($m, n \in \mathbb{N}$).

Next, for each $k \in \mathbb{N}$ denote $B_k = \{f \in C_b(X) : \|f\|_A \leq 1/k\}$ and choose a countable set G_k in $C_b(X)$ so that $C_b(X) = G_k + B_k$. Let G be the union of all these sets G_k ($k \in \mathbb{N}$).

We are going to show that the countable set $H = H_{A,W}$ of all functions in $C_b(X) \otimes E$ of the form $h = \sum_{i=1}^{r} g_i \otimes d_i$, where $g_i \in G$ and $d_i \in D$ ($i = 1, \ldots, r$, $r \in \mathbb{N}$), is as required.

Take any $f \in C_b(X) \otimes E$. Then $f = \sum_{i=1}^{n} f_i \otimes a_i$ for some $f_1, \ldots, f_n \in C_b(X)$ and $a_1, \ldots, a_n \in E$. Let $m \in \mathbb{N}$ be such that $\|f_i\|_A \leq m$ for $i = 1, \ldots, n$, and next choose $k \in \mathbb{N}$ so that $k^{-1}a_i \in U_{m,n}$ for $i = 1, \ldots, n$. By the definitions of $D_{m,n}$ and G_k, there are $d_1, \ldots, d_n \in D_{m,n}$ and $g_1, \ldots, g_k \in G_k$ such that

$$a_i - d_i \in U_{m,n} \quad \text{and} \quad \|f_i - g_i\|_A \leq 1/k \qquad \text{for } i = 1, \ldots, n.$$

Now, for $i = 1, \ldots, n$ and $x \in A$,

$$(*) \quad f_i(x)a_i - g_i(x)d_i = (f_i(x) - g_i(x))a_i + f_i(x)(a_i - d_i) + (g_i(x) - f_i(x))(a_i - d_i).$$

Hence (using the fact that $U_{m,n}$ is balanced)

$$f_i(x)a_i - g_i(x)d_i \in U_{m,n} + mU_{m,n} + U_{m,n} = V_{m,n}.$$

In consequence, setting $h = \sum_{i=1}^{n} g_i \otimes d_i$, we have $h \in H$ and for every $x \in A$,

$$f(x) - h(x) = \sum_{i=1}^{n} (f_i(x)a_i - g_i(x)d_i) \in W$$

so that $f - h \in N(A, W)$. ■

REMARK (I). A somewhat more transparent variant of the above proof that (c) implies (a) can be based on Proposition 4. Thus, one has to show that for any σ-compact subset A of X and any continuous F-seminorm p on E, the space $(C_b(X) \otimes E, p_A)$ is separable , where $p_A(f) = \sup_{x \in A} p(f(x))$. Now, let G be a countable subset dense in $(C_b(X), \|\cdot\|_A)$, and D a countable set dense in (E, p). Take any $f = \sum_{i=1}^{n} f_i \otimes a_i$ in $C_b(X) \otimes E$, and choose $m \in \mathbb{N}$ so that $\|f_i\|_A \leq m$ for each i. Given $\varepsilon > 0$, let $g = \sum_{i=1}^{n} g_i \otimes d_i$, where $g_i \in G$ and $d_i \in D$. Assume that $\|f_i - g_i\|_A \leq \delta$ for all i and some as yet unspecified $0 < \delta < 1$. Then, making use of $(*)$, it can be easily seen that

$$\begin{aligned} p_A(f-g) &\leq \sum_{i=1}^{n} (p(\|f_i - g_i\|_A a_i) \\ &\quad + p(\|f_i\|_A (a_i - d_i)) + p(\|f_i - g_i\|_A (a_i - d_i))) \\ &\leq \sum_{i=1}^{n} (p(\delta a_i) + (m+1) p((a_i - d_i). \end{aligned}$$

But this can be made less than ε by taking δ sufficiently small and choosing the d_i's in D sufficiently close to the a_i's. It follows that the countable set of all g's of the above form is dense in $(C_b(X) \otimes E, p_A)$.

(II). If $C_b(X) \otimes E$ is u-dense in $C_{rc}(X, E)$ (cf. Theorem 1), then clearly Theorem 5 holds with $C_b(X) \otimes E$ replaced by $C_{rc}(X, E)$. We do not know of any general assumption on X other than its compactness in Theorem 1 ensuring that $C_b(X) \otimes E$ is σ- or σ_0-dense in $C_b(X, E)$. However, without this density assumption, the following holds.

THEOREM 6. *$(C_b(X, E), \sigma)$ is trans-separable $\Leftrightarrow$ $(C_b(X, E), \sigma_0)$ is so.*

Proof. $[\Rightarrow]$ This follows from the fact that $\sigma_0 \leq \sigma$.

$[\Leftarrow]$ Suppose $(C_b(X, E), \sigma)$ is not separable. We show that $(C_b(X, E), \sigma_0)$ is also not trans-separable, i.e., if $H = \{f_n\}$ is any countable set in $C_b(X, E)$, then there exist a countable set $B = \{x_n\} \subseteq X$ and a $W \in \mathcal{W}$ such that $C_b(X, E) \neq H + N(B, W)$. Since $(C_b(X, E), \sigma)$ is not separable, there exists a σ-compact set $A \subseteq X$ and a $W \in \mathcal{W}$ such that $C_b(X, E) \neq H + N(B, W)$. Then there exists an $f \in C_b(X, E)$ such that $f - f_n \notin N(A, W)$ for all $n \geq 1$. So, for each $n \geq 1$, there exists an $x_n \in A$ such that

$$f(x_n) - f_n(x_n) \notin W.$$

Thus, if $B = \{x_n\}$, $f - f_n \notin N(B, W)$ for all $n \geq 1$. Hence $C_b(X, E) \neq H + N(B, W)$, and so $(C_b(X, E), \sigma_0)$ is not trans-separable, as desired.

Next, using Theorem 2 and the method of the proof of Theorem 5, we obtain:

THEOREM 7. *Let E be a TVS. Then:*

(a) *$(C_b(X) \otimes E, u)$ is trans-separable iff X is a compact metric space and E is trans-separable.*

(b) *Suppose X is locally compact. Then $(C_0(X) \otimes E, u)$ is trans-separable iff X is a σ-compact metric space and E is trans-separable.*

Again we remark that, if $C_b(X) \otimes E$ *(resp.* $C_0(X) \otimes E$) *is* u-dense in $C_b(X, E)$ $(C_0(X, E))$, Theorem 6 remains valid with $C_b(X) \otimes E$ $(C_0(X) \otimes E)$ replaced by $C_b(X, E)$ $(C_0(X, E))$.

Finally, we give some examples concerning the above results and remarks.

EXAMPLES. For $0 < p < 1$, let $l_p = l_p(\mathbb{N})$ denote the usual space of scalar sequences, and let $h_p = h_p(D)$ denote te Hardy space of certain harmonic functions on the unit disc D of complex plane (see [17]). l_p and h_p are not locally convex but are locally bounded (hence locally pseudo-convex or semi-convex [8]) and their duals separate points; l_p is separable while h_p is not since it contains a copy of l_∞ ([17], Theorem 3.5). Let ω be the first uncountable ordinal and ω_0 the first countable infinite ordinal. Let $\Omega = [0, \omega]$ and $\Omega_0 = [0, \omega_0]$, each endowed with the order topology, and let $T = \Omega \times \Omega_0 \backslash \{(\omega, \omega_0)\}$, the *deleted Tychonoff plank*, with the product topology ([19]). Then:

1. $C_b(X) \otimes l_p$ is u-dense (hence also σ-, σ_0-dense) in $C_{rc}(X, l_p)$ for any Hausdorff space X since l_p is admissable.
2. $(C_b(\Omega) \otimes l_p, u)$ and $(C_b(\Omega_0) \otimes h_p, u)$ are not trans-separable since Ω is not merizable and h_p is not (trans-) separable.
3. $(C_b(\Omega_0, l_p), u)$ is trans-separable since Ω_0 is compact and metrizable.
4. $(C_b(T) \otimes l_p, \sigma)$ is not trans-separable since, if $A = \cup_{i=1}^{\infty} \Omega \times \{n\}$, A is σ-compact and $\bar{A}^{\beta T} = \beta T = \Omega \times \Omega_0$ which is not metrizable (see [5], p. 259).

Acknowledgement. Theorems 5 and 6 were originally proved for E a locally pseudo-convex space with nontrivial dual E'. The author is greatly obliged to the referee for several useful suggestions including the present general proofs of Theorems 5 and 6 which holds for any non-zero TVS E.

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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KING ABDULAZIZ UNIVERSITY
P. O. BOX 80203
JEDDAH-21589, SAUDI ARABIA
E-mail: akliaqat@hotmail.com

Received May 24, 2002; revised last version January 31, 2004.