DEMONSTRATIO MATHEMATICA
Vol. XXXVII No 3 2004

Włodzimierz Łenski, Bogdan Roszak

ON THE RATE OF POINTWISE STRONG SUMMABILITY OF FOURIER SERIES

Abstract. There is introduced a modified local modulus of continuity as a measure of pointwise strong summability. The approximation versions of known results FuTraing Wang [6] and A. A. Zakharov [8] are obtained.

1. Introduction

Let L^p $(1 \leq p \leq \infty)$ $[resp.C]$ be the class of all 2π-periodic real-valued functions integrable in the Lebesgue sense with p–th power [continuous] over $Q = [-\pi, \pi]$ and let $X = X^p$ where $X^p = L^p$ when $1 \leq p < \infty$ or $X^p = C$ when $p = \infty$. Let us define the norm of $f \in X^p$ as

$$\|f\|_{X^p} = \|f(x)\|_{X^p} = \begin{cases} \left(\int_Q \mid f(x) \mid^p dx\right)^{1/p} & \text{when } 1 \leq p < \infty, \\ \sup_{x\in Q} \mid f(x) \mid & \text{when } p = \infty. \end{cases}$$

Consider the trigonometric Fourier series

$$Sf(x) = \frac{a_o(f)}{2} + \sum_{k=0}^{\infty}(a_k(f)\cos kx + b_k(f)\sin kx) = \sum_{k=0}^{\infty} C_k f(x)$$

and denote by $S_k f$, the partial sums of Sf, and let

$$H_n^q f(x) := \left\{\frac{1}{n+1}\sum_{k=0}^{n} |S_k f(x) - f(x)|^q\right\}^{\frac{1}{q}}, \quad (q > 0).$$

The pointwise characteristic

$$\overline{w}_x^p f(\delta) := \sup_{0<h\leq\delta}\left\{\frac{1}{h}\int_0^h |\varphi_x(t)|^p \, dt\right\}^{1/p},$$

where $\varphi_x(t) := f(x+t) + f(x-t) - 2f(x)$ constructed on the base of definition of Lebesgue points (*L-points*) was firstly used as a measure of approximation, by S. Aljančič, R. Bojanic and M. Tomić [1]. This characteristic was very often used, but it appears that such approximation cannot be comparable with the norm approximation beside when $X = C$. In [4] there was introduced the slight modified quantity

$$w_x^p f(\delta) := \left\{ \frac{1}{\delta} \int_0^\delta |\varphi_x(t)|^p \, dt \right\}^{1/p}.$$

On the base of definition of the indefinite integral differentiability points (*D-points*) it was considered the quantity

$$\overline{w}_x^\star f(\delta) := \sup_{0<h\leq\delta} \left| \frac{1}{h} \int_0^h \varphi_x(t) \, dt \right|, \text{ (see [2])}$$

and in [5] was introduced the following also slight modified quantity

$$w_x^\star f(\delta) := \sup_{0<h\leq\delta} \left| \frac{1}{\delta} \int_0^h \varphi_x(t) \, dt \right|.$$

We can observe that for $p \in [1, \infty)$ and $f \in C$

$$w_x^p f(\delta) \leq \overline{w}_x^p f(\delta) \leq \omega_C f(\delta)$$

and

$$w_x^\star f(\delta) \leq \overline{w}_x^\star f(\delta) \leq \omega_C f(\delta)$$

and also, with $\widetilde{p} > p$ for $f \in X^{\widetilde{p}}$, by the Minkowski inequality

$$\text{(1)} \qquad \|w_{\cdot}^\star f(\delta)\|_{X^{\widetilde{p}}} \leq \|w_{\cdot}^p f(\delta)\|_{X^{\widetilde{p}}} \leq \omega_{X^{\widetilde{p}}} f(\delta),$$

and

$$\text{(2)} \qquad \|w_{\cdot}^p f(\delta)\|_{X^{\widetilde{p}}} \leq \|\overline{w}_{\cdot}^p f(\delta)\|_{X^{\widetilde{p}}} \leq \omega_{X^{\widetilde{p}}} f(\delta),$$

where $\omega_X f$ is the modulus of continuity of f in the space $X = X^{\widetilde{p}}$ defined by the formula

$$\omega_X f(\delta) := \sup_{0<|h|\leq\delta} \|f(\cdot + h) - f(\cdot)\|_X .$$

It is well-known that $H_n^q f(x)$ means tend to 0 at the $D-$*points* of $f \in L^p$ $(1 < p \leq 2)$. In [2] this fact was presented in the approximation version with the quantity $\overline{w}_x^\star f$ as a measure of such approximation. Here for estimation of the $H_n^q f(x)$ means both the new measures of approximation are used in the approximation version of the result of A. A. Zaharov [8]. We will also consider the case when $p = 1$ and prove the approximation version of the result of Fu Triang Wang [6] with the following characteristic

$$(*) \qquad w_x^{\log} f(\delta) = \sup_{0<h\leq\delta} \frac{\log^\alpha 1/h}{h} \int_0^h |\varphi_x(t)| \, dt.$$

By K we shall designate either an absolute constant or a constant depending on the indicated parameters, not necessarily the same of each occurrence.

2. Statement of the results

We start with a theorem, for the case $p = 1$.

THEOREM 1. *If $f \in L^1$, then with $\alpha > \frac{1}{2}$,*

$$H_n^2 f(x) \le K\left\{w_x f(\delta_n) + w_x^{\log} f(\gamma_n)\right\} + K\left\{\frac{1}{n+1}\sum_{k=0}^{n}\left[\omega_{L^1} f(\frac{\pi}{k+1})\right]^2\right\}^{1/4}$$

where $\delta_n = \frac{\pi}{n+1}$, $\gamma_n = \sqrt[4]{(n+1)^{-1}\sum_{k=0}^{n}\left[\omega_{L^1} f(\frac{\pi}{k+1})\right]^2}$, and $w_x^{\log}$ is defined by $(*)$ for all real x and every positive integer n.

Next, we consider the case $p > 1$.

THEOREM 2. *If $f \in L^p$ $(1 < p \le 2)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$H_n^q f(x) \le K\left\{(n+1)^{1-p}\sum_{k=0}^{[\pi/\delta_n]}\frac{(w_x^p f(\frac{\pi}{k+1}))^p}{(k+1)^{2-p}}\right\}^{\frac{1}{p}} + K\left\{(n+1)\sum_{k=[\pi/\delta_n]}^{\infty}\frac{w_x^\star f(\frac{\pi}{k})}{k^2}\right\}$$

for some positive δ_n tending to zero, all real x and every positive integer n.

From [8] we can deduce the possibility of existence of a sequence δ_n for which we have

REMARK 1. If $f \in L^1$ and $x \in \mathbf{R}$ is such that

$$w_x^\star f(t) = o_x(1), \tag{3}$$

then there exists $\delta_n \longrightarrow 0_+$ such that

$$(n+1)\,\delta_n \nearrow \infty \tag{4}$$

and

$$\delta_n \sup_{0<\delta\le\delta_n} w_x^\star f(\delta_n) = o_x(1/n). \tag{5}$$

The construction of δ_n goes in the following way: Condition (3) implies that there exists a minimal $n_1 \in \mathbf{N}$ such that for $n \ge n_1$ we have $w_x^\star [f]\left(\frac{\pi}{n+1}\right) < 1$ and also it follows that we can find a minimal $n_2 \in \mathbf{N}$ such that $\frac{2}{n_2} < \frac{1}{2}$ and for $n \ge n_2$ it is $w_x^\star [f]\left(\frac{2\pi}{n+1}\right) < \frac{1}{2^2}$. If now we put $\delta_n = \frac{\pi}{n+1}$ for $n_1 \le n \le n_2 - 1$, we obtain for these n the relation $(n+1)\,\delta_n \sup_{0<\delta\le\delta_n} w_x^\star [f](\delta) < 1$. Now, again by (3) we can point at such a minimal n_3 that $\frac{3}{n_3} < \frac{1}{3}$ and for $n \ge n_3$ it is $w_x^\star [f]\left(\frac{3\pi}{n+1}\right) < \frac{1}{3^2}$ and if we put $\delta_n = \frac{2\pi}{n+1}$ for $n_2 \le n \le n_3 - 1$ we will have for these n also

$(n+1)\,\delta_n \sup_{0<\delta\leq\delta_n} w_x^\star[f](\delta) < \frac{1}{2}$. Repeating this argument we can find a subsequence of natural numbers $n_k \to \infty$ such that for n fulfilling $n_k \leq n \leq n_{k+1}-1$ we will have $(n+1)\,\delta_n = k\pi$ and $(n+1)\,\delta_n \sup_{0<\delta\leq\delta_n} w_x^\star[f](\delta) < \frac{1}{k}$. Thus, the sequence δ_n, satisfying (4) and (5), is constructed.

With such a sequence we can estimate the second term in Theorem 2 in the following way:

$$(n+1)\sum_{k=[\pi/\delta_n]}^{\infty}\frac{w_x^\star f\left(\frac{\pi}{k}\right)}{k^2} \leq (n+1)\sup_{0<\delta\leq\delta_n} w_x^\star[f](\delta)\sum_{k=[\pi/\delta_n]}^{\infty}\frac{1}{k^2}$$
$$\leq \frac{(n+1)\,\delta_n}{\pi}\sup_{0<\delta\leq\delta_n} w_x^\star[f](\delta) = o_x(1) \quad \text{as } n\to\infty$$

and if $|w_x^p[f](\delta)| \leq K$ for $\delta > 0$, the first one can be estimated as follows:

$$\left\{(n+1)^{1-p}\sum_{k=0}^{[\pi/\delta_n]}\frac{(w_x^p f(\frac{\pi}{k+1}))^p}{(k+1)^{2-p}}\right\}^{\frac{1}{p}} \leq K\left\{(n+1)^{1-p}\sum_{k=0}^{[\pi/\delta_n]-1}\frac{1}{(k+1)^{2-p}}\right\}^{1/p}$$
$$\leq \frac{K}{\{(n+1)\,\delta_n\}^{1/q}} = o_x(1) \quad \text{as } n\to\infty$$

by condition (4). Thus, from our theorems the results of [8] and [6] follow.

Corollary 1. *Let $f \in L^p\,(1<p\leq 2)$ and $\frac{1}{p}+\frac{1}{q}=1$. If $x\in\mathbf{R}$ is such that*

$$w_x^\star f(t) = o_x(1) \ \text{ and } w_x^p f(t) = O_x(1) \ \text{ as } t \longrightarrow 0_+,$$

then

$$H_n^q f(x) = o_x(1) \ \text{ as } n\to\infty$$

and if $f\in L^1$, $w_x^{\log} f(\frac{\pi}{k+1}) = o_x(1)$, with $\alpha > \frac{1}{2}$ as $t \longrightarrow 0_+$, then

$$H_n^2 f(x) = o_x(1) \ \text{ as } n\to\infty.$$

Now, using the relations (1) and (2), we can derive norm approximation corollaries from Theorem 2.

Corollary 2. *If $f \in X = X^{\widetilde{p}}\,(\widetilde{p}\in(1,\infty)$ and $\widetilde{p}\geq p\in(1,2]$, then*

$$\|H_n^q f\|_X \leq K\{(n+1)\delta_n\omega_X f(\delta_n)\} + K\left\{(n+1)^{1-p}\sum_{k=0}^{[\pi/\delta_n]}\frac{(\omega_X f(\frac{\pi}{k+1}))^p}{(k+1)^{2-p}}\right\}^{1/p}$$

and, when $\delta_n = \frac{\pi}{n+1}$, then

$$\|H_n^q f\|_X \leq K\left\{(n+1)^{1-p}\sum_{k=0}^{n}\frac{\left(\omega_X f\left(\frac{\pi}{k+1}\right)\right)^p}{(k+1)^{2-p}}\right\}^{1/p}$$

for every positive integer n.

In the proof of Theorem 1 we will need the following modified result of E. Hille and G. Klein [3].

LEMMA 1. *If $g \in L^1$ and ν is a continuous decreasing function in $[0, \pi]$, then*

$$\int_0^\pi g(t)\,\nu(t)\sin p_n t\,dt \le K\,\nu(0)\left(1 + \|g\|_{L^1}\right)\omega_{L^1} g\left(\frac{\pi}{n+1}\right),$$

where e.g. $p_n = n$ or $p_n = n + \frac{1}{2}$ and $n = 0, 1, 2, \ldots$

Putting

$$\nu(t) = \begin{cases} \left(2\sin\frac{\delta}{2}\right)^{-1} & \text{for } t \in [0, \delta] \\ \left(2\sin\frac{t}{2}\right)^{-1} & \text{for } t \in [\delta, \pi] \end{cases},$$

and $p_n = n + \frac{1}{2}$ with φ_x instead of g we obtain

COROLLARY 3. *If $f \in L^1$ then for all real x and $\delta \in (0, \pi]$,*

$$\frac{1}{\pi}\int_\delta^\pi \varphi_x(t)\,\frac{\sin\frac{(2n+1)t}{2}}{2\sin\frac{t}{2}}\,dt \le K\frac{1}{\delta}\left(1 + \|f\|_{L^1}\right)\omega_{L^1} f\left(\frac{\pi}{n+1}\right),$$

where $n = 0, 1, 2, \ldots$

3. Proofs of the results

3.1. Proof of Theorem 1. Let as usually

$$H_n^2 f(x) = \left\{\frac{1}{n+1}\sum_{k=0}^{n}\left|\frac{1}{\pi}\int_0^\pi \varphi_x(t)\,\frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}\,dt\right|^2\right\}^{1/2} \le A_n + B_n + C_n,$$

where

$$A_n = \left\{\frac{1}{n+1}\sum_{k=0}^{n}\left|\frac{1}{\pi}\int_0^{\delta_n} \varphi_x(t)\,\frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}\,dt\right|^2\right\}^{1/2},$$

$$B_n = \left\{\frac{1}{n+1}\sum_{k=0}^{n}\left|\frac{1}{\pi}\int_{\delta_n}^{\gamma_n} \varphi_x(t)\,\frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}\,dt\right|^2\right\}^{1/2}$$

and

$$C_n = \left\{\frac{1}{n+1}\sum_{k=0}^{n}\left|\frac{1}{\pi}\int_{\gamma_n}^{\pi} \varphi_x(t)\,\frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}\,dt\right|^2\right\}^{1/2},$$

with $\delta_n = \frac{\pi}{n+1}$ and $\gamma_n = \sqrt[4]{(n+1)^{-1}\sum_{k=0}^{n}\left[\omega_{L^1} f\left(\frac{\pi}{k+1}\right)\right]^2}$.

Then

$$(n+1)(A_n)^2 \le \frac{(n+1)^3}{4\pi^2}\left|\int_0^{\delta_n} |\varphi_x(t)|\,dt\right|^2 = \frac{n+1}{4}\left[w_x^1 f\left(\frac{\pi}{n+1}\right)\right]^2$$
$$\le \frac{n+1}{4}\left[w_x^{\log} f(\gamma_n)\right]^2 \quad \text{for } \delta_n \le \gamma_n.$$

For the quantity B_n we construct the special estimate

$$(n+1)B_n^2 \le \frac{1}{\pi^2}\int_{\delta_n}^{\gamma_n}\int_{\delta_n}^{\gamma_n} \frac{\varphi_x(u)\,\varphi_x(v)}{8\sin\frac{u}{2}\sin\frac{v}{2}}$$
$$\times \sum_{\nu=0}^{n}\left[\cos\left(\nu+\frac{1}{2}\right)(u-v) - \cos\left(\nu+\frac{1}{2}\right)(u+v)\right] dudv$$
$$= \frac{2}{8\pi^2}\int_{\delta_n}^{\gamma_n}\int_{\delta_n}^{u} \frac{\varphi_x(u)\,\varphi_x(v)}{\sin\frac{u}{2}\sin\frac{v}{2}}$$
$$\times \sum_{\nu=0}^{n}\left[\cos\nu(u-v)\cos\frac{u-v}{2} - \sin\nu(u-v)\sin\frac{u-v}{2}\right.$$
$$\left. + -\cos\nu(u+v)\cos\frac{u+v}{2} + \sin\nu(u+v)\sin\frac{u+v}{2}\right] dudv$$
$$= \frac{1}{4\pi^2}\int_{\delta_n}^{\gamma_n}\int_{\delta_n}^{u} \frac{\varphi_x(u)\,\varphi_x(v)}{\sin\frac{u}{2}\sin\frac{v}{2}}\left[\left(\frac{1}{2} + \frac{\sin\frac{(2n+1)(u-v)}{2}}{2\sin\frac{u-v}{2}}\right)\cos\frac{u-v}{2}\right.$$
$$- \left(\frac{1}{2}\cot\frac{u-v}{2} - \frac{\cos\frac{(2n+1)(u-v)}{2}}{2\sin\frac{u-v}{2}}\right)\sin\frac{u-v}{2}$$
$$+ \left(\frac{1}{2} + \frac{\sin\frac{(2n+1)(u+v)}{2}}{2\sin\frac{u+v}{2}}\right)\cos\frac{u+v}{2}$$
$$\left. - \left(\frac{1}{2}\cot\frac{u+v}{2} - \frac{\cos\frac{(2n+1)(u+v)}{2}}{2\sin\frac{u+v}{2}}\right)\sin\frac{u+v}{2}\right] dudv$$
$$= \frac{1}{4\pi^2}\int_{\delta_n}^{\gamma_n}\int_{\delta_n}^{u} \frac{\varphi_x(u)\varphi_x(v)}{\sin\frac{u}{2}\sin\frac{v}{2}}\left[\frac{\sin(n+1)(u-v)}{2\sin\frac{u-v}{2}} + \frac{\sin(n+1)(u+v)}{2\sin\frac{u+v}{2}}\right] dudv$$
$$= B_n' + B_n''.$$

If we observe that (cf. [8] p. 78)

$$\frac{1}{v(u-v)} = \left|\frac{1}{v(u-v)}\right| = \frac{1}{u}\left|\frac{1}{u-v} + \frac{1}{v}\right| \le \left|\frac{1}{u(u-v)}\right| + \frac{1}{uv}$$

then

$$\left|B_n'\right| \le \frac{1}{4\pi^2} \int\limits_{\delta_n}^{\gamma_n} \int\limits_{\delta_n}^{u} \frac{|\varphi_x(u)|\,|\varphi_x(v)|}{\frac{2}{\pi}\frac{u}{2}\frac{2}{\pi}\frac{v}{2}} \frac{|\sin(n+1)(u-v)|}{2\frac{2}{\pi}\frac{u-v}{2}} dudv$$

$$= \frac{\pi}{8} \int\limits_{\delta_n}^{\gamma_n} \frac{|\varphi_x(u)|}{u} \int\limits_{\delta_n}^{u} |\varphi_x(v)|\,|\sin(n+1)(u-v)| \frac{1}{v(u-v)} dudv$$

$$\le \frac{\pi}{8} \int\limits_{\delta_n}^{\gamma_n} \frac{|\varphi_x(u)|}{u} \int\limits_{\delta_n}^{u} \frac{|\varphi_x(v)|}{u} \left(\frac{|\sin(n+1)(u-v)|}{u-v} + \right.$$

$$\left. + \frac{|\sin(n+1)(u-v)|}{v} \right) dudv$$

$$\le \frac{\pi}{8} \int\limits_{\delta_n}^{\gamma_n} \frac{|\varphi_x(u)|}{u} \int\limits_{\delta_n}^{u} \frac{|\varphi_x(v)|}{u} \left(\frac{(n+1)(u-v)}{u-v} + \frac{1}{\delta_n} \right) dudv$$

$$\le \frac{\pi(n+1)}{8} \int\limits_{\delta_n}^{\gamma_n} \frac{|\varphi_x(u)|}{u^2} \int\limits_{\delta_n}^{u} |\varphi_x(v)|\, dudv$$

and

$$\left|B_n''\right| \le \frac{1}{4\pi^2} \int\limits_{\delta_n}^{\gamma_n} \int\limits_{\delta_n}^{u} \frac{|\varphi_x(u)|\,|\varphi_x(v)|}{\frac{2}{\pi}\frac{u}{2}\frac{2}{\pi}\frac{v}{2}} \frac{|\sin(n+1)(u+v)|}{2\frac{2}{\pi}\frac{u+v}{2}} dudv$$

$$\le \frac{\pi}{8} \int\limits_{\delta_n}^{\pi/2} \int\limits_{\delta_n}^{u} \frac{|\varphi_x(u)|\,|\varphi_x(v)|}{u^2 v} dudv$$

$$\le \frac{\pi(n+1)}{8} \int\limits_{\delta_n}^{\pi/2} \frac{|\varphi_x(u)|}{u^2} \int\limits_{\delta_n}^{u} |\varphi_x(v)|\, dudv.$$

Consequently,

$$B_n \le \frac{\pi(n+1)}{4} \int\limits_{\delta_n}^{\gamma_n} \frac{|\varphi_x(u)|}{u^2} \int\limits_{\delta_n}^{u} |\varphi_x(v)|\, dudv$$

$$\le \frac{\pi(n+1)}{4} \int\limits_{\delta_n}^{\gamma_n} \frac{|\varphi_x(t)|\, w_x^{\log} f(t)}{t \log^{\alpha} 1/t} dt$$

$$\le \frac{\pi(n+1)}{4} w_x^{\log} f(\gamma_n) \int\limits_{m_n}^{n+1} \frac{|\varphi_x(\pi/u)|}{u \log^{\alpha} u/\pi} du$$

$$= K(n+1)\, w_x^{\log} f(\gamma_n) \int\limits_{\pi/(n+1)}^{\pi/m_n} \frac{|\varphi_x(t)|}{t \log^{\alpha} 1/t} dt$$

$$= K\,(n+1)\,w_x^{\log} f(\gamma_n) \left\{ \left[\frac{w_x f(t)}{\log^\alpha 1/t} \right]_{\pi/(n+1)}^{\pi/m_n} + \right.$$

$$\left. + \int_{\pi/(n+1)}^{\pi/m_n} \frac{w_x^{\log} f(t)}{t \log^{2\alpha} 1/t} dt - \alpha \int_{\pi/(n+1)}^{\pi/m_n} \frac{w_x^{\log} f(t)}{t \log^{2\alpha+1} 1/t} dt \right\}$$

$$\leq K\,(n+1) \left[w_x^{\log} f(\gamma_n) \right]^2 \left\{ \frac{1}{\log^\alpha m_n/\pi} + \int_{\pi/(n+1)}^{\pi/m_n} \frac{1}{t \log^{2\alpha} 1/t} dt \right\}$$

$$\leq K\,(n+1) \left[w_x^{\log} f(\gamma_n) \right]^2,$$

where $m_n = [\pi/\gamma_n]$.

Finally, by Corollary 3,

$$(n+1)\,(C_n)^2 \leq \sum_{k=0}^{n} \left[K \frac{1}{\gamma_n} \left(1 + \|f\|_{L^1}\right) \omega_{L^1} f \left(\frac{\pi}{k+1} \right) \right]^2$$

$$\leq K \frac{1}{\gamma_n^2} \left(1 + \|f\|_{L^1}\right)^2 \sum_{k=0}^{n} \left[\omega_{L^1} f \left(\frac{\pi}{k+1} \right) \right]^2,$$

with $\gamma_n = \sqrt[4]{(n+1)^{-1} \sum_{k=0}^{n} [\omega_{L^1} f(\frac{\pi}{k+1})]^2}$.

Collecting our estimates we obtain the thesis in the case $\delta_n \leq \gamma_n$.

If $\gamma_n \leq \delta_n$ then, again by Corollary 3, we get

$$(n+1)(H_n^2 f(x))^2 \leq (n+1)(A_n)^2 + K \frac{1}{\delta_n^2} (1 + \|f\|_{L^1})^2 \sum_{k=0}^{n} \left[\omega_{L^1} f \left(\frac{\pi}{k+1} \right) \right]^2$$

$$\leq \frac{n+1}{4} [w_x^1 f(\delta_n)]^2 + K \frac{1}{\gamma_n^2} (1 + \|f\|_{L^1})^2 \sum_{k=0}^{n} \left[\omega_{L^1} f \left(\frac{\pi}{k+1} \right) \right]^2$$

and our thesis follows.

3.2. Proof of Theorem 2. Let

$$H_n^q f\,(x) = \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left| \frac{1}{\pi} \int_0^\pi \varphi_x\,(t)\, \frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}} dt \right|^q \right\}^{1/q} \leq A_n + D_n,$$

where

$$A_n = \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left| \frac{1}{\pi} \int_0^{\delta_n} \varphi_x\,(t)\, \frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}} dt \right|^q \right\}^{1/q} = \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |T_k|^q \right\}^{1/q}$$

and

$$D_n = \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left| \frac{1}{\pi} \int_{\delta_n}^{\pi} \varphi_x\,(t)\, \frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}} dt \right|^q \right\}^{1/q}$$

with some $\delta_n > 0$.

Applying estimate

$$\pi T_k \le \left| \frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}} \int_0^t \varphi_x(u)\,du \right|_0^{\delta_n} + \left| \int_0^{\delta_n} \frac{d}{dt}\left(\frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}\right) \int_0^t \varphi_x(u)\,dudt \right|$$

$$\le 2w_x^\star f(\delta_n) + \left| \frac{1}{2\pi} \int_0^{\delta_n} \frac{\frac{2k+1}{2}\cos\frac{2k+1}{2}t \sin\frac{1}{2}t - \frac{1}{2}\cos\frac{1}{2}t \sin\frac{2k+1}{2}t}{(\sin\frac{1}{2}t)^2} \int_0^t \varphi_x(u)dudt \right|$$

$$\le 2w_x^\star f(\delta_n) + \frac{\pi}{2}\int_0^{\delta_n} \frac{k+1}{t}\left|\int_0^t \varphi_x(u)\,du\right| dt \le 2w_x^\star f(\delta_n)$$

$$+ + \frac{\pi(k+1)}{2}\int_0^{\delta_n} w_x^\star f(t)\,dt$$

$$\le 2w_x^\star f(\delta_n) + \frac{\pi^2(k+1)}{2}\int_{\pi/\delta_n}^{\infty} \frac{1}{u^2} w_x^\star f\left(\frac{\pi}{u}\right) du$$

$$\le 2w_x^\star f(\delta_n) + \frac{\pi^2(k+1)}{2}\sum_{k=[\pi/\delta_n]}^{\infty} \int_k^{k+1} \frac{w_x^\star f\left(\frac{\pi}{u}\right)}{u^2} du$$

$$\le 2w_x^\star f(\delta_n) + K(k+1)\sum_{k=[\pi/\delta_n]}^{\infty} \frac{w_x^\star f\left(\frac{\pi}{k}\right)}{k^2}$$

we obtain immediately that

$$A_n \le K\left\{(n+1)\sum_{k=[\pi/\delta_n]}^{\infty} \frac{w_x^\star f\left(\frac{\pi}{k+1}\right)}{(k+1)^2}\right\}.$$

The second term we give as a sum

$$D_n \le \left\{\frac{1}{n+1}\sum_{k=0}^{n}\left|\frac{1}{\pi}\int_{\delta_n}^{\pi} \varphi_x(t)\frac{\cos\frac{t}{2}}{2\sin\frac{t}{2}}\sin ktdt\right|^q\right\}^{1/q}$$

$$+\left\{\frac{1}{n+1}\sum_{k=0}^{n}\left|\frac{1}{\pi}\int_{\delta_n}^{\pi} \frac{\varphi_x(t)}{2}\cos ktdt\right|^q\right\}^{1/q}$$

$$= P_n + R_n,$$

and using the Hausdorff-Young inequality we obtain

$$R_n \le K(n+1)^{-\frac{1}{q}}\left\{\int_{\delta_n}^{\pi} |\varphi_x(t)|^p\,dt\right\}^{1/p} \le K(n+1)^{-\frac{1}{q}} w_x^p f(\pi)$$

$$\leq K\left\{(n+1)^{1-p}\left(w_x^p f(\pi)\right)^p\right\}^{\frac{1}{p}} \leq K\left\{(n+1)^{1-p}\sum_{k=0}^{[\pi/\delta_n]}\frac{\left(w_x^p f\left(\frac{\pi}{k+1}\right)\right)^p}{(k+1)^{2-p}}\right\}^{\frac{1}{p}}$$

and similarly

$$P_n \leq K(n+1)^{-\frac{1}{q}}\left\{\int_{\delta_n}^{\pi}\left|\frac{\varphi_x(t)}{t}\right|^p dt\right\}^{1/p}$$

$$\leq K\left\{(n+1)^{1-p}\sum_{k=0}^{[\pi/\delta_n]}\frac{\left(w_x^p f\left(\frac{\pi}{k+1}\right)\right)^p}{(k+1)^{2-p}}\right\}^{\frac{1}{p}},$$

by partial integration. Thus our result is proved.

References

[1] S. Aljančič, R. Bojanic and M. Tomić, *On the degree of convergence of Féjer-Lebesgue sums*, L'Enseignement Mathematique, Geneve, Tome XV (1969), 21–28.

[2] Hsiang Wen-Hang, *On the degree of convergence of Abel and conjugate Abel series*, Funct. Approx. Comment. Math. 12 (1982), 83–103.

[3] E. Hille, G. Klein, *Riemann's localization theorem for Fourier series*, Duke Math J. 21 (1954), 587–591.

[4] W. Łenski, *On the rate of pointwise strong (C,α) summability of Fourier series*, Colloq. Math. Soc. János Bolyai, 58 Approx.Theory, Kecskemét (Hungary), (1990), 453–486.

[5] W. Łenski, *On the rate of pointwise summability of Fourier series*, Applied Mathematics E-Notes 1 (2001), 143–148.

[6] Fu Traing Wang, *Note on H_2 summability of Fourier series*, J. London Math. Soc. 19 (1944), 208–209.

[7] Fu Traing Wang, *Strong summability of Fourier series*, Duke Math. J. 12 (1945), 77–87.

[8] A. A. Zakharov, *O silnoi sumirujemosti riadov Furie*, Sibirsk. Mat. Zh. 23 No 5 (1982), 92–99.

UNIVERSITY OF ZIELONA GÓRA
INSTITUTE OF MATHEMATICS
ul. Szafrana 4a
65-516 ZIELONA GÓRA, POLAND

Received October 21, 2003.