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GENERALIZED I-NONEXPANSIVE MAPS AND BEST APPROXIMATIONS IN BANACH SPACES

Abstract. A noncommutative version of a best approximation result for generalized I-nonexpansive maps is obtained.

Let $E=(E, \|.\|)$ be a Banach space and C a subset of E. Let $T, I : E \to E$. Then T is called I-nonexpansive on C if $\|Tx - Ty\| \leq \|Ix - Iy\|$ for all $x, y \in C$. The set of fixed points of T (resp. I) is denoted by $F(T)$ (resp. $F(I)$). The set C is called p-starshaped with $p \in C$ if the segment $[x,p]$ joining x to p is contained in C for all $x \in C$ (that is, $kx + (1-k)p \in C$ for all $x \in C$ and all k with $0 \leq k \leq 1$). The mappings T and I are said to be:

(1) commuting on C if $ITx = TIx$ for all $x \in C$;

(2) R-weakly commuting on C [3] if there exists a real number $R > 0$ such that $\|TIx - ITx\| \leq R\|Tx - Ix\|$ for all $x \in C$.

Suppose C is p-starshaped with $p \in F(I)$ and is both T- and I-invariant. Then T and I are called

(3) R-subcommuting on C [6, 7] if there exists a real number $R > 0$ such that $\|TIx - ITx\| \leq (R/k)\|(kTx + (1-k)p) - Ix\|$ for all $x \in C$ and all $k \in (0,1)$. Clearly commutativity implies R-subcommutativity but the converse is not true in general (see, [6]).

The set $P_C(\hat{x}) = \{y \in C : \|y - \hat{x}\| = dist(\hat{x}, C)\}$ is called the set of best approximants to $\hat{x} \in X$ out of C, where $dist(\hat{x}, C) = \inf\{\|y - \hat{x}\| : y \in C\}$.

The space E is said to satisfy Opial's condition [2] if for every sequence $\{x_n\} \subset E$ converging weakly to $y \in E$,

$$\liminf_{n\to\infty} \|x_n - y\| < \liminf_{n\to\infty} \|x_n - x\|$$

holds for all $x \neq y$.

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A map $T : C \to X$ is called demiclosed if the conditions $\{x_n\} \subseteq C$, $x_n \to x$ weakly, $Tx_n \to y$ strongly imply that $Tx = y$.

In 1995, Jungck and Sessa [1] proved the following result in best approximation theory using the concept of commuting maps.

THEOREM 1. *Let $E = (E, \|.\|)$ be a Banach space and $T, I : E \to E$. Let C be a subset of E such that $T(\partial C) \subset C$ and $\hat{x} \in F(T) \cap F(I)$. Suppose I is affine and continuous in the weak and strong topology on $P_C(\hat{x})$, T is I-nonexpansive on $P_C(\hat{x}) \cup \{\hat{x}\}$, and $I(P_C(\hat{x})) = P_C(\hat{x})$. If $P_C(\hat{x})$ is nonempty, weakly compact and p-starshaped with $p \in F(I)$, T and I are commuting on $P_C(\hat{x})$, and if E satisfies Opial's condition, then $P_C(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$.*

Recently, several results regarding invariant approximation for noncommuting maps have been obtained (see [4–8]). Shahzad [7] extended Theorem 1 to a class of noncommuting maps as follows.

THEOREM 2. *Let $E = (E, \|.\|)$ be a Banach space and $T, I : E \to E$. Let C be a subset of E such that $T(\partial C) \subset C$ and $\hat{x} \in F(T) \cap F(I)$. Suppose I is affine and continuous in the weak topology on $P_C(\hat{x})$, T is continuous in the strong topology on $P_C(\hat{x})$ and I-nonexpansive on $P_C(\hat{x}) \cup \{\hat{x}\}$, and $I(P_C(\hat{x})) = P_C(\hat{x})$. If $P_C(\hat{x})$ is nonempty, weakly compact and p-starshaped with $p \in F(I)$, T and I are R-subcommuting on $P_C(\hat{x})$ and if E satisfies Opial's condition, then $P_C(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$.*

The aim of this note is to show the validity of Theorem 2 for generalized I-nonexpansive maps.

We need the following lemma, which is a special case of Theorem 1 of Shahzad [4].

LEMMA 3. *Let (X, d) be a complete metric space and $T, I : X \to X$ R-weakly commuting mappings such that $T(X) \subset I(X)$. Suppose T is continuous. If there exists $k \in [0, 1)$ such that*

$$d(Tx, Ty) \leq k \max\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), \frac{1}{2}[d(Ix, Ty) + d(Iy, Tx)]\}$$

for all $x, y \in X$, then $F(T) \cap F(I)$ is a singleton.

The following result contains Theorem 6 of [1].

THEOREM 4. *Let $E = (E, \|.\|)$ be a Banach space and C a weakly compact subset of E. Let $T, I : C \to C$ be such that $T(C) \subset I(C)$. Suppose C is p-starshaped with $p \in F(I)$, I is affine and continuous in the weak topology, and T is continuous in the strong topology. If $I - T$ is demiclosed, T and I are R-subcommuting and satisfy*

$$\begin{aligned}||Tx - Ty|| \le \max\{&||Ix - Iy||, dist(Ix, [Tx, p]), dist(Iy, [Ty, p]),\\ &\frac{1}{2}[dist(Ix, [Ty, p]) + dist(Iy, [Tx, p])]\}\end{aligned}$$

for all $x, y \in C$, then $F(T) \cap F(I) \neq \emptyset$.

Proof. Choose a sequence $\{k_n\} \subset (0,1)$ with $k_n \to 1$ as $n \to \infty$. Define for each $n \ge 1$, a mapping T_n by $T_n x = k_n Tx + (1 - k_n)p$. Since C is p-starshaped and $T(C) \subset I(C)$, it follows that each T_n maps C into itself and $T_n(C) \subset I(C)$ for each n. Since I is affine, $p \in F(I)$, and T and I are R-subcommuting, we have

$$\begin{aligned}||T_n Ix - IT_n x|| &= k_n ||TIx - ITx|| \le R||(k_n Tx + (1 - k_n)p) - Ix||\\ &= R||T_n x - Ix||\end{aligned}$$

for all $x \in C$. Thus T_n and I are R-weakly commuting. Moreover,

$$\begin{aligned}||T_n x - T_n y|| &= k_n ||Tx - Ty||\\ &\le k_n \max\{||Ix - Iy||, dist(Ix, [Tx, p]), dist(Iy, [Ty, p]),\\ &\quad \frac{1}{2}[dist(Ix, [Ty, p]) + dist(Iy, [Tx, p])]\}\\ &\le k_n \max\{||Ix - Iy||, ||Ix - T_n x||, ||Iy - T_n y||,\\ &\quad \frac{1}{2}[||Ix - T_n y|| + ||Iy - T_n x||]\}\end{aligned}$$

for all $x, y \in C$. Furthermore C is complete since the weak topology is Hausdorff and C is weakly compact. Now Lemma 3 guarantees that $F(T_n) \cap F(I) = \{x_n\}$ for each n. Since C is weakly compact, there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to x_0 \in C$ weakly as $m \to \infty$. By the weak continuity of I, we have $x_0 \in F(I)$. Also

$$(I - T)x_m = (1 - (k_m)^{-1})(x_m - p).$$

This implies that $(I - T)x_m \to 0$ strongly as $m \to \infty$ since $\{x_m\}$ is bounded and $k_m \to 1$ as $m \to \infty$. Now the demiclosedness of $I - T$ guarantees that $(I - T)x_0 = 0$, that is, $Tx_0 = x_0$. Hence $F(T) \cap F(I) \neq \emptyset$.

The following theorem shows the validity of Theorem 2 for generalized I-nonexpansive maps.

THEOREM 5. *Let $E = (E, ||.||)$ be a Banach space and $T, I : E \to E$. Let C a subset of E such that $T(\partial C) \subset C$ and $\hat{x} \in F(T) \cap F(I)$. Suppose I is affine and continuous in the weak topology on $P_C(\hat{x})$, T is continuous in the strong topology on $P_C(\hat{x})$, and $I(P_C(\hat{x})) = P_C(\hat{x})$. If $P_C(\hat{x})$ is nonempty, weakly compact and p-starshaped with $p \in F(I)$, T and I are R-subcommuting on*

$P_C(\hat{x})$, $I - T$ is demiclosed, and if T and I satisfy for all $x \in P_C(\hat{x}) \cup \{\hat{x}\}$

$$||Tx - Ty|| \leq \begin{cases} ||Ix - I\hat{x}|| & \text{if } y = \hat{x} \\ \max\{||Ix - Iy||, dist(Ix, [Tx, p]), dist(Iy, [Ty, p]), \\ \frac{1}{2}[dist(Ix, [Ty, p]) + dist(Iy, [Tx, p])]\} & \text{if } y \in P_C(\hat{x}), \end{cases}$$

then $P_C(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

Proof. Let $x \in P_C(\hat{x})$. Then $x \in \partial C$ and so $Tx \in C$ since $T(\partial C) \subset C$. Now

$$||Tx - \hat{x}|| = ||Tx - T\hat{x}|| \leq ||Ix - I\hat{x}|| = ||Ix - \hat{x}|| = dist(\hat{x}, C).$$

This shows that $Tx \in P_C(\hat{x})$. Consequently, $T(P_C(\hat{x})) \subset P_C(\hat{x}) = I(P_C(\hat{x}))$. Now Theorem 4 guarantees that $P_C(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

References

[1] G. Jungck and S. Sessa, *Fixed point theorems in best approximation theory*, Math. Japonica 42 (1995), 249–252.

[2] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. 73 (1967), 591–597.

[3] R. P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl. 188 (1994), 436–440.

[4] N. Shahzad, *Invariant approximations, generalized I-contractions and R-subweakly commuting maps*, submitted.

[5] N. Shahzad, *Invariant approximation and R-subweakly commuting maps*, J. Math. Anal. Appl. 257 (2001), 39–45.

[6] N. Shahzad, *Noncommuting maps and best approximations*, Radovi Mat. 10 (2001), 77–83.

[7] N. Shahzad, *On R-subcommuting maps and best approximations in Banach spaces*, Tamkang J. Math. 32 (2001), 51–53.

[8] N. Shahzad, *A result on best approximation*, Tamkang J. Math. 29 (1998), 223–226; Corrections 30 (1999), 165.

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