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ON GENERALIZED SINE AND COSINE FUNCTIONS, II

1. In [1] we have presented some addition formulas for functions f, g which satisfy the functional equation

$$(f(x))^n + (g(x))^n = 1 \tag{I}$$

in the case where $n \in N$ is fixed and even; f, g are supposed to be real functions on a given group $(X, +)$, N being the set of all positive integers. Those addition formulas coincide with the well known representations of $\cos(x+y)$ and $\sin(x+y)$ in the case where $(X,+) = (\mathfrak{R}, +)$ and $n = 2$, where $\mathfrak{R}$ stands for the set of all real numbers. In the present paper we consider analogous formulas in the case where $n \in N$ is odd. However, we were able to do this assuming that the domain yields a 2-divisible Abelian group (in [1] we had no assumption whatsoever upon the given group). But even in this situation it turns out that some "strange" (=unexpected) solutions appear which was not the case where n was supposed to be even.

2. In the sequel we denote the set of all integers by Z and the set of all complex numbers by C. The symbol $(T, \cdot)$ will stand for the multiplicative group of the unit circle.

Let $(X, +)$ be a uniquely 2-divisible Abelian group. By $H(X, T)$ we denote the family of all homomorphisms between X and T. In the sequel, for any functions $f, g : X \to \mathfrak{R}$ we define the function $w : X^2 \to \mathfrak{R}$ by the formula:

$$w(x,y) = \sqrt[n]{(f(x)f(y) - g(x)g(y))^n + (f(x)g(y) + f(y)g(x))^n}, \quad (x,y) \in X^2. \tag{1}$$

Moreover, if f, g do not vanish simultaneously, that is $f(x)^2 + g(x)^2 > 0$ for all $x \in X$, then $m : X \to C$ will denote the function such that

$$m(x) = f(x) + ig(x), \quad x \in X. \tag{2}$$

With the aid of this function we define a mapping $k : X \to T$ by the formula:

$$k(x) = \frac{(m(x))^2}{|m(x)|^2}, \quad x \in X. \tag{3}$$

Moreover, we put $z_1 := \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$ and $z_2 := -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$.

LEMMA 1. *If functions $f, g : X \to \mathfrak{R}$ do not vanish simultaneously and $x, y \in X$, then the following conditions are pairwise equivalent:*

$w(x,y) = 0$;

$\dfrac{m(x)\ m(y)}{|m(x)|\ |m(y)|} \in \{z_1, z_2\}$;

$k(x)k(y) = -i$.

Proof. Since n is odd we have

$$w(x,y) = 0 \Leftrightarrow f(x)f(y) - g(x)g(y) = -(f(x)g(y) + f(y)g(x)) \Leftrightarrow$$
$$\mathrm{Re}\left(\frac{m(x)\ m(y)}{|m(x)|\ |m(y)|}\right) = -\mathrm{Im}\left(\frac{m(x)\ m(y)}{|m(x)|\ |m(y)|}\right) \Leftrightarrow \frac{m(x)\ m(y)}{|m(x)|\ |m(y)|} \in \{z_1, z_2\}$$
$$\Leftrightarrow k(x)k(y) = -i.$$

LEMMA 2. *If functions $f, g : X \to \mathfrak{R}$ do not vanish simultaneously and satisfy the following conditional system of functional equations*

(II) $w(x,y) \neq 0$ *implies:*

$$f(x+y) = \frac{f(x)f(y) - g(x)g(y)}{w(x,y)},$$
$$g(x+y) = \frac{f(x)g(y) + f(y)g(x)}{w(x,y)}$$

for all $x, y \in X$ and $g(0) = 0$, then (I) *holds for every $x \in X$ such that $k(x) \neq -i$. Moreover $f(0) = 1 = k(0)$.*

Proof. Since $g(0) = 0$, we have $k(0) = 1$. Hence, if $k(x) \neq -i$ for any $x \in X$, then $k(x)k(0) \neq -i$, and, consequently, $w(x,0) \neq 0$ by Lemma 1. Now, system (II) implies (I). Therefore, $k(0) = 1 \neq -i$ implies $(f(0))^n + (g(0))^n = 1$, whence $f(0) = 1$.

On account of R. Ger's Theorem 3 from [3] (cf. also J. G. Dhombres, R. Ger [2]) we get the following

COROLLARY 1. *If the function $s : X \to T$ satisfies the conditional functional equation*

(III) $\quad s(x)s(y) \neq -i \quad$ *implies* $\quad s(x)s(y) = s(x+y)$

for all $x, y \in X$ *and* $s(0) = 1$, *then* $s \in H(X,T)$ *or*

$$s(x) = \begin{cases} 1 & \text{for } x \in G \\ z_1 & \text{for } x \in X \backslash G \end{cases} \quad \text{or} \quad s(x) = \begin{cases} 1 & \text{for } x \in G \\ z_2 & \text{for } x \in X \backslash G, \end{cases}$$

where $(G,+)$ *is an arbitrary subgroup of* $(X,+)$.

THEOREM. *Let* $(X,+)$ *be a uniquely 2-divisible Abelian group and let an odd* $n \in N$ *be fixed. Suppose that two functions* $f, g : X \to \Re$ *do not vanish simultaneously. Then* f, g *satisfy the conditional system* (II) *of functional equations on* X *and* $g(0) = 0$ *if and only if* f, g *have one of the following forms*:

(4) *there exists a homomorphism* $h \in H(X,T)$ *with* $u = Re h$, $v = Im\, h$ *such that*:

(i) $$f(x) = \frac{u(x)}{\sqrt[n]{(u(x))^n + (v(x))^n}}, \quad g(x) = \frac{v(x)}{\sqrt[n]{(u(x))^n + (v(x))^n}},$$

for all $x \in X \backslash h^{-1}(\{z_1, z_2\})$ *and*

(ii) $f(x) = -g(x) \neq 0$ *for all* $x \in h^{-1}(\{z_1, z_2\})$;

(5) $$f(x) = \begin{cases} 1 & \text{for } x \in G \\ \dfrac{1}{\sqrt[n]{1 + (1 - \sqrt{2})^n}} & \text{for } x \in X \backslash G, \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{for } x \in G \\ \dfrac{1 - \sqrt{2}}{\sqrt[n]{1 + (1 - \sqrt{2})^n}} & \text{for } x \in X \backslash G; \end{cases}$$

(6) $$f(x) = \begin{cases} 1 & \text{for } x \in G \\ \dfrac{1}{\sqrt[n]{1 + (1 + \sqrt{2})^n}} & \text{for } x \in X \backslash G, \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{for } x \in G \\ \dfrac{1 + \sqrt{2}}{\sqrt[n]{1 + (1 + \sqrt{2})^n}} & \text{for } x \in X \backslash G; \end{cases}$$

where $(G,+)$ *is an arbitrary subgroup of* $(X,+)$.

Proof. Necessity. From Lemma 1 it follows that, if $x, y \in X$ and $k(x)k(y) \neq -i$, then $w(x,y) \neq 0$. On account of (II) one has

$$k(x+y) = \frac{(m(x+y))^2}{|m(x+y)|^2} = \frac{(f(x+y) + ig(x+y))^2}{(f(x+y))^2 + (g(x+y))^2}$$

$$= \frac{((f(x)f(y) - g(x)g(y)) + i(f(x)g(y) + f(y)g(x)))^2}{(f(x)f(y) - g(x)g(y))^2 + (f(x)g(y) + f(y)g(x))^2}$$
$$= \frac{m(x)m(y)}{|m(x)|^2|m(y)|^2} = k(x)k(y).$$

Hence k satisfies the conditional functional equation (III) on X. Moreover, by Lemma 2, we have $k(0) = 1$. Now, Corollary 1 implies that k has one of the following forms:

(7) $k \in H(X, T)$;

(8) $k(x) = \begin{cases} 1 & \text{for } x \in G \\ z_1 & \text{for } x \in X \backslash G; \end{cases}$

(9) $k(x) = \begin{cases} 1 & \text{for } x \in G \\ z_2 & \text{for } x \in X \backslash G; \end{cases}$

where $(G, +)$ is an arbitrary subgroup of $(X, +)$.

Assume (7). Let $h : X \to T$ be a function defined by the formula

$$h(x) = k\left(\frac{x}{2}\right), \quad x \in X.$$

Since $(X, +)$ is uniquely 2-divisible Abelian group, we have $h \in H(X, T)$, as well. Therefore

$$(h(x))^2 = h(2x) = k(x) = \frac{(m(x))^2}{|m(x)|^2}, \quad x \in X,$$

and hence

$$h(x) = \frac{m(x)}{|m(x)|} \quad \text{or} \quad h(x) = -\frac{m(x)}{|m(x)|} \tag{10}$$

for all $x \in X$. Put $u = \text{Re}h$, $v = \text{Im}h$. Condition (10) implies that for every $x \in X$ one has

$$f(x) = |m(x)|u(x) \quad \text{and} \quad g(x) = |m(x)|v(x) \tag{11}$$

or

$$f(x) = -|m(x)|u(x) \quad \text{and} \quad g(x) = -|m(x)|v(x). \tag{12}$$

Let $x \in X$ be arbitrarily fixed. Assume that $h(x) \notin \{z_1, z_2\}$. Then $k(x) = (h(x))^2 \neq -i$. By Lemma 2, equation (I) holds true. Moreower, from the assumption that $h(x) \notin \{z_1, z_2\}$ it follows that $u(x) \neq -v(x)$, and, consequently, $\sqrt[n]{(u(x))^n + (v(x))^n} \neq 0$. Hence, if (11) holds, then, by (I), we have

$$1 = (f(x))^n + (g(x))^n = |m(x)|^n((u(x))^n + (v(x))^n)$$

and therefore

$$|m(x)| = \frac{1}{\sqrt[n]{(u(x))^n + (v(x))^n}}.$$

Consequently, $f(x), g(x)$ have form (i). Analogously, if (12) holds true, then

$$|m(x)| = \frac{-1}{\sqrt[n]{(u(x))^n + (v(x))^n}}$$

and therefore condition (i) is satisfied as well.

Now assume that $h(x) \in \{z_1, z_2\}$. Then $u(x) = -v(x)$, whence, by (11) or (12), we obtain $f(x) = -g(x)$. Obviously $f(x) = -g(x) \neq 0$, because f, g do not vanish simultaneously. Consequently, condition (ii) holds true.

Thus, we have proved that condition (4) is satisfied provided k is of the form (7).

Suppose now that k has the form (8), where $(G, +)$ is an arbitrarily fixed subgroup of $(X, +)$. Since $k(x) \neq -i$, f, g satisfy equation (I) for all $x \in X$, because of Lemma 2. Moreover, for every $x \in G$ we have

$$1 = k(x) = \mathrm{Re}k(x) = \frac{(f(x))^2 - (g(x))^2}{|m(x)|^2} \tag{13}$$

and

$$0 = \mathrm{Im}k(x) = \frac{2f(x)g(x)}{|m(x)|^2}. \tag{14}$$

From (13) it follows that $f(x) \neq 0$ for all $x \in G$. This jointly with (14) implies $g(x) = 0$ for all $x \in G$. Hence, by (I), we have $f(x) = 1$ for all $x \in G$.

However, for every $x \in X \backslash G$ we get

$$\frac{\sqrt{2}}{2} = \mathrm{Re}k(x) = \frac{(f(x))^2 - (g(x))^2}{|m(x)|^2} \tag{15}$$

and

$$-\frac{\sqrt{2}}{2} = \mathrm{Im}k(x) = \frac{2f(x)g(x)}{|m(x)|^2}. \tag{16}$$

Conditions (15) and (16) imply that

$$g(x) = (1 - \sqrt{2})f(x), \quad x \in X \backslash G. \tag{17}$$

By (17) and (I) we obtain

$$f(x) = \frac{1}{\sqrt[n]{1 + (1 - \sqrt{2})^n}} \quad \text{and} \quad g(x) = \frac{1 - \sqrt{2}}{\sqrt[n]{1 + (1 - \sqrt{2})^n}}, \quad x \in X \backslash G.$$

Thus we have proved that condition (5) is satisfied provided k is of form (8). Analogously, one may show that condition (6) holds true whenever k is of form (9).

Sufficiency. Assume that f, g satisfy condition (4). Put

$$Y := h^{-1}(\{z_1, z_2\}).$$

Obviously $0 \notin Y$. Therefore, by (i), we get $g(0) = 0$.

Let a pair $(x, y) \in X^2$ be arbitrarily fixed. Since $(X, +)$ is Abelian group, it suffices to consider merely one of the following five cases:

(18) $x, y, x + y \in X;$

(19) $x, y \in X\backslash Y, \ \ x + y \in Y;$

(20) $x, y \in Y, \ \ x + y \in X\backslash Y;$

(21) $x \in Y, \ \ y, x + y \in X\backslash Y;$

(22) $x \in X\backslash Y, \ \ y, x + y \in Y.$

Assume (18); then we have $u(x) \neq -v(x), u(y) \neq -v(y)$ and $u(x + y) \neq -v(x + y)$. Therefore, by (i) we get

$$(w(x,y))^n = \frac{(u(x+y))^n + (v(x+y))^n}{((u(x))^n + (v(x))^n)((u(y))^n + (v(y))^n)} \neq 0,$$

whence

$$\frac{f(x)f(y) - g(x)g(y)}{w(x,y)} = \frac{u(x+y)}{\sqrt[n]{(u(x+y))^n + (v(x+y))^n}} = f(x+y)$$

and

$$\frac{f(x)g(y) + f(y)g(x)}{w(x,y)} = \frac{v(x+y)}{\sqrt[n]{(u(x+y))^n + (v(x+y))^n}} = g(x+y),$$

which states that system (II) is satisfied.

Condition (19) implies that $u(x) \neq -v(x), u(y) \neq -v(y)$ and $u(x+y) = -v(x + y)$. Hence, in this case $w(x, y) = 0$ and, therefore, (II) is trivially satisfied.

Assuming (20), by means of (ii), we obtain $f(x) = -g(x) \neq 0$ and $f(y) = -g(y) \neq 0$.

Consequently, $(w(x,y))^n = -(2f(x)f(y))^n \neq 0$, $\dfrac{f(x)f(y) - g(x)g(y)}{w(x,y)} = 0$ and $\dfrac{f(x)g(y) + f(y)g(x)}{w(x,y)} = 1$. Moreover, in this case $u(x) = -v(x), u(y) = -v(y)$, whence, by (i) we get

$$f(x+y) = \frac{u(x+y)}{\sqrt[n]{(u(x+y))^n + (v(x+y))^n}} = \frac{u(x)u(y) - v(x)v(y)}{\sqrt[n]{(u(x+y))^n + (v(x+y))^n}} = 0.$$

Thus $u(x+y)=0$, and, therefore, by (i), we obtain $g(x+y)=1$. This shows that (II) holds true.

Assume that condition (21) is satisfied. Then, by (ii), one has $f(x) = -g(x) \neq 0$, and, consequently, we have

$$(w(x,y))^n = (f(x))^n((f(y)+g(y))^n - (f(y)-g(y))^n).$$

Observe, that $w(x,y) \neq 0$ if and only if $g(y) \neq 0$. Hence, if $w(x,y) \neq 0$, then

$$\frac{f(x)f(y)-g(x)g(y)}{w(x,y)} = \frac{f(y)+g(y)}{\sqrt[n]{(f(y)+g(y))^n-(f(y)-g(y))^n}}$$

and

$$\frac{f(x)g(y)+f(y)g(x)}{w(x,y)} = \frac{-(f(y)-g(y))}{\sqrt[n]{(f(y)+g(y))^n-(f(y)-g(y))^n}}.$$

Moreover, from (21) we deduce that $u(x) = -v(x) \neq 0$, whence, by (i)

$$\begin{aligned} f(x+y) &= \frac{u(x+y)}{\sqrt[n]{(u(x+y))^n+(v(x+y))^n}} \\ &= \frac{u(y)+v(y)}{\sqrt[n]{(u(y)+v(y))^n-(u(y)-v(y))^n}} \\ &= \frac{f(y)+g(y)}{\sqrt[n]{(f(y)+g(y))^n-(f(y)-g(y))^n}}. \end{aligned}$$

Analogously, we obtain

$$g(x+y) = \frac{-(f(y)-g(y))}{\sqrt[n]{(f(y)+g(y))^n-(f(y)-g(y))^n}}.$$

This proves that (21) implies (II).

Finally, suppose that (22) holds. Then $u(y) = -v(y) \neq 0$, and $u(x+y) = -v(x+y) \neq 0$. These conditions imply that $u(x)+v(x) = u(x)-v(x)$, and, consequently, $v(x)=0$. By (i), we get $g(x)=0$. Moreover, by (ii), we have $f(y) = -g(y) \neq 0$. Therefore $w(x,y)=0$ and (II) is trivially satisfied.

Thus, the sufficiency has been proved whenever (4) is fulfilled. It remains to prove the sufficiency in the case where f, g have one of forms (5) or (6), where $(G,+)$ is an arbitrarily fixed subgroup of $(X,+)$. Obviously, this implies the equality $g(0)=0$.

Let a pair $(x,y) \in X^2$ be arbitrarily fixed. Since $(X,+)$ is an Abelian group, it suffices to consider one of the following four cases:

(23) $x, y, x+y \in G$;

(24) $x \in G, y, x+y \in X \setminus G$;

(25) $x, y \in X \setminus G, x+y \in G$;

(26) $x, y, x+y \in X \setminus G$.

Both (23) and (24) imply the equality $w(x,y)=1$, and some easy calculations show that (II) holds true in these cases. In cases (25) and (26) we have $w(x,y)=0$ and, consequently, (II) is satisfied, as well.

This completes the proof.

COROLLARY 2. *Let* $(X,+)$ *be a uniquely 2-divisible Abelian group and let an odd* $n \in N$ *be fixed. If function* $f,g : X \to \Re$ *do not vanish simultaneously and satisfy the conditional system* (II) *on* X *with* $g(0)=0$, *then equation* (I) *is satisfied for every* $x \in X$ *such that* $f(x) \neq -g(x)$.

Proof. Note that, if $x \in X$, then $k(x)=-i$ if and only if $f(x)=-g(x)$. The assertion follows now from Lemma 2.

EXAMPLE. Let $(X,+)$ be a uniquely 2-divisible Abelian group and let an odd $n \in N$ be fixed. Let $a : X \to \Re$ be an additive function.

Put $Y := a^{-1}\left(\left\{-\frac{\pi}{4}, \frac{3\pi}{4}\right\} + 2\pi Z\right)$.

Suppose that $f,g : X \to \Re$ are functions such that

$$f(x) = \frac{\cos(x)}{\sqrt[n]{\cos^n a(x) + \sin^n a(x)}} \quad \text{and} \quad g(x) = \frac{\sin a(x)}{\sqrt[n]{\cos^n a(x) + \sin^n a(x)}}$$

for every $x \in X \backslash Y$ and $f(x) = -g(x) \neq 0$ for all $x \in Y$. Then $g(0)=0, f, g$ do not wanish simultaneously and satisfy the conditional system (II) on X and equation (I) on $X \backslash Y$.

Proof. Let $h : X \to T$ be a function such that $h(x) = \exp(ia(x))$ for all $x \in X$. Then $h \in H(X,T)$. Moreover, $h^{-1}(\{z_1, z_2\}) = Y$. Condition (4) of Theorem 1 jointly with Corollary 2 imply the assertion.

References

[1] I. Adamaszek, *On generalized sine and cosine functions*, I, Demonstratio Math. 28 (1995), 263–270.

[2] J. G. Dhombres, R. Ger, *Conditional Cauchy equations*, Glasnik Mat. 13 (33) (1978), 39-62.

[3] R. Ger, *O pewnych równaniach funkcyjnych z obciętą dziedziną*, (in Polish) Prace Naukowe Uniwersytetu Śląskiego w Katowicach, Nr 132 (1976).

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