DEMONSTRATIO MATHEMATICA
Vol. XXXVII No 3 2004

Jarosław Łazuka

GLOBAL SOLUTION TO THE INITIAL VALUE PROBLEM FOR NONLINEAR SYSTEM OF EQUATIONS OF THERMODIFFUSION WITHOUT DISPLACEMENTS

Abstract. In the paper we shall present the proof of global-in-time solution to the initial value problem for nonlinear partial differential equations describing physical proccesses of thermodiffusion without displacements. Time decay of global solution will be also shown.

1. Introduction

The aim of this paper is to prove the existence and uniqueness of global-in-time solution to the initial value problem for a nonlinear partial differential equation (pde) system describing a special case of thermodiffusion of solids in three-dimensional space. In these solids we have the field of temperature θ_1 and chemical potential θ_2 without displacements [8], [9].

The equations describing this type of solid have form

$$(1.1) \qquad (k\Delta - c\partial_t)\,\theta_1 = d\partial_t\theta_2,$$

$$(1.2) \qquad (D\Delta - n\partial_t)\,\theta_2 = d\partial_t\theta_1$$

with the initial conditions

$$(1.3) \qquad \theta_1(0,x) = \theta_1^0(x),$$

$$(1.4) \qquad \theta_2(0,x) = \theta_2^0(x),$$

where θ_1, θ_2 are temperature and chemical potential respectively, both depending on $t \in \mathbf{R}_+$ and $x \in \mathbf{R}^3$.

The system (1.1)–(1.2) is nonlinear because the physical parameters k, D, c, n, d depend on actual states (θ_1, θ_2) of described material. They denote respectively: the coefficient of thermal conductivity, the coefficient of

Key words and phrases: matrix of fundamental solution, linear and nonlinear parabolic systems, thermodiffusion, local solutions, global solutions, $L^p - L^q$ time decay estimates.
1991 *Mathematics Subject Classification*: 35E05, 35K45, 35B40, 35B65, 35K57.

diffusion and quantities n, c, d are the coefficients of thermodiffusion. These parameters satisfy the following inequality $cn - d^2 > 0$.

The pde system (1.1)–(1.2) may be easily rewritten in the equivalent form

$$\partial_t\theta_1 - \bar{k}_1\Delta\theta_1 + \bar{k}_2\Delta\theta_2 = 0, \tag{1.5}$$

$$\partial_t\theta_2 + \bar{k}_3\Delta\theta_1 - \bar{k}_4\Delta\theta_2 = 0, \tag{1.6}$$

$$\theta_1(0,x) = \theta_1^0(x), \tag{1.7}$$

$$\theta_2(0,x) = \theta_2^0(x), \tag{1.8}$$

where

$$\bar{k}_1 = \frac{nk}{cn-d^2}, \bar{k}_2 = \frac{dD}{cn-d^2}, \bar{k}_3 = \frac{dk}{cn-d^2}, \bar{k}_4 = \frac{cD}{cn-d^2}.$$

Because of $cn - d^2 > 0$, that we have

$$\bar{k}_1\bar{k}_4 - \bar{k}_2\bar{k}_3 = \frac{kD}{cn-d^2} > 0. \tag{1.9}$$

We assume that given functions k_i have the following property

(1.10) $\bar{k}_i \in C^\infty(\mathbf{R}^2)$, $\bar{k}_i(U) - k_i = O(|U|^\lambda)$ as $|\mathrm{U}| \to 0, \lambda \in \mathbf{N}$, $i = 1,2,3,4$,

where

$$k_i = \bar{k}_i(0,0). \tag{1.11}$$

In the paper we will use notations:

$$\partial_t = \frac{\partial}{\partial_t}, \quad \partial_i = \frac{\partial}{\partial_{x_i}}, \quad \Delta = \sum_{i=1}^{3}\partial_i^2, \quad \nabla = (\partial_1, \partial_2, \partial_3), \quad D^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3},$$

$$U = (\theta_1,\theta_2)^T, \quad U^0 = \left(\theta_1^0,\theta_2^0\right)^T, \quad \bar{A} = \begin{pmatrix} \bar{k}_1 & -\bar{k}_2 \\ -\bar{k}_3 & \bar{k}_4 \end{pmatrix}, \quad A = \begin{pmatrix} k_1 & -k_2 \\ -k_3 & k_4 \end{pmatrix},$$

$$\mathrm{A}_2^0 = \begin{pmatrix} 1 & 0 \\ 0 & \bar{k}_2/\bar{k}_3 \end{pmatrix}, \quad B = \begin{pmatrix} \bar{k}_1 & -\bar{k}_2 \\ -\bar{k}_2 & \bar{k}_2\bar{k}_4/\bar{k}_3 \end{pmatrix}.$$

Using the above notations we may write the system (1.5)–(1.8) in the form

$$\partial_t U - \bar{A}\Delta U = 0, \tag{1.12}$$

$$U(0,x) = U^0(x) \tag{1.13}$$

or in the equivalent form

$$\partial_t U - A\Delta U = F, \tag{1.14}$$

$$U(0,x) = U^0(x), \tag{1.15}$$

where $F = (\bar{A} - A)\,\Delta U$.

THEOREM 1. (Main theorem—global existence and asymptotic behaviour). *Let* $s, \lambda, N_p \in \mathbf{N}$, $s \geq 8$, $\frac{1}{\lambda}\left(1+\frac{1}{\lambda}\right) < \frac{3}{2}$, $q = 2\lambda+2, \frac{1}{p}+\frac{1}{q} = 1$, $N_p > 3\left(1-\frac{2}{q}\right)$, *if* $U^0 \in W^{s,2}(\mathbf{R}^3) \cap W^{N_p,p}(\mathbf{R}^3)$ *and* $\|U^0\|_{s,2} + \|U^0\|_{N_p,p} < \delta$ (δ *sufficiently small*), *then problem* (1.1)–(1.4) (or (1.14)–(1.15)) *has a unique solution* $U \in C^0([0,\infty), W^{s,2}(\mathbf{R}^3)) \cap C^1([0,\infty), W^{s-2,2}(\mathbf{R}^3)$.) *Moreover*

$$\begin{aligned} &\|U(t)\|_q = O(t^{-3\lambda/2\lambda+2}) \\ &\|U(t)\|_2 = O(1) \end{aligned} \qquad \text{hold as} \quad t \to \infty.$$

This paper is organized as follows. In section 2 we prove the $L^p - L^q$ time decay estimates. Section 3 presents the existence and uniqueness of local solution. In sections 4 and 5 we prove the energy estimates and a priori estimates. Main theorem 1 is proved in section 6. The procedure indicated below has been applied for example in [2], [3], [5] and [10], for the nonlinear wave equations, the nonlinear heat equation, the nonlinear thermoelasticity.

2. L^p–L^q time decay estimates of solution to the linearized problem

In this section we construct a solution to the Cauchy problem (2.1), (2.2) for linear system of equations describing linear termodiffusion without displacements

$$\partial_t U - A\Delta U = 0, \tag{2.1}$$

$$U(0,x) = U^0. \tag{2.2}$$

Using this solution we present the $L^p - L^q$decay estimates to the linearized problem.

In the paper [6] there was constructed the matrix of fundamental solution to the differential operator

$$P(\partial_t, \partial) = \begin{pmatrix} \partial_t - k_1\Delta & k_2\Delta \\ k_3\Delta & \partial_t - k_4\Delta \end{pmatrix} \tag{2.3}$$

which has form $E(t,x) = (E_{ij}(t,x))_{2\times 2}$, where

$$E_{ij}(t,x) = \frac{(-1)^{j+i}}{\sigma} H(t) \sum_{m=1}^{2} \frac{(-1)^{m+1}(\frac{1}{2}\gamma_m\delta_{ij} + \varphi_{ij})}{(2\pi\gamma_m t)^{\frac{3}{2}}} \exp\left(\frac{-|x|^2}{2\gamma_m t}\right) \tag{2.4}$$

$i, j = 1, 2$, $H(t)$—the Heaviside function,

$$\gamma_m = \eta - (-1)^m\sigma,$$

$$\varphi_{ij} = \delta_{1i}(k_2\delta_{2j} - k_4\delta_{1j}) + \delta_{2i}(k_3\delta_{1j} - k_1\delta_{2j}),$$

$$\sigma = \frac{1}{cn-d^2}\sqrt{(nk-cD)^2 + 4d^2Dk}, \quad \eta = \frac{1}{cn-d^2}(nk-cD)^2.$$

If $U^0 \in C_0^\infty(R^3)$ then problem (2.1), (2.2) has unique solution

$$U(t,x) = E(t, x-\cdot) * U^0(\cdot), \tag{2.5}$$

where $(u * v)(x) = \int\limits_{R^3} u(x-y)v(y)dy$ is the convolution of the functions u and v.

From (2.4) and (2.5) we get

$$U_i(t,x) = \sum_{j=1}^{2} E_{ij}(t, x-\cdot) * \theta_j^0(\cdot) = \sum_{m=1}^{2}\sum_{j=1}^{2} c_{ijm} E_m(t, x-\cdot) * \theta_j^0(\cdot), \quad i = 1,2, \tag{2.6}$$

$$U_i(t,x) = \sum_{m=1}^{2}\sum_{j=1}^{2} c_{ijm} W_{jm}(t,x), \quad i = 1,2, \tag{2.7}$$

where

$$E_m(t,x) = \frac{1}{(2\pi\gamma_m t)^{3/2}} \exp\left(\frac{-|x|^2}{2\gamma_m t}\right) \quad \text{for } t>0,\ x \in \mathbf{R}^3, \tag{2.8}$$

$$W_{jm}(t,x) = E_m(t, x-\cdot) * \theta_j^0(\,\cdot\,);\ j,m = 1,2,$$

c_{ijm} denoted some constants depending on physical parameters. We have the following representations for functions $W_{jm}(t,x) = E_m(t, x-\cdot) * \theta_j^0(\cdot)$

$$W_{jm}(t,x) = \frac{1}{(2\pi\gamma_m t)^{3/2}} \int\limits_{R^3} \exp\left(\frac{-|x-y|^2}{2\gamma_m t}\right) \theta_j^0(y)dy \quad \text{for } t>0 \tag{2.9}$$

or

$$W_{jm}(t,x) = \frac{1}{(2\pi\gamma_m)^{3/2}} \int\limits_{R^3} \exp\left(\frac{-|z|^2}{2\gamma_m}\right) \theta_j^0(x-\sqrt{t}z)dz \quad \text{for } t \geq 0. \tag{2.10}$$

Let $\langle u, v\rangle = \int\limits_{R^3} u(x)\bar{v}(x)dx$ be the standard inner product in the Hilbert space $L^2(\mathbf{R}^3)$.

We first shall derive the $L^p - L^q$ estimates for functions W_{jm}. It is easy to show that functions W_{jm} satisfy equations

$$\partial_t W_{jm} - \frac{\gamma_m}{2}\Delta W_{jm} = 0. \tag{2.11}$$

Taking in (2.11) the inner product in L^2 with W_{jm} we obtain

$$\langle W_{jm}, \partial_t W_{jm}\rangle - \frac{\gamma_m}{2}\langle W_{jm}, \Delta W_{jm}\rangle = 0,$$

$$\frac{1}{2}\frac{d}{dt} < W_{jm}, W_{jm} > -\frac{\gamma_m}{2} < \nabla W_{jm}, \nabla W_{jm} >= 0,$$

$$\frac{1}{2}\frac{d}{dt}\|W_{jm}\|_2^2 + \frac{\gamma_m}{2}\|\nabla W_{jm}\|_2^2 = 0,$$

or

$$\|W_{jm}(t)\|_2^2 + \gamma_m \int_0^t \|\nabla W_{jm}(\tau)\|_2^2 d\tau = \left\|\theta_j^0\right\|_2^2,$$

which implies

$$\|W_{jm}(t)\|_2 \leq \left\|\theta_j^0\right\|_2 \quad \text{for } t \geq 0. \tag{2.12}$$

For $t > 0$ we obtain from (2.9)

$$|W_{jm}(t,x)| \leq ct^{-3/2}\left\|\theta_j^0\right\|_1. \tag{2.13}$$

From (2.10) we also have for $t \geq 0$

$$\|W_{jm}(t)\|_\infty \leq \left\|\theta_j^0\right\|_\infty \frac{1}{(2\pi\gamma_m)^{3/2}} \int_{R^3} \exp\left(\frac{-|z|^2}{2\gamma_m}\right) dz \leq c\left\|\theta_j^0\right\|_\infty. \tag{2.14}$$

Using Sobolev imbedding theorem [1] we get from (2.14)

$$\|W_{jm}(t)\|_\infty \leq c\left\|\theta_j^0\right\|_\infty \leq c\left\|\theta_j^0\right\|_{3,1} \quad \text{for } t \geq 0. \tag{2.15}$$

From (2.13) and (2.15) we conclude

$$\|W_{jm}(t)\|_\infty \leq c\,(1+t)^{-3/2}\left\|\theta_j^0\right\|_{3,1} \quad \text{for } t \geq 0. \tag{2.16}$$

Thus we have obtained $L^2 - L^2$ and $W^{3,1} - L^\infty$ estimation for functions W_{jm}, if $U^0 \in C_0^\infty(\mathbf{R}^3)$. Notice that formula (2.5) has sense for $U^0 \in W^{3,1}(\mathbf{R}^3)$. From the imbedding $W^{3,1}(\mathbf{R}^3) \hookrightarrow W^{1,2}(\mathbf{R}^3)$ and above remark we can obtain corresponding results for $U^0 \in W^{3,1}(\mathbf{R}^3)$ by approximation with $(U_n^0) \subset C_0^\infty(\mathbf{R}^3)$. By interpolation (cf. [10, 11]) we get from (2.12) and (2.16) $L^p - L^q$ decay for W_{jm}

$$\|W_{jm}(t)\|_q \leq c\,(1+t)^{-\frac{3}{2}\left(1-\frac{2}{q}\right)}\left\|\theta_j^0\right\|_{N_p,p} \quad \text{for } t \geq 0, \tag{2.17}$$

where $\frac{1}{p} + \frac{1}{q} = 1, N_p > 3\left(1 - \frac{2}{q}\right)$ or $N_p = 3\,(1 - 2/q)$ if $p \in \{1, 2\}$.

From (2.17) and formula (2.7) we get $L^p - L^q$ decay estimations for solution to the problem (2.1) and (2.2).

THEOREM 2. *Let $U^0 \in W^{N_p,p}(\mathbf{R}^3)$, then solution to the Cauchy problem (2.1), (2.2) has the following $L^p - L^q$ decay estimations*

$$(2.18) \qquad \|U(t)\|_q \le C\,(1+t)^{-\frac{3}{2}\left(1-\frac{2}{q}\right)} \left\|U^0\right\|_{N_p,p} \quad \text{for } t \ge 0,$$

where $\frac{1}{p}+\frac{1}{q}=1, N_p > 3\left(1-\frac{2}{q}\right)$ *or* $N_p = 3\,(1-2/q)$ *if* $p \in \{1,2\}$.

3. Local solution

Now, we present the local existence theorem to the initial value problem for nonlinear thermodiffusion without displacements. It will be proved that the system is a particular case of well known second-order symmetric hiperbolic-parabolic system for which a local solution exists (cf. [4]).

THEOREM 3. *Let $s \in \mathbf{N}$, $s \ge 3$, $U^0 \in W^{s,2}(\mathbf{R}^3)$, then problem*

$$(3.1) \qquad \partial_t U - \bar{A}(U)\Delta U = 0,$$

$$(3.2) \qquad U(0,x) = U^0$$

has unique local solution

$$(3.3) \qquad U \in C^0\left([0,T]; W^{s,2}(\mathbf{R}^3)\right) \cap C^1\left([0,T]; W^{s-2,2}(\mathbf{R}^3)\right) \cap L^2\left([0,T]; W^{s+1,2}(\mathbf{R}^3)\right),$$

$$(3.4) \qquad \sup_{0\le\tau\le t} \|U(t)\|^2_{s,2} + \int_0^t \|U(\tau)\|^2_{s+1,2} d\tau \le C\left\|U^0\right\|^2_{s,2} \quad \forall t \in [0,T].$$

Proof. Now we will show that the system (3.1)–(3.2) has properties which are described in [4]. Multiplying equation (3.1) by symmetric and positive defined matrix $A_2^0 = \begin{pmatrix} 1 & 0 \\ 0 & \bar{k}_2/\bar{k}_3 \end{pmatrix}$ we obtain

$$(3.5) \qquad A_2^0 \partial_t U - B(U)\Delta U = 0,$$

where

$$B(U) = A_2^0 \bar{A}(U) = \begin{pmatrix} 1 & 0 \\ 0 & \bar{k}_2/\bar{k}_3 \end{pmatrix} \begin{pmatrix} \bar{k}_1 & -\bar{k}_2 \\ -\bar{k}_3 & \bar{k}_4 \end{pmatrix} = \begin{pmatrix} \bar{k}_1 & -\bar{k}_2 \\ -\bar{k}_2 & \bar{k}_2\bar{k}_4/\bar{k}_3 \end{pmatrix}.$$

Since $\bar{k}_1 > 0, \det B = \frac{\bar{k}_2}{\bar{k}_3}\left(\bar{k}_1\bar{k}_4 - \bar{k}_2\bar{k}_3\right) > 0$, the matrix B(U) is symmetric and positive definite.

If we take $B_2^{jk}(U) = \delta_{jk}B(U)$, $f_2(U, D_xU) = 0$ we conclude that system (3.5), (3.2) has the properties described in [4]. Thus we obtain that problem (3.1)–(3.2) equivalent to (3.5), (3.2) has unique local solution with properties (3.3) and (3.4). q.e.d.

4. Higher energy estimate

In this section we establish a priori estimates for higher-order derivatives of the solution to the problem (1.14), (1.15) by an energy method.

THEOREM 4. *Let $s \in \mathbf{N}$ and $s \geq 4$, $U^0 \in W^{s,2}(\mathbf{R}^3)$. Then there is a constant C which is independent of T and U^0 such that the local solution U satisfies:*

$$\|U(t)\|_{s,2} \leq c\|U^0\|_{s,2} \exp\left\{c\int_0^t \|U(\tau)\|_{2,\infty}^{\lambda} d\tau\right\} \quad \forall t \in [0,T]. \tag{4.1}$$

Proof. We have $\bar{A}(U) = A + A(U)$ and then the equation

$$\partial_t U - \bar{A}(U)\Delta U = 0 \tag{4.2}$$

can be written in the form

$$\partial_t U - A\Delta U = A(U)\Delta U, \tag{4.3}$$

where $A(U) = \bar{A}(U) - A = (a_{ij})_{2\times 2}, a_{ij} \in C^{\infty}(\mathbf{R}^2), a_{ij}(U) = O(|U|^{\lambda})$ as $|U| \to 0$, $\lambda \in \mathbf{N}$. We use the standard energy method with the help of mollification. Here we use following notations

$$U_\varepsilon = J_\varepsilon(U) = j_\varepsilon * U = \int_{R^3} j_\varepsilon(x-y)U(y)dy, \quad U_\varepsilon^\alpha = D^\alpha U_\varepsilon.$$

Proving some energy estimate for U_ε and finally letting ε tend to zero we obtain the estimate for U. Applying the Friedrichs mollifier to the both sides of equations and D_x^α, $|\alpha| \leq s$, we get

$$\begin{aligned} \partial_t U_\varepsilon^\alpha - A_0\Delta U_\varepsilon^\alpha \\ = D_\varepsilon^\alpha\{J_\varepsilon * (A(U)\Delta U) - A(U_\varepsilon)\Delta U_\varepsilon\} + D_x^\alpha(A(U_\varepsilon)\Delta U_\varepsilon) \end{aligned} \tag{4.4}$$

and then taking the inner product in $L^2(R^3)$ with U_ε^α we obtain

$$\begin{aligned} \langle \partial_t U_\varepsilon^\alpha, U_\varepsilon^\alpha\rangle - \langle A_0\Delta U_\varepsilon^\alpha, U_\varepsilon^\alpha\rangle \\ = \langle G_x^\alpha, U_\varepsilon^\alpha\rangle + \langle D_\varepsilon^\alpha(A(U_\varepsilon)\Delta U_\varepsilon), U_\varepsilon^\alpha\rangle \equiv I + II \end{aligned} \tag{4.5}$$

hence

$$\tfrac{1}{2}\tfrac{d}{dt}\|U_\varepsilon^\alpha\|_2^2 + \langle A_0\nabla U_\varepsilon^\alpha, \nabla U_\varepsilon^\alpha\rangle \leq |I| + |II|. \tag{4.6}$$

The first term I is estimated as follows:

$$\begin{aligned} |I| &= \left|\left\langle G_x^\beta, U_\varepsilon^\gamma\right\rangle\right| \leq \left\|G_x^\beta\right\|_2 \|U_\varepsilon^\gamma\|_2 \\ &\leq \|J_\varepsilon * (A(U)\Delta U) - A(U_\varepsilon)\Delta U_\varepsilon\|_{s-1,2}\|U_\varepsilon\|_{s+1,2} \\ &= \|H_\varepsilon\|_{s-1,2}\|U_\varepsilon\|_{s+1,2}, \end{aligned} \tag{4.7}$$

where $|\gamma| = |\alpha| + 1$, $|\beta| = |\alpha| - 1$.

The second term II is estimated with the help of Moser type inequalities (cf. [2, 10])

$$(4.8)\quad \begin{aligned} |II| &\le |\langle D_x^\alpha(A(U_\varepsilon)\Delta U_\varepsilon), U_\varepsilon^\alpha\rangle| = \left|\left\langle D_x^\beta(A(U_\varepsilon)\Delta U_\varepsilon), U_\varepsilon^\gamma\right\rangle\right| \\ &\le \left|\left\langle D_x^\beta(A(U_\varepsilon)\Delta U_\varepsilon) - A(U_\varepsilon)\Delta U_\varepsilon^\beta, U_\varepsilon^\gamma\right\rangle\right| + \left|\left\langle A(U_\varepsilon)\Delta U_\varepsilon^\beta, U_\varepsilon^\alpha\right\rangle\right| \\ &\equiv |II.a| + |II.b|, \end{aligned}$$

$$(4.9)\quad |II.b| \le \|A(U_\varepsilon)\|_\infty \left\|\Delta U_\varepsilon^\beta\right\|_2 \|U_\varepsilon^\gamma\|_2 \le \|U_\varepsilon\|_\infty^\lambda \|U_\varepsilon\|_{s+1,2}^2,$$

$$(4.10)\quad \begin{aligned} |II.a| &= \left|\left\langle D_x^\beta(A(U_\varepsilon)\Delta U_\varepsilon) - A(U_\varepsilon)\Delta U_\varepsilon^\beta, U_\varepsilon^\gamma\right\rangle\right| \\ &\le \left\|D_x^\beta(A(U_\varepsilon)\Delta U_\varepsilon) - A(U_\varepsilon)\Delta U_\varepsilon^\beta\right\|_2 \|U_\varepsilon^\gamma\|_2 \\ &\le C\Big\{\|\nabla A(U_\varepsilon)\|_\infty \left\|\nabla^{|\beta|-1}\Delta U_\varepsilon\right\|_2 + \left\|\nabla^{|\beta|}A\right\|_2 \|\Delta U_\varepsilon\|_\infty\Big\}\|U_\varepsilon^\gamma\|_2 \\ &\le C\Big\{\|\nabla_U A(U_\varepsilon)\|_\infty \|U_\varepsilon\|_\infty^{\lambda-1} \|\nabla U_\varepsilon\|_\infty \|U_\varepsilon\|_{s,2} \\ &\qquad + (1 + \|U_\varepsilon\|_\infty)^{s-\lambda} \|U_\varepsilon\|_\infty^{\lambda-1} \|U_\varepsilon\|_{s,2} \|U_\varepsilon\|_{2,\infty}\Big\} \|U_\varepsilon\|_{s+1,2} \\ &\le C\|U_\varepsilon\|_{2,\infty}^\lambda \|U_\varepsilon\|_{s,2} \|U_\varepsilon\|_{s+1,2} \\ &\le C\|U_\varepsilon\|_{2,\infty}^\lambda \Big\{\tfrac{1}{2\sigma}\|U_\varepsilon\|_{s,2}^2 + \sigma\|U_\varepsilon\|_{s+1,2}^2\Big\}. \end{aligned}$$

The inequalities (4.7)–(4.10) together with (4.6) imply

$$(4.11)\quad \begin{aligned} \|U_\varepsilon\|_{s,2}^2 + c\int_0^t \|U_\varepsilon(\tau)\|_{s+1,2}^2 d\tau &\le \left\|U_\varepsilon^0\right\|_{s,2}^2 + C(\sigma,\eta)\int_0^t \|U_\varepsilon(\tau)\|_{s+1,2}^2 d\tau \\ &+ C\int_0^t \|H_\varepsilon(\tau)\|_{s-1,2}\|U_\varepsilon(\tau)\|_{s+1,2} d\tau + \int_0^t \|U_\varepsilon(\tau)\|_{2,\infty}^\lambda \|U_\varepsilon(\tau)\|_{s,2}^2 d\tau, \end{aligned}$$

where we use $\|U\|_{2,\infty} \le \eta$.

Choosing σ and η $(resp. \|U^0\|_{s,2})$ sufficiently small we can achieve $C(\sigma,\eta) < c$. Therefore we obtain from (4.11) and above conclusion that

$$(4.12)\quad \begin{aligned} \|U_\varepsilon\|_{s,2}^2 \le \left\|U_\varepsilon^0\right\|_{s,2}^2 &+ C\int_0^t \|H_\varepsilon(\tau)\|_{s-1,2}\|U_\varepsilon(\tau)\|_{s+1,2} d\tau \\ &+ \int_0^t \|U_\varepsilon(\tau)\|_{2,\infty}^\lambda \|U_\varepsilon(\tau)\|_{s,2}^2 d\tau. \end{aligned}$$

Hence, using Gronwall's inequality we get

$$\|U_\varepsilon\|_{s,2}^2 \le \left\{ \|U_\varepsilon^0\|_{s,2}^2 + C\int_0^t \|H_\varepsilon(\tau)\|_{s-1,2}\,\|U_\varepsilon(\tau)\|_{s+1,2} \right. \tag{4.13}$$
$$\left. \times \left(\int_0^\tau - \|U_\varepsilon(\xi)\|_{2,\infty}^\lambda \, d\xi \right) d\tau \right\} \cdot \exp\left\{ \int_0^t \|U_\varepsilon(\tau)\|_{2,\infty}^\lambda \, d\tau \right\}.$$

From mollification properties we have

$$\lim_{\varepsilon\to 0} \|H_\varepsilon(t)\|_{s-1,2} = 0 \quad \forall t \in [0,T]. \tag{4.14}$$

Using these properties and Lebesgue's dominated convergence we get

$$\lim_{\varepsilon\to 0} \left\{ \int_0^t \|H_\varepsilon(\tau)\|_{s-1,2} \|U_\varepsilon(\tau)\|_{s+1,2} \left(\int_0^\tau - \|U_\varepsilon(\xi)\|_{2,\infty}^\lambda d\xi \right) d\tau \right\} = 0. \tag{4.15}$$

Letting tend ε to zero in (4.13) and using (4.15) we obtain

$$\|U(t)\|_{s,2}^2 \le C\left\|U^0\right\|_{s,2}^2 \exp\left\{ \int_0^t \|U(\tau)\|_{2,\infty}^\lambda d\tau \right\} \quad \forall t \in [0,T]. \qquad \text{q.e.d.} \tag{4.16}$$

5. Weighted a priori estimates

Besides the energy estimate which we proved in Theorem 4 we shall prove a priori estimate which is essential for the proof of global existence theorem.

THEOREM 5. *Let* $s_1 > \left[\frac{s_1+N_p}{2}\right]$, $s \ge s_1 + N_p + 2$, $s_1, s, N_p \in \mathbf{N}$, $d = 3/2\left(1 - \frac{2}{q}\right)$, $p = \frac{2\lambda+2}{2\lambda+1}$, $q = 2\lambda + 2$,

$$U^0 \in W^{s,2}(\mathbf{R}^3) \cap W^{s,p}(\mathbf{R}^3), \qquad \left\|U^0\right\|_{s,2} + \left\|U^0\right\|_{s,q} \le \delta_1.$$

Then there exist $M_0 > 0$ *and* $\delta_1 > 0$, *both independent of* T *and* U^0, *such that for the local solution* U *the following estimate holds*

$$M(T) = \sup_{t\in[0,T]} (1+t)^d \|U(t)\|_{s_1,q} \le M_0. \tag{5.1}$$

Proof. Using classical formula we can represent the solution of problem (1.1)–(1,2) or (1.14)–(1.15) in the form

$$U(t,x) = E(t, x - \cdot) * U^0(\cdot) + \int_0^t E(t-\tau, x - \cdot) * F(U(\tau,\cdot))d\tau. \tag{5.2}$$

From (5.2) and Theorem 2 we get

$$\|U(t)\|_{s_1,q} \le \left\|E(t,\cdot) * U^0(\cdot)\right\|_{s_1,q} + \int_0^t \|E(t-\tau,\cdot) * F(U(\tau))\|_{s_1,q} d\tau \tag{5.3}$$

Now we prove Lemma 1, which we use in this section.

LEMMA 1. *Let* $a : \mathbf{R}^2 \to \mathbf{R}$, $a \in C^\infty(\mathbf{R}^2)$, $a(U) = O(|U|^\lambda)$ *for* $|U| \to 0$,

$$s_1 > \left[\frac{s_1 + N_p}{2}\right], \quad s \geq s_1 + N_p + 2, \quad \frac{1}{p} + \frac{1}{q} = 1 \text{ and } \frac{\lambda}{q} + \frac{1}{2} = \frac{1}{p},$$

then for all $U \in W^{s,2}(\mathbf{R}^3) \cap W^{s,p}(\mathbf{R}^3)$ *we have*

$$\|a(U)\partial_i\partial_j U\|_2 \leq C\|U\|_{s_1,q}^\lambda \|U\|_{s,2}.$$

Proof. Let $1 \leq |\alpha| \leq s_1 + N_p$, then

$$D^\alpha \left(a(U)\partial_i\partial_j U\right) = D^\alpha\left(\int_0^1 \nabla_U a(r_1 U) U dr_1 \partial_i\partial_j U\right)$$

$$= D^\alpha\left(\int_0^1\int_0^1 \nabla_U^2 a(r_2 \cdot r_1 U) r_1 U \cdot U dr_2 dr_1 \partial_i\partial_j U\right)$$

$$= D^\alpha\Big(\int_0^1 \ldots \int_0^1 \nabla_U^{\lambda-1} a(r_{\lambda-1} \cdot \ldots \cdot r_1 U) r_{\lambda-2} \cdot \ldots \cdot r_1 \cdot$$

$$\cdot r_{\lambda-3} \cdot \ldots \cdot r_1 \cdot \ldots \cdot r_1 U^{\lambda-1} dr_{\lambda-2} \cdot \ldots \cdot dr_1 \partial_i\partial_j U\Big)$$

$$= \int_0^1 \ldots \int_0^1 r_{\lambda-2} \cdot \ldots \cdot r_1 D^\alpha\left(\nabla_U^{\lambda-1} a(r_{\lambda-1} \cdot \ldots \cdot r_1 U) \cdot U^{\lambda-1}\partial_i\partial_j U\right) dr_{\lambda-1} \ldots dr_1$$

$$= \int_0^1 \ldots \int_0^1 r_{\lambda-2} \cdot \ldots \cdot r_1 \Big(\sum_{1 \leq |\alpha_0| + \ldots + |\alpha_\lambda| \leq s_1 + N_p} c_\alpha \nabla^{\alpha_0}(\nabla_U^{\lambda-1} a(r_{\lambda-1} \cdot \ldots \cdot r_1 U) \cdot$$

$$\cdot \nabla^{\alpha_1} U \cdot \ldots \cdot \nabla^{\alpha_{\lambda-1}} U \cdot \nabla^{\alpha_\lambda}\partial_i\partial_j U)\Big) dr_{\lambda-1} \cdot \ldots \cdot dr_1.$$

It is easy to see, that only one of the components of multiindex α is higher or equal to s_1. Let it be contrary $\alpha_i \geq s_1$ and $\alpha_j \geq s_1$ for some $i, j \in \{0, 1, \ldots, \lambda\}$, $i \neq j$, then

$$|\alpha| \geq \alpha_i + \alpha_j \geq 2s_1 > s_1 + N_p \geq |\alpha|,$$

that is false. We consider the cases:

1. $\alpha_0 \geq s_1$, using the above representations and Hölder inequality with $\frac{\lambda}{q} + \frac{1}{2} = \frac{1}{p}$, we get

$$\|D^\alpha \left(a(U)\partial_i\partial_j U\right)\|_p \leq c\left\|\nabla^{\alpha_0}\left(\nabla_U^{\lambda-1} a(U)\right)\right\|_2 \|\nabla^{\alpha_1} U\|_q \cdot \ldots$$

$$\cdot \|\nabla^{\alpha_{\lambda-1}} U\|_q \|\nabla^{\alpha_\lambda}\partial_i\partial_j U\|_q$$

$$\leq c\|U\|_{s,2}\|U\|_{s_1,q}^\lambda.$$

2. $\alpha_i \geq s_1$ for $1 \leq i \leq \lambda - 1$, then $\alpha_0 < s_1$, $\alpha_1 < s_1$,

$$\begin{aligned}\|D^\alpha (a(U)\partial_i\partial_j U)\|_p &\leq c\left\|\nabla^{\alpha_0}\left(\nabla_U^{\lambda-1}a(U)\right)\right\|_q \|\nabla^{\alpha_1}U\|_q \cdot \ldots \cdot \|\nabla^{\alpha_i}U\|_2 \cdot \ldots \\ &\quad \cdot \|\nabla^{\alpha_{\lambda-1}}U\|_q \|\nabla^{\alpha_\lambda}\partial_i\partial_j U\|_q \\ &\leq c\|U\|_{s,2}\|U\|_{s_1,q}^\lambda .\end{aligned}$$

3. $\alpha_\lambda \geq s_1$

$$\begin{aligned}\|D^\alpha (a(U)\partial_i\partial_j U)\|_p &\leq c\left\|\nabla^{\alpha_0}\left(\nabla_U^{\lambda-1}a(U)\right)\right\|_q \|\nabla^{\alpha_1}U\|_q \cdot \ldots \\ &\quad \cdot \|\nabla^{\alpha_{\lambda-1}}U\|_q \|\nabla^{\alpha_\lambda}\partial_i\partial_j U\|_2 \\ &\leq c\|U\|_{s,2}\|U\|_{s_1,q}^\lambda . \end{aligned}$$

q.e.d.

From (5.3) and Lemma 1 we get

$$\begin{aligned}(5.4)\qquad &\|U(t)\|_{s_1,q} \\ &\leq C_1(1+t)^{-d}\left\|U^0\right\|_{s,p} + C_2\int_0^t (1+t-\tau)^{-d}\|U(\tau)\|_{s_1,q}^\lambda \|U(\tau)\|_{s,2}d\tau.\end{aligned}$$

Multiplying both sides of (5.4) by $(1+t)^d$ and using (4.1) we get

$$\begin{aligned}(5.5)\qquad &(1+t)^d\|U(t)\|_{s_1,q} \\ &\leq C_1\left\|U^0\right\|_{s,p} + C_2\int_0^t (1+t-\tau)^{-d}(1+t)^d\|U(\tau)\|_{s_1,q}^\lambda \left\|U^0\right\|_{s,2} \\ &\quad \times \exp\left\{\int_0^\tau \|U(\sigma)\|_{2,\infty}^\lambda d\sigma\right\} d\tau \\ &\leq C_1\left\|U^0\right\|_{s,p} + C_2\int_0^t (1+t-\tau)^{-d}(1+t)^d(1+\tau)^{-\lambda d}M(t)^\lambda \left\|U^0\right\|_{s,2} \\ &\quad \times \exp\left\{\int_0^\tau \|U(\sigma)\|_{2,\infty}^\lambda d\sigma\right\} d\tau \\ &\leq C_1\left\|U^0\right\|_{s,p} + C_2\left\|U^0\right\|_{s,2} M(t)^\lambda \left\{\int_0^t (1+t-\tau)^{-d}(1+t)^d(1+\tau)^{-\lambda d}d\tau\right\} \\ &\quad \times \exp\left\{\int_0^t \|U(\sigma)\|_{2,\infty}^\lambda \, d\sigma\right\}.\end{aligned}$$

Now we use the inequality

$$\int_0^t (1+t-\tau)^{-d}(1+t)^d(1+\tau)^{-\lambda d}d\tau < C \quad \forall t > 0$$

where C is independent of t (cf. [10], p. 88).

Thus from (5.5) we get

$$(5.6)\qquad (1+t)^d\|U(t)\|_{s_1,q} \le \delta_1\Big\{C_1 + C_2M(t)^\lambda \exp\Big[\int_0^t \|U(\tau)\|_{2,\infty}^\lambda d\tau\Big]\Big\}.$$

Since $s_1 > 2 + \frac{3}{q}$, there is an imbedding of $W^{s_1,q}(\mathbf{R}^3) \hookrightarrow C^2(\mathbf{R}^3)$ (cf. [1]) and then

$$(5.7)\qquad (1+t)^d\|U(t)\|_{s_1,q} \le \delta_1\Big\{C_1 + C_2\kappa M(t)^\lambda \exp\Big[\int_0^t \|U(\tau)\|_{s_1,q}^\lambda d\tau\Big]\Big\}.$$

Therefore

$$(5.8)\ (1+t)^d\|U(t)\|_{s_1,q} \le \delta_1\Big\{C_1 + C_2\kappa M(t)^\lambda \exp\Big[cM(t)^\lambda\int_0^t (1+\tau)^{-\lambda d} d\tau\Big]\Big\}.$$

For $\lambda d > 1$ we have $\int_0^t (1+\tau)^{-\lambda d}d\tau < \frac{1}{\lambda d - 1}$, $\forall t \in [0,\infty)$, that from (5.8) we get

$$(5.9)\qquad (1+t)^d\|U(t)\|_{s_1,q} \le \delta_1\Big\{C_1 + C_2\kappa M(t)^\lambda \exp\Big[cM(t)^\lambda\Big]\Big\},$$

$$(5.10)\ (1+t)^d\|U(t)\|_{s_1,q} \le C\delta_1\Big\{1 + \kappa M(t)^\lambda \exp\Big[cM(t)^\lambda\Big]\Big\} \quad \forall t \in [0,T].$$

Therefore

$$(5.11)\quad M(t) \le C\delta_1\Big\{1 + M(t)^\lambda \exp\Big[cM(t)^\lambda\Big]\Big\}$$
$$\le \mathrm{C}\delta_1\Big\{1 + M(t)^\lambda \exp\Big[cM(t)^\lambda\Big]\Big\} \quad \forall \mathrm{t} \in [0,\mathrm{T}].$$

Now we consider the function $\varphi_\varepsilon : [0,\infty) \to \mathbf{R}$ defined by

$$(5.12)\qquad \varphi_\varepsilon(x) = C\varepsilon\Big\{1 + x^\lambda \exp\Big[cx^\lambda\Big]\Big\} - x.$$

Since φ_ε is continuous and $\varphi_\varepsilon(0) > 0$, that $x_0 > 0$ can be chosen sufficiently small so that

$$(5.13)\qquad \varphi_\varepsilon(x) > 0 \quad \forall x \in [0, x_0].$$

If we fixed $x = x_0$ in (5.12), then $\varphi_\varepsilon(x_0)$ it is linear function of parameter ε, and it follows from (5.13) and above remark, that $\exists\delta_1\ 0 < \delta_1 < \varepsilon$, and $\varphi_{\delta_1}(x_0) = 0$ then

$$(5.14)\qquad \varphi_{\delta_1}(x) = C\delta_1\Big\{1 + x^\lambda \exp\Big[cx^\lambda\Big]\Big\} - x > 0\ \ \forall x \in [0, x_0),$$

$$(5.15)\qquad M(0) = \Big\|U^0\Big\|_{s_1,q} \le \kappa\Big\|U^0\Big\|_{s,2} \le \kappa\delta_1 < x_0.$$

The relation (5.11) implies

$$(5.16)\qquad \varphi_{\delta_1}(M(t)) \ge 0,\ \forall t \in [0,T],$$

which, together with (5.15) and a continuous dependence argument, leads to

$$M(t) \le x_0, \quad \forall t \in [0,T]. \tag{5.17}$$

This yields the claim of Theorem 5 with $M_0 := x_0$ independent of T and U^0.

q.e.d.

6. Global solutions to initial problem (1.1) and (1.2)

Using results from section 2-5 we now prove the main theorem.

THEOREM 6. *For the local solution described in sections*: 3, 4 *and* 5, *there exists a constant K independent of T and U^0, such that*

$$\|U(t)\|_{s,2} \le K\left\|U^0\right\|_{s,2} \quad \forall t \in [0,T]. \tag{6.1}$$

Proof. By using Theorem 4, imbedding theorems and Theorem 5 we successively obtain the following inequalities

$$\begin{aligned} \|U(t)\|_{s,2} &\le C\|U^0\|_{s,2}\exp\left\{c\int_0^t \|U(\tau)\|_{2,\infty}^{\lambda} d\tau\right\} \\ &\le C\|U^0\|_{s,2}\exp\left\{c\int_0^t \|U(\tau)\|_{s_1,q}^{\lambda} d\tau\right\}. \end{aligned} \tag{6.2}$$

Hence

$$\begin{aligned} \|U(t)\|_{s,2} &\le C\left\|U^0\right\|_{s,2}\exp\left\{cM(t)^{\lambda}\int_0^t (1+\tau)^{-\lambda d} d\tau\right\} \\ &\le C\left\|U^0\right\|_{s,2}\exp\left\{C_3 M_0^{\lambda}\right\} \le K\left\|U^0\right\|_{s,2}. \end{aligned} \tag{6.3}$$

This leads to

$$\|U(t)\|_{s,2} \le K\left\|U^0\right\|_{s,2} \quad \forall t \in [0,T] \tag{6.4}$$

with $K := C\exp\left\{C_3 M_0^{\lambda}\right\}$ independent of T and U^0. q.e.d.

If we choose δ such that $0 < \delta < \frac{\delta_1}{K}$ we obtain

$$\|U(T)\|_{s,2} \le K\left\|U^0\right\|_{s,2} = K\delta < \delta_1. \tag{6.5}$$

Applying the local existence theorem (Theorem 3) at initial time T, we conclude that $\exists U \in C^0([T,T+T_1], W^{s,2}) \cap C^1([T,T+T_1], W^{s-2,2}) \cap L^2([T,T+T_1], W^{s+1,2})$ for some positive number $T_1 = T_1(\delta_1)$. This leads to the conclusion that the solution exists on $[0, T+T_1]$. The inequality (6.4) applied for $t \in [0, T+T_1]$ implies

$$\|U(T+T_1)\|_{s,2} \le K\left\|U^0\right\|_{s,2} = K\delta < \delta_1. \tag{6.6}$$

Hence we may apply the same arguments once more at initial time $t = T+T_1$ and proceeding like above we can continue solution onto $[T + T_1, T + 2T_1]$.

Proceeding in this way we prove the existence of a global solution

(6.7) $U \in C^0\left([0,\infty], W^{s,2}\right) \cap C^1\left([0,\infty], W^{s-2,2}\right) \cap L^2([0,\infty], W^{s+1,2}).$

The applied method of proof gives two additional results

(6.8) $\forall t \in [0,\infty) : \|U(t)\|_{s,2} \le K\delta < \delta_1$

and, with Sobolev's imbedding theorem and Theorem 5

(6.9) $\forall t \in [0,\infty) : \|U(t)\|_\infty \le c\|U(t)\|_{s_1,q} \le c(1+t)^{-d}M(t) \le cM_0(1+t)^{-d}.$

q.e.d.

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INSTITUTE OF MATHEMATICS AND CRYPTOLOGY
FACULTY OF CYBERNETICS
MILITARY UNIVERSITY OF TECHNOLOGY
ul. Kaliskiego 2
00-908 WARSAW, POLAND
E-mail: j.lazuka@imbo.wat.edu.pl

Received August 21, 2003.