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FAMILIES OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE WRIGHT GENERALIZED HYPERGEOMETRIC FUNCTION

Abstract. By introducing a new class of analytic functions with negative coefficients which involves the Wright's generalized hypergeometric function, we investigate the coefficient bounds, distortion theorems, extreme points and radii of convexity and starlikeness for this class of functions.

1. Introduction & preliminaries

Let $\mathcal{A}$ denote the class of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are *analytic* in $\mathcal{U} := \mathcal{U}(1)$, where

$$\mathcal{U}(r) := \{z : z \in \mathbf{C} \text{ and } |z| < r\}.$$

A function f belonging to the class $\mathcal{A}$ is said to be *convex* in $\mathcal{U}(r)$ if and only if

$$\mathbf{R}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathcal{U}(r);\ 0 < r \leqq 1).$$

A function f belonging to the class $\mathcal{A}$ is said to be *starlike* in $\mathcal{U}(r)$ if and only if

$$\mathbf{R}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathcal{U}(r);\ 0 < r \leqq 1). \tag{2}$$

We denote by $\mathcal{S}^c$ the class of all functions in $\mathcal{A}$ which are convex in $\mathcal{U}$ and by $\mathcal{S}^*$ we denote the class of all functions in $\mathcal{A}$ which are starlike in $\mathcal{U}$.

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For analytic functions

$$f(z)=\sum_{n=0}^{\infty}a_nz^n \qquad \text{and} \qquad g(z)=\sum_{n=0}^{\infty}b_nz^n,$$

by $f*g$ we denote the *Hadamard product* (*or convolution*) of f and g, defined by

$$(f*g)(z):=\sum_{n=0}^{\infty}a_nb_nz^n.$$

Let $\mathcal{B}$ be a subclass of the class $\mathcal{A}$. We define the radius of starlikeness $R^*(\mathcal{B})$ and the radius of convexity $R^c(\mathcal{B})$ for the class $\mathcal{B}$ by

$$R^*(\mathcal{B})=\inf_{f\in\mathcal{B}}\left(\sup\{r\in(0,1]: f \text{ is starlike in } \mathcal{U}(r)\}\right)$$

and

$$R^c(\mathcal{B})=\inf_{f\in\mathcal{B}}\left(\sup\{r\in(0,1]: f \text{ is convex in } \mathcal{U}(r)\}\right),$$

respectively.

Let $\alpha_1, A_1, \dots, \alpha_q, A_q$ and $\beta_1, B_1, \dots, \beta_s, B_s$ $(q,s\in\mathbb{N}:=\{1,2,\dots\})$ be positive real parameters such that

$$1+\sum_{n=1}^{s}B_n-\sum_{n=1}^{q}A_n\geq 0.$$

The Wright generalized hypergeometric function [12] (see also [11])

$$\begin{aligned}&{}_q\Psi_s[(\alpha_1,A_1),\dots,(\alpha_q,A_q);(\beta_1,B_1),\dots,(\beta_s,B_s);z]\\ &\qquad\qquad={}_q\Psi_s[(\alpha_k,A_k)_{1,q};(\beta_k,B_k)_{1,s};z]\end{aligned}$$

is defined by

$$\begin{aligned}&{}_q\Psi_s\left[(\alpha_k,A_k)_{1,q};(\beta_k,B_k)_{1,s};z\right]\\ &\quad:=\sum_{n=0}^{\infty}\left\{\prod_{k=1}^{q}\Gamma(\alpha_k+A_k n)\right\}\left\{\prod_{k=1}^{s}\Gamma(\beta_k+B_k n)\right\}^{-1}\frac{z^n}{n!}\quad (z\in\mathcal{U}).\end{aligned}$$

If $A_k=1$ $(k=1,\dots,q)$ and $B_k=1$ $(k=1,\dots,s)$, we have the relationship:

$$\omega\,{}_q\Psi_s\left[(\alpha_k,1)_{1,q};(\beta_k,1)_{1,s};z\right]={}_qF_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z), \tag{3}$$

where ${}_qF_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)$ is the generalized hypergeometric function and

$$\omega=\left(\prod_{k=1}^{q}\Gamma(\alpha_k)\right)^{-1}\left(\prod_{k=1}^{s}\Gamma(\beta_k)\right). \tag{4}$$

The Wright generalized hypergeometric functions were invoked in the geometric function theory (see [6]–[9]).

In [4] Dziok and Srivastava, using the generalized hypergeometric function, introduced a linear operator. Now we extend the linear operator by using the Wright generalized hypergeometric functions. First we define a function ${}_q\phi_s\left[(\alpha_k, A_k)_{1,q};(\beta_k, B_k)_{1,s}; z\right]$ by

$$ {}_q\phi_s\left[(\alpha_k, A_k)_{1,q};(\beta_k, B_k)_{1,s}; z\right] = \omega\ z\ {}_q\Psi_s\left[(\alpha_k, A_k)_{1,q};(\beta_k, B_k)_{1,s}; z\right] $$

and consider the following linear operator

$$ \Theta\left[(\alpha_k, A_k)_{1,q};(\beta_k, B_k)_{1,s}\right]\ :\ \mathcal{A} \to \mathcal{A}, \tag{5} $$

defined by the convolution

$$ \Theta\left[(\alpha_k, A_k)_{1,q};(\beta_k, B_k)_{1,s}\right] f(z) = {}_q\phi_s\left[(\alpha_k, A_k)_{1,q};(\beta_k, B_k)_{1,s}; z\right] * f(z). $$

It readily follows from (1) that

$$ \Theta\left[(\alpha_k, A_k)_{1,q};(\beta_k, B_k)_{1,s}\right] f(z) = z + \sum_{n=2}^{\infty} \omega\ \sigma_n\, a_n z^n, \tag{6} $$

where

$$ \sigma_n := \frac{\Gamma\left(\alpha_1 + A_1\ (n-1)\right) \cdots \Gamma\left(\alpha_q + A_q\ (n-1)\right)}{\Gamma\left(\beta_1 + B_1\ (n-1)\right) \cdots \Gamma\left(\beta_s + B_s\ (n-1)\right)(n-1)!}. \tag{7} $$

Equation (6) yields the following relationship after some elementary calculations

$$ \alpha_1 \Theta\left[\alpha_1 + 1\right] f(z) = z\, A_1 \Theta'\left[\alpha_1\right] f(z) + (\alpha_1 - A_1)\, \Theta\left[\alpha_1\right] f(z) \tag{8} $$

where, for the sake of convenience we denote

$$ \Theta\left[\alpha_1\right] f(z) = \Theta\left[(\alpha_1, A_1), \ldots, (\alpha_q, A_q); (\beta_1, B_1), \ldots, (\beta_s, B_s)\right] f(z). $$

In view of the relationship (3), the linear operator (5) includes the Dziok-Srivastava linear operator [4] (see also [3] and [5]). Further, the linear operator defined by Raina [6] is contained in (5).

Let us denote by $W(q, s; A, B)$ the class of functions f of the form:

$$ f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0;\ n = 2, 3 \ldots), \tag{9} $$

which also satisfy the following condition:

$$ \alpha_1 \frac{\Theta[\alpha_1 + 1] f(z)}{\Theta[\alpha_1] f(z)} + A_1 - \alpha_1 \prec A_1 \frac{1 + Az}{1 + Bz} \qquad (0 \leqq B \leqq 1,\ -B \leqq A < B). \tag{10} $$

In particular, for $q = s + 1$ and $\alpha_{s+1} = A_{s+1} = 1$, we write

$$ W(s; A, B) = W(s + 1, s; A, B). $$

Classes of functions of the form (9) defined by some linear operators were investigated by (among others) Choi, Kim and Srivastava [1], Srivastava and Aouf [10] and Dziok [2].

Now using techniques due to Dziok and Srivastava [4] we investigate the coefficient estimates, distortion properties and radii of convexity and starlikeness for the class $W\ (q,s;A,B)$.

2. Coefficient estimates

The following lemmas follow easily:

LEMMA 1. *If* $\alpha_k = \beta_k,\ A_k = B_k\ (k = 1,\dots,s)$, *then*

$$W(s;1,-1) \subset S^*.$$

LEMMA 2. *If* $A_1 \le A_2$ *and* $B_1 \ge B_2$, *then*

$$W(q,s;A_1,B_1) \subset W(q,s;A_2,B_2) \subset W(q,s;1,-1).$$

THEOREM 1. *A function* f *of the form* (9) *belongs to the class* $W\ (q,s;A,B)$ *if and only if*

$$\sum_{n=2}^{\infty} \delta_n a_n \le \Omega \quad \left(\delta_n = ((B+1)n - A - 1)\,\sigma_n;\ \ \Omega = \frac{B-A}{\omega}\right), \tag{11}$$

where ω *and* σ_n *are defined by* (4) *and* (7).

Proof. Let a function f of the form (9) belong to the class $W(q,s;A,B)$. By (10) and the definition of subordination, we have

$$\alpha_1 \frac{\Theta\,[\alpha_1+1]\,f(z)}{\Theta\,[\alpha_1]\,f(z)} + A_1 - \alpha_1 = A_1 \frac{1+Au(z)}{1+Bu(z)},$$

where $u(0) = 0$ and $|u(z)| < 1$ for $z \in \mathcal{U}$. Thus we obtain (for $z \in \mathcal{U}$)

$$\left| \frac{\alpha_1 \{\Theta(\alpha_1+1)f(z) - \Theta(\alpha_1)f(z)\}}{\alpha_1 B\Theta(\alpha_1+1)f(z) - (AA_1 + (\alpha_1 - A_1)B)\Theta(\alpha_1)f(z)} \right| < 1. \tag{12}$$

Hence, by (6), we have

$$\left| \frac{\sum\limits_{n=2}^{\infty}(n-1)\sigma_n a_n z^{n-1}}{\Omega - \sum\limits_{n=2}^{\infty}(Bn-A)\sigma_n a_n z^{n-1}} \right| < 1 \qquad (z \in \mathcal{U}),$$

where ω, σ_n are defined by (4) and (7), respectively. Putting $z = r\ (0 \le r < 1)$, we obtain

$$\sum_{n=2}^{\infty}(n-1)\sigma_n a_n r^{n-1} < \Omega - \sum_{n=2}^{\infty}(Bn-A)\sigma_n a_n r^{n-1},$$

which, upon letting $r \to 1-$, readily yields the assertion (11).

In order to prove the converse, let a function f of the form (9) satisfy the condition (11). Then, in view of (12), it is sufficient to prove that

$$\alpha_1 \left|\Theta(\alpha_1+1)f(z)-\Theta(\alpha_1)f(z)\right|$$

$$-\left|\alpha_1 B\Theta(\alpha_1+1)f(z)-(AA_1+(\alpha_1-A_1)B)\Theta(\alpha_1)f(z)\right|<0 \quad (z\in\mathcal{U}).$$

Indeed, letting $|z|=r$ $(0<r<1)$, we have

$$\begin{aligned}
&\alpha_1 \left|\Theta(\alpha_1+1)f(z)-\Theta(\alpha_1)f(z)\right| \\
&\qquad -\ \left|\alpha_1 B\Theta(\alpha_1+1)f(z)-(AA_1+(\alpha_1-A_1)B)\Theta(\alpha_1)f(z)\right| \\
&= A_1\Big|\sum_{n=2}^{\infty}(n-1)\omega\sigma_n a_n z^n\Big| - A_1\Big|\Omega-\sum_{n=2}^{\infty}(Bn-A)\omega\sigma_n a_n z^n\Big| \\
&\le A_1\Big(\sum_{n=2}^{\infty}(n-1)\omega\sigma_n a_n r^n - \Omega + \sum_{n=2}^{\infty}(Bn-A)\omega\sigma_n a_n r^n\Big) \\
&= rA_1\Big(\sum_{n=2}^{\infty}\delta_n a_n r^{n-1}-\Omega\Big) < A_1\Big(\sum_{n=2}^{\infty}\delta_n a_n-\Omega\Big)\le 0,
\end{aligned}$$

whence $f\in W(q,s;A,B)$.

Since the expression δ_n defined with (11) is a decreasing function with respect to β_k $(k=1,\dots,s)$ and an increasing function with respect to α_l $(l=1,\dots,q)$, from Theorem 1 we obtain

COROLLARY 1. *If $l\in\{1,\dots,q\}$, $j\in\{1,\dots,s\}$, $0<\alpha_l'\le\alpha_l, 0<A_l'\le A_l$ and $\beta_j'\ge\beta_j, 0<B_l'\le B_l$ then the class $W(q,s;A,B)$ (for the parameters $(\alpha_k,A_k)_{1,q};(\beta_k,B_k)_{1,s}$) is included in the class $W(q,s;A,B)$ for the parameters*

$$(\alpha_1,A_1),\dots(\alpha_{l-1},A_{l-1}),(\alpha_l',A_l')(\alpha_{l+1},A_{l+1}),\dots,(\alpha_q,A_q);$$
$$(\beta_1,B_1),\dots,(\beta_{j-1},B_{j-1}),(\beta_j',B_j'),(\beta_{j+1},B_{j+1}),\dots,(\beta_s,B_s).$$

By Theorem 1, we also have

COROLLARY 2. *If a function f of the form (9) belongs to the class $W(q,s;A,B)$, then*

$$a_n\le\frac{\Omega}{\delta_n}\quad (n=2,3,\dots),$$

where δ_n and Ω are defined by (11). The result is sharp, the functions f_n of the form:

$$f_n(z)=z-\frac{\Omega}{\delta_n}z^n\quad (n=2,3,\dots) \tag{13}$$

being the extremal functions.

3. Distortion theorems

THEOREM 2. *Let a function f of the form* (9) *belong to the class* $W(q,s;A,B)$. *If the sequence* $\{\delta_n\}$ *is nondecreasing, then*

$$r-\frac{\Omega}{\delta_2}r^2\le|f(z)|\le r+\frac{\Omega}{\delta_2}r^2\quad(|z|=r<1). \tag{14}$$

If the sequence $\left\{\frac{\delta_n}{n}\right\}$ *is nondecreasing, then*

$$1-\frac{2\Omega}{\delta_2}r\le|f'(z)|\le 1+\frac{2\Omega}{\delta_2}r\quad(|z|=r<1), \tag{15}$$

where δ_n and Ω are defined by (11). *The result is sharp, with the extremal function f_2 of the form* (13).

Proof. Let a function f of the form (9) belong to the class $W(q,s;A,B)$. If the sequence $\{\delta_n\}$ is nondecreasing and positive, by Corollary 2 we have

$$\sum_{n=2}^{\infty}a_n\le\frac{\Omega}{\delta_2} \tag{16}$$

and if the sequence $\left\{\frac{\delta_n}{n}\right\}$ is nondecreasing and positive, by Corollary 2 we have

$$\sum_{n=2}^{\infty}na_n\le\frac{2\Omega}{\delta_2}. \tag{17}$$

Using conditions (9) and (16) we can write

$$\begin{aligned}|f(z)|&=\left|z-\sum_{n=2}^{\infty}a_nz^n\right|\le r+\sum_{n=2}^{\infty}a_nr^n=r+r^2\sum_{n=2}^{\infty}a_nr^{n-2}\\&\le r+r^2\sum_{n=2}^{\infty}a_n\le r+\frac{\Omega}{\delta_2}r^2\end{aligned}$$

and

$$\begin{aligned}|f(z)|&=\left|z-\sum_{n=2}^{\infty}a_nz^n\right|\ge r-\sum_{n=2}^{\infty}a_nr^n=r-r^2\sum_{n=2}^{\infty}a_nr^{n-2}\\&\ge r-r^2\sum_{n=2}^{\infty}a_n\ge r-\frac{\Omega}{\delta_2}r^2.\end{aligned}$$

Thus we have (14). Using conditions (9) and (17) we have

$$\begin{aligned}|f'(z)| &= \left|1-\sum_{n=2}^{\infty} na_n z^{n-1}\right| \le 1+\sum_{n=2}^{\infty} na_n r^{n-1} = 1+r\sum_{n=2}^{\infty} na_n r^{n-2} \\ &\le 1+r\sum_{n=2}^{\infty} na_n \le 1+\frac{2\Omega}{\delta_2} r\end{aligned}$$

and

$$\begin{aligned}|f'(z)| &= \left|1-\sum_{n=2}^{\infty} na_n z^{n-1}\right| \ge 1-\sum_{n=2}^{\infty} na_n r^{n-1} = 1-r\sum_{n=2}^{\infty} na_n r^{n-2} \\ &\ge 1-r\sum_{n=2}^{\infty} na_n \le 1-\frac{2\Omega}{\delta_2} r.\end{aligned}$$

Thus we have (15).

COROLLARY 3. *Let a function f of the form* (9) *belong to the class $W(s; A, B)$. If $\beta_k \le \alpha_k, B_k \le A_k$ $(k = 1, 2, \ldots, s)$, then the assertion* (14) *and* (15) *holds true.*

Proof. If $q = s$, and $\beta_k \le \alpha_k$, $B_k \le A_k$ $(k = 1, 2, \ldots, s)$, then the sequences $\{\delta_n\}$ and $\{\frac{\delta_n}{n}\}$ are nondecreasing. Thus, by Theorem 2, we have Corollary 3.

4. The radii of convexity and starlikeness

THEOREM 4. *The radius of starlikeness for the class $W(q, s; A, B)$ is given by*

$$R^*\left(W(q,s;A,B)\right) = \inf_{n\ge 2}\left(\frac{\delta_n}{n\Omega}\right)^{\frac{1}{n-1}}, \tag{18}$$

where δ_n and Ω are defined by (11). *The result is sharp.*

Proof. By (2), the function f of the form (9) is starlike in the disk $\mathcal{U}(r)$, if

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \quad (z \in \mathcal{U}(r);\ 0 < r \le 1). \tag{19}$$

Since

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{\sum\limits_{n=2}^{\infty}(n-1)a_n z^n}{z+\sum\limits_{n=2}^{\infty} a_n z^n}\right| \le \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n |z|^{n-1}}{1-\sum\limits_{n=2}^{\infty} a_n |z|^{n-1}},$$

putting $|z| = r$, the condition (19) is true if

$$\sum_{n=2}^{\infty} na_n r^{n-1} \le 1. \tag{20}$$

By Theorem 1, we have

$$\sum_{n=2}^{\infty} \frac{\delta_n a_n}{\Omega} \leq 1,$$

where δ_n and Ω are defined by (11). Thus the condition (20) is true if

$$nr^{n-1} \leq \frac{\delta_n}{\Omega} \qquad (n = 2, 3 \ldots),$$

that is, if

$$r \leq \left(\frac{\delta_n}{n\Omega}\right)^{\frac{1}{n-1}} \qquad (n = 2, 3 \ldots).$$

It follows that any function $f \in W(q, s; A, B)$ is starlike in the disk $\mathcal{U}(R^*(W(q, s; A, B)))$, where $R^*(W(q, s; A, B))$ is defined by (18).

COROLLARY 5.

$$R^*\left(W(s; A, B)\right) = \begin{cases} 1 & (\alpha_k \geq \beta_k,\ A_k \geq B_k;\ k = 1, \ldots, s) \\ \min\limits_{n \geq 2} \left(\frac{\delta_n}{n\Omega}\right)^{\frac{1}{n-1}} & (\alpha_k < \beta_k,\ A_k < B_k;\ k = 1, \ldots, s), \end{cases}$$

where δ_n and Ω are defined with (11). *The result is sharp.*

Proof. By Corollary 1, Lemma 1, and Lemma 2, we have

$$W(s; A, B) \subset S^* \quad (\alpha_k \geq \beta_k,\ A_k \geq B_k;\ k = 1, \ldots, s).$$

By Theorem 4, any function $f \in W(s; A, B)$ is starlike in the disk $\mathcal{U}(r)$, where

$$r = \inf_{n \geq 2} (d_n)^{\frac{1}{n-1}} \qquad \left(d_n = \frac{\delta_n}{n\Omega}\right).$$

Since, for $\alpha_k < \beta_k$, $A_k < B_k$ $(k = 1, \ldots, s)$, we have

$$\lim_{n \to \infty} d_n = d < 1, \ \lim_{n \to \infty} (d_n)^{\frac{1}{n-1}} = 1, \text{ and } d_n > 0 \ (n = 2, 3 \ldots),$$

the infimum of the set $\left\{(d_n)^{\frac{1}{n-1}} : n \geq 2\right\}$ is realized for an element of this set for some $n = n_0$. Moreover, the function

$$f_{n_0}(z) = z - \frac{\Omega}{\delta_{n_0}} z^{n_0},$$

belongs to the class $W(s; A, B)$, and for $z = (d_{n_0})^{\frac{1}{n_0-1}}$, we have

$$\operatorname{Re}\left(\frac{z f_{n_0}'(z)}{f_{n_0}(z)}\right) = 0.$$

Thus the result is sharp.

THEOREM 5. *The radius of convexity for the class* $W(q,s;A,B)$ *is given by*

$$R^c\left(W(q,s;A,B)\right)=\inf_{n\geq 2}\left(\frac{\delta_n}{n^2\Omega}\right)^{\frac{1}{n-1}},$$

where δ_n *and* Ω *are defined with* (11). *The result is sharp.*

Proof. The proof is analogous to that of Theorem 4, and we omit the details.

REMARK. The results presented in this paper extend the results obtained earlier by Dziok and Srivastava [4].

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