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HADAMARD INEQUALITIES FOR WRIGHT-CONVEX FUNCTIONS

Abstract. In this paper we establish several inequalities of Hadamard's type for Wright-convex functions.

1. Introduction

If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, then

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

is known as the Hadamard inequality ([5]).

For some results which generalize, improve, and extend this famous integral inequality see [1] – [8], [10] – [15].

In [2], Dragomir established the following theorem which is a refinement of the first inequality of (1.1).

THEOREM 1. *If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, and H is defined on $[0, 1]$ by*

$$(1.2) \quad H(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx,$$

then H is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have

$$(1.3) \quad f\left(\frac{a+b}{2}\right) = H(0) \leq H(t) \leq H(1) = \frac{1}{b-a} \int_a^b f(x) dx.$$

In [10], Yang and Hong established the following theorem which is a refinement of the second inequality of (1.1).

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THEOREM 2. *If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, and F is defined on $[0, 1]$ by*

$$(1.4) \quad F(t) = \frac{1}{2(b-a)} \int_a^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right) \right] dx,$$

then F is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have

$$(1.5) \quad \frac{1}{b-a} \int_a^b f(x) dx = F(0) \leq F(t) \leq F(1) = \frac{f(a) + f(b)}{2}.$$

We recall the definition of a Wright-convex function:

DEFINITION 1 (see [9, p. 223]). We say that $f : [a, b] \rightarrow \mathbb{R}$ is a Wright-convex function, if, for all $x, y + \delta \in [a, b]$ with $x < y$ and $\delta \geq 0$, we have

$$(1.6) \quad f(x + \delta) + f(y) \leq f(y + \delta) + f(x).$$

Let $C([a, b])$ be the set of all convex functions on $[a, b]$ and $W([a, b])$ be the set of all Wright-convex functions on $[a, b]$. Then $C([a, b]) \subsetneq W([a, b])$. That is, a convex function must be a Wright-convex function but not conversely (see [9, p. 224]).

In this paper, we shall establish several inequalities of Hadamard's type for Wright-convex functions.

2. Main results

In order to prove our main theorems, we need the following lemma:

LEMMA 1. *If $f : [a, b] \rightarrow \mathbb{R}$, then the following statements are equivalent:*

- (1) $f \in W([a, b])$;
- (2) *for all $s, t, u, v \in [a, b]$ with $s \leq t \leq u \leq v$ and $t + u = s + v$, we have*

$$(2.1) \quad f(t) + f(u) \leq f(s) + f(v).$$

Proof. Suppose $f \in W([a, b])$. If $s, t, u, v \in [a, b]$, and $s \leq t \leq u \leq v$, where $t + u = s + v$, then we can write $x = s$, $x + \delta = t$, $y = u$, $y + \delta = v$, it follows from (1.6) that

$$f(t) + f(u) \leq f(s) + f(v).$$

Conversely, if $x, y + \delta \in [a, b]$, $x < y$ and $\delta \geq 0$. We may have

$$x \leq x + \delta \leq y \leq y + \delta$$

or

$$x \leq y \leq x + \delta \leq y + \delta.$$

In either case we have, by (2.1), that

$$f(x + \delta) + f(y) \leq f(x) + f(y + \delta).$$

Thus $f \in W([a, b])$. ■

THEOREM 3. *Let $f \in W([a, b]) \cap L_1[a, b]$. Then (1.1) holds.*

Proof. For (2.1), we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= \frac{1}{(b-a)} \int_a^{\frac{a+b}{2}} \left[f\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}\right) \right] dx \\ &\leq \frac{1}{b-a} \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)] dx \left(x \leq \frac{a+b}{2} \leq \frac{a+b}{2} \leq a+b-x \right) \\ &= \frac{1}{b-a} \left[\int_a^{\frac{a+b}{2}} f(x) dx + \int_{\frac{a+b}{2}}^b f(x) dx \right] = \frac{1}{b-a} \int_a^b f(x) dx, \end{aligned}$$

and

$$\begin{aligned} \frac{f(a) + f(b)}{2} &= \frac{1}{b-a} \int_a^{\frac{a+b}{2}} [f(a) + f(b)] dx \\ &\geq \frac{1}{b-a} \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)] dx \quad (a \leq x \leq a+b-x \leq b) \\ &= \frac{1}{b-a} \left[\int_a^{\frac{a+b}{2}} f(x) dx + \int_{\frac{a+b}{2}}^b f(x) dx \right] = \frac{1}{b-a} \int_a^b f(x) dx. \end{aligned}$$

This completes the proof. ■

THEOREM 4. *Let $f \in W([a, b]) \cap L_1[a, b]$ and let H be defined as in (1.2). Then $H \in W([0, 1])$ is increasing on $[0, 1]$, and (1.3) holds for all $t \in [0, 1]$.*

Proof. If $s, t, u, v \in [0, 1]$ and $s \leq t \leq u \leq v$, $t + u = s + v$, then for $x \in [a, \frac{a+b}{2}]$ we have

$$\begin{aligned} b &\geq sx + (1-s) \frac{a+b}{2} \geq tx + (1-t) \frac{a+b}{2} \\ &\geq ux + (1-u) \frac{a+b}{2} \geq vx + (1-v) \frac{a+b}{2} \geq a, \end{aligned}$$

and if $x \in [\frac{a+b}{2}, b]$, then

$$\begin{aligned} a &\leq sx + (1-s) \frac{a+b}{2} \leq tx + (1-t) \frac{a+b}{2} \\ &\leq ux + (1-u) \frac{a+b}{2} \leq vx + (1-v) \frac{a+b}{2} \leq b, \end{aligned}$$

where

$$\begin{aligned} &\left[tx + (1-t) \frac{a+b}{2} \right] + \left[ux + (1-u) \frac{a+b}{2} \right] \\ &= \left[sx + (1-s) \frac{a+b}{2} \right] + \left[vx + (1-v) \frac{a+b}{2} \right]. \end{aligned}$$

By Lemma 1, we have

$$\begin{aligned} &f\left(tx + (1-t) \frac{a+b}{2}\right) + f\left(ux + (1-u) \frac{a+b}{2}\right) \\ &\leq f\left(sx + (1-s) \frac{a+b}{2}\right) + f\left(vx + (1-v) \frac{a+b}{2}\right), \end{aligned}$$

for all $x \in [a, b]$. Integrating this inequality over x on $[a, b]$, and dividing both sides by $b - a$, yields

$$H(t) + H(u) \leq H(s) + H(v).$$

Hence, $H \in W([0, 1])$.

Next, if $0 \leq s \leq t \leq 1$ and $x \in [a, \frac{a+b}{2}]$, then

$$\begin{aligned} tx + (1-t) \frac{a+b}{2} &\leq sx + (1-s) \frac{a+b}{2} \\ &\leq s(a+b-x) + (1-s) \frac{a+b}{2} \leq t(a+b-x) + (1-t) \frac{a+b}{2}, \end{aligned}$$

where

$$\begin{aligned} &\left[sx + (1-s) \frac{a+b}{2} \right] + \left[s(a+b-x) + (1-s) \frac{a+b}{2} \right] \\ &= \left[tx + (1-t) \frac{a+b}{2} \right] + \left[t(a+b-x) + (1-t) \frac{a+b}{2} \right]. \end{aligned}$$

By Lemma 1, we have

$$H(s) = \frac{1}{b-a} \int_a^b f\left(sx + (1-s) \frac{a+b}{2}\right) dx$$

$$\begin{aligned}
&= \frac{1}{b-a} \int_a^{\frac{a+b}{2}} \left[f\left(sx + (1-s)\frac{a+b}{2}\right) + f\left(s(a+b-x) + (1-s)\frac{a+b}{2}\right) \right] dx \\
&\leq \frac{1}{b-a} \int_a^{\frac{a+b}{2}} \left[f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) \right] dx \\
&= \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx = H(t).
\end{aligned}$$

Thus, H is increasing on $[0, 1]$, and (1.3) holds for all $t \in [0, 1]$.

This completes the proof. ■

THEOREM 5. Let $f \in W([a, b]) \cap L_1[a, b]$ and let F be defined as in (1.4). Then $F \in W([0, 1])$ is increasing on $[0, 1]$, and (1.5) holds for all $t \in [0, 1]$.

Proof. If $s, t, u, v \in [0, 1]$ and $s \leq t \leq u \leq v$, $t + u = s + v$, then

$$\begin{aligned}
a &\leq \left(\frac{1+v}{2}\right)a + \left(\frac{1-v}{2}\right)x \leq \left(\frac{1+u}{2}\right)a + \left(\frac{1-u}{2}\right)x \\
&\leq \left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x \leq \left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x \leq b \text{ for all } x \in [a, b],
\end{aligned}$$

and

$$\begin{aligned}
a &\leq \left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x \leq \left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x \\
&\leq \left(\frac{1+u}{2}\right)b + \left(\frac{1-u}{2}\right)x \leq \left(\frac{1+v}{2}\right)b + \left(\frac{1-v}{2}\right)x \leq b \text{ for all } x \in [a, b],
\end{aligned}$$

where

$$\begin{aligned}
&\left[\left(\frac{1+u}{2}\right)a + \left(\frac{1-u}{2}\right)x\right] + \left[\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right] \\
&= \left[\left(\frac{1+v}{2}\right)a + \left(\frac{1-v}{2}\right)x\right] + \left[\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left[\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right] + \left[\left(\frac{1+u}{2}\right)b + \left(\frac{1-u}{2}\right)x\right] \\
&= \left[\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right] + \left[\left(\frac{1+v}{2}\right)b + \left(\frac{1-v}{2}\right)x\right].
\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned} & f\left(\left(\frac{1+u}{2}\right)a + \left(\frac{1-u}{2}\right)x\right) + f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) \\ & \quad + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right) + f\left(\left(\frac{1+u}{2}\right)b + \left(\frac{1-u}{2}\right)x\right) \\ & \leq f\left(\left(\frac{1+v}{2}\right)a + \left(\frac{1-v}{2}\right)x\right) + f\left(\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x\right) \\ & \quad + f\left(\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right) + f\left(\left(\frac{1+v}{2}\right)b + \left(\frac{1-v}{2}\right)x\right), \end{aligned}$$

for all $x \in [a, b]$. Integrating this inequality over x on $[a, b]$, and dividing both sides by $2(b-a)$, we have

$$F(t) + F(u) \leq F(s) + F(v),$$

hence, $F \in W([0, 1])$.

Next, if $0 \leq s \leq t \leq 1$ and $x \in [a, b]$, then

$$\begin{aligned} & \left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x \leq \left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x \\ & \leq \left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)(a+b-x) \leq \left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)(a+b-x), \end{aligned}$$

and

$$\begin{aligned} & \left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)(a+b-x) \leq \left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)(a+b-x) \\ & \leq \left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x \leq \left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x, \end{aligned}$$

where

$$\begin{aligned} & \left[\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x\right] + \left[\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)(a+b-x)\right] \\ & = \left[\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right] + \left[\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)(a+b-x)\right], \end{aligned}$$

and

$$\begin{aligned} & \left[\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)(a+b-x)\right] + \left[\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right] \\ & = \left[\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)(a+b-x)\right] + \left[\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right]. \end{aligned}$$

Thus

$$\begin{aligned}
 F(s) &= \frac{1}{2(b-a)} \int_a^b \left[f\left(\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x\right) \right. \\
 &\quad \left. + f\left(\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right) \right] dx \\
 &= \frac{1}{4(b-a)} \int_a^b \left\{ \left[f\left(\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x\right) \right. \right. \\
 &\quad \left. + f\left(\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)(a+b-x)\right) \right] \\
 &\quad + \left[f\left(\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)(a+b-x)\right) \right. \\
 &\quad \left. \left. + f\left(\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right) \right] \right\} dx \\
 &\leq \frac{1}{4(b-a)} \int_a^b \left\{ \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) \right. \right. \\
 &\quad \left. + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)(a+b-x)\right) \right] \\
 &\quad + \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)(a+b-x)\right) \right. \\
 &\quad \left. \left. + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right) \right] \right\} dx.
 \end{aligned}$$

Hence, F is increasing on $[0, 1]$ and (1.5) holds for all $t \in [0, 1]$.

This completes the proof. ■

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