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Tamás Glavosits, Árpád Száz

ON THE EXISTENCE OF NONNEGATIVITY DOMAINS OF SUBSETS OF GROUPS

Abstract. A subset D of a subset A of an additive group X is called a nonnegativity domain of A if D is total and antisymmetric in the sense that $A = D \cup (-D)$ and $D \cap (-D) \subset \{0\}$. Moreover, the nonnegativity domain D of A is called additive and normal if $D + D \subset D$ and $D + x \subset x + D$ for all $x \in A$, respectively.

The most important particular cases of the main results of this paper say that: A subset A of the group X has a nonnegativity domain if and only if A is symmetric and 2-cancellable. Moreover, A has an additive nonnegativity domain if and only if A is symmetric and perfectly cancellable. In addition, it is shown that a commutative subset of X is perfectly cancellable if and only if it is infinitely cancellable.

Here, the set A is called n-cancellable for some natural n if $nx = 0$ implies $x = 0$ for all $x \in A$. In particular, A is called infinitely cancellable if it is n-cancellable for all n. Moreover, A is called perfectly cancellable if for any additive antisymmetric subset B of A and any $x \in A \setminus (B \cup (-B))$ we have either $0 \notin F_B(x)$ or $0 \notin F_B(-x)$, where $F_B(x)$ is the additive hull of $(B \cup \{x\}) \setminus \{0\}$.

The results obtained are illustrated here only with the help of additive and multiplicative groups of complex numbers. Their applications to odd, additive and translation relations are postponed to subsequent papers. We do not compare our results with the more complicated results of P. Conrad, R. Botto Mura and A. Rhemtulla on the extensions of partial right-orders to total ones in multiplicative groups.

1. A few basic facts on families of sets and subsets of groups

A family $\mathcal{A}$ of sets is called chained if for any $A, B \in \mathcal{A}$ we have either $A \subset B$ or $B \subset A$. The family $\mathcal{A}$ is called directed if for any $A, B \in \mathcal{A}$ there exists $C \in \mathcal{A}$ such that $A \subset C$ and $B \subset C$.

A subfamily $\mathcal{B}$ of a family of sets $\mathcal{A}$ is called bounded above in $\mathcal{A}$ if there exists $A \in \mathcal{A}$ such that $B \subset A$ for all $B \in \mathcal{B}$. Moreover, the family $\mathcal{A}$ is called inductive if each chained subfamily of $\mathcal{A}$ is bounded above in $\mathcal{A}$.

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An element A of a family of sets $\mathcal{A}$ is called maximal if $A \subset B$ implies $A = B$ for all $B \in \mathcal{A}$. A particular case and an equivalent of Zorn's lemma [4, p. 4] says that each nonvoid inductive family of sets has at least one maximal element.

If X is a group, then for any $x \in X$ and $n \in \mathbb{N}$, we define $nx = x$ if $n = 1$ and $nx = (n-1)x + x$ if $n > 1$. Moreover, for any sequence $(x_i)_{i=1}^{\infty}$ in X, we define $\sum_{i=1}^{n} x_i = x_1$ if $n = 1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n-1} x_i + x_n$ if $n > 1$.

For any $A, B \subset X$, we define $-A = \{-x : x \in A\}$ and $A + B = \{x + y : x \in A,\ y \in B\}$. Moreover, if $x \in X$, then we simply write $x + A$ and $A + x$ in place of $\{x\} + A$ and $A + \{x\}$, respectively.

A subset A of the group X is called symmetric and antisymmetric if $-A \subset A$ and $A \cap (-A) \subset \{0\}$, respectively. Moreover, a subset B of A is called total in A if $A = B \cup (-B)$.

A subset A of X is called commutative if $x + y = y + x$ for all $x, y \in A$. Moreover, a subset B of A is called normal in A if $B + x \subset x + B$ for all $x \in A$. Thus, each subset of a commutative set is normal.

A subset A of X is called additive if $A+A \subset A$. Moreover, for any $A \subset X$, we denote by $A^{\#}$ the intersection of all additive subsets of X containing A. Thus, $A^{\#}$ is an additive subset of X

Moreover, for any $x \in X$, we have $x \in A^{\#}$ if and only if $x = \sum_{i=1}^{k} n_i x_i$ for some finite families $(n_i)_{i=1}^{k}$ and $(x_i)_{i=1}^{k}$ in $\mathbb{N}$ and A, respectively. And if X is commutative, then $\big(A \cup \{x\}\big)^{\#} = \big\{na + mx : a \in A,\ \ n,\ m \in \mathbb{N} \cup \{0\}\big\}$.

2. Nonnegativity domains of subsets of groups

DEFINITION 2.1. A total and antisymmetric subset D of a subset A of a group X is called a nonnegativity domain of A.

EXAMPLE 2.2. Clearly, $\mathbb{R}_{\oplus} = \mathbb{R}_{+} \cup \{0\}$, where $\mathbb{R}_{+} =]0, +\infty[$, is an additive nonnegativity domain of the additive group $\mathbb{R}$ of all real numbers.

EXAMPLE 2.3. Moreover, we can easily see that $D = \big(\mathbb{R}_{\oplus}\big)^2 \cup (-\mathbb{R}_{+}) \times \mathbb{R}_{+}$ is an additive nonnegativity domain of the additive group $\mathbb{C}$ of all complex numbers.

To check the additivity of D, note that if $(x, y), (z, w) \in D$, then because of the obvious additivity of $\big(\mathbb{R}_{\oplus}\big)^2$ and $(-\mathbb{R}_{+}) \times \mathbb{R}_{+}$ we may, for instance, assume that $(x, y) \in \big(\mathbb{R}_{\oplus}\big)^2$ and $(z, w) \in (-\mathbb{R}_{+}) \times \mathbb{R}_{+}$. Hence, in particular, it follows that $0 \leq y$ and $0 < w$, and thus $0 < y + w$. This already implies that $(x, y) + (z, w) = (x + z, y + w) \in D$.

Concerning nonnegativity domains, we can also easily establish the following basic theorems.

THEOREM 2.4. *If D is a nonnegativity domain of a subset A of a group X, then $0 \in D$ if and only if $0 \in A$. Therefore, if $0 \in A$, then $D \cap (-D) = \{0\}$. Moreover, if in addition D is additive, then $D + D = D$.*

Proof. If $0 \in D$, then since $D \subset A$ we also have $0 \in A$. On the other hand, if $0 \in A$, then since $A = D \cup (-D)$ we have either $0 \in D$ or $0 \in -D$. However, $0 \in D$ implies $0 = -0 \in -D$, and $0 \in -D$ implies $0 = -0 \in -(-D) = D$. Therefore, $\{0\} \subset D \cap (-D)$. And, thus by the antisymmetry of D, the corresponding equality is also true. Moreover, if $0 \in D$, then we also have $D = 0 + D \subset D + D$. Therefore, if D is in addition additive, then the corresponding equality is also true.

THEOREM 2.5. *If D is a nonnegativity domain of a subset A of a group X, then A is a symmetric subset of X. Moreover, if in addition D is normal in A, then $D + x = x + D$ and $A + x = x + A$ for all $x \in A$.*

Proof. By using the function ϕ defined by $\phi(u) = -u$ for all $u \in X$, we can at once see that $-A = \phi(A) = \phi\big(D \cup (-D)\big) = \phi(D) \cup \phi(-D) = -D \cup D = D \cup (-D) = A$. Therefore, A is symmetric.

Moreover, if D is in addition normal in A, i. e., $D+x \subset x+D$ for all $x \in A$, then by using the symmetry of A we can also easily see that $D + (-x) \subset -x + D$, and hence $x+D = x+\big(D+(-x)\big)+x \subset x+(-x+D)+x = D+x$ for all $x \in A$. Therefore, the corresponding equality is also true.

Hence, it is clear that for each $x \in A$ we also have $-D+x = -(-x+D) = -\big(D+(-x)\big) = x+(-D)$. Moreover, by using the functions ψ_1 and ψ_2 defined by $\psi_1(u) = u + x$ and $\psi_2(u) = x + u$ for all $u \in X$, we can at once see that $A + x = \psi_1(A) = \psi_1\big(D \cup (-D)\big) = \psi_1(D) \cup \psi_1(-D) = \psi_2(D) \cup \psi_2(-D) = \psi_2\big(D \cup (-D)\big) = \psi_2(A) = x + A$.

THEOREM 2.6. *If D is a nonnegativity domain of a subset A of a group X, B is a symmetric subset of A and $E = D \cap B$, then E is a nonnegativity domain of B. If D and B are additive (normal in B), then E is also additive (normal in B).*

Proof. By using the injective function ϕ defined above, we can at once see that $-E = \phi(E) = \phi(D \cap B) = \phi(D) \cap \phi(B) = -D \cap (-B) = -D \cap B$, and thus $B = A \cap B = \big(D \cup (-D)\big) \cap B = (D \cap B) \cup (-D \cap B) = E \cup (-E)$ and $E \cap (-E) = (D \cap B) \cap (-D \cap B) = \big(D \cap (-D)\big) \cap B \subset \{0\} \cap B \subset \{0\}$. Therefore, E is a nonnegativity domain of B.

If D and B are additive, then we can at once see that $E+E = (D \cap B) + (D \cap B) \subset (D + D) \cap (B + B) \subset D \cap B = E$. Therefore, E is also additive. While, if D and B are normal in B, then for any fixed $x \in B$ by using the injective functions ψ_1 and ψ_2 defined above we can easily see that $E+x=\psi_1(E)=\psi_1(D\cap B)=\psi_1(D)\cap\psi_1(B)=\psi_2(D)\cap\psi_2(B)=\psi_2(D\cap B)=\psi_2(E)=x+E$. Therefore, E is also normal in B.

EXAMPLE 2.7. By Example 2.2 and Theorem 2.6, it is clear that $\mathbb{Z}_\oplus=\mathbb{Z}\cap\mathbb{R}_\oplus$ and $\mathbb{Q}_\oplus=\mathbb{Q}\cap\mathbb{R}_\oplus$ are additive nonnegativity domains of the additive groups $\mathbb{Z}$ and $\mathbb{Q}$ of all integer and rational numbers, respectively.

THEOREM 2.8. *If A and B are symmetric subsets of the groups X and Y, respectively, E is a nonnegativity domain of B and f is an odd function of A into B such that $0\notin f(A\setminus\{0\})$, then $D=f^{-1}(E)$ is a nonnegativity domain of A.*

Proof. Since f is odd, for any $x\in A$, we have $x\in f^{-1}(-E)\iff f(x)\in -E\iff -f(x)\in E\iff f(-x)\in E\iff -x\in f^{-1}(E)\iff x\in -f^{-1}(E)$. Since, $0\notin f(A\setminus\{0\})$, we can see that $x\in f^{-1}(\{0\})\Rightarrow 0=f(x)\Rightarrow x\notin A\setminus\{0\}\Rightarrow x=0$. Therefore, $f^{-1}(-E)=-f^{-1}(E)$ and $f^{-1}(\{0\})\subset\{0\}$.

Since E is a nonnegativity domain of B, it is already quite obvious that $A=f^{-1}(B)=f^{-1}(E\cup(-E))=f^{-1}(E)\cup f^{-1}(-E)=f^{-1}(E)\cup(-f^{-1}(E))=D\cup(-D)$ and $D\cap(-D)=f^{-1}(E)\cap(-f^{-1}(E))=f^{-1}(E)\cap f^{-1}(-E)=f^{-1}(E\cap(-E))\subset f^{-1}(\{0\})\subset\{0\}$. Therefore, D is a nonnegativity domain of A.

Now, as a useful consequence of this theorem, we can also state

COROLLARY 2.9. *If A and B are symmetric subsets of the groups X and Y, respectively, D is a nonnegativity domain of A and f is an odd injective function of A onto B such that $f(0)=0$ if $0\in A$, then $E=f(D)$ is a nonnegativity domain of B.*

THEOREM 2.10. *If D is a nonnegativity domain of a subset A of a group X and f is an injective additive function of X into a group Y, then $E=f(D)$ is a nonnegativity domain of $B=f(A)$. Moreover, if D is additive (normal in A), then E is additive (normal in B).*

Proof. Since f is additive, we have $f(0)+f(0)=f(0)$, and hence $f(0)=0$. We also have $f(x)+f(-x)=f(0)=0$, and hence $f(-x)=-f(x)$ for all $x\in X$. Now, since $-B=-f(A)=f(-A)=f(A)=B$, by Corollary 2.9 it is clear that E is a nonnegativity domain of B.

To prove the remaining assertions, note that if D is additive, then $E+E=f(D)+f(D)=f(D+D)\subset f(D)=E$. Therefore, E is also additive. While, if D is normal in A, then $E+f(x)=f(D)+f(x)=f(D+x)=f(x+D)=f(x)+f(D)=f(x)+E$ for all $x\in A$. Hence, since $B=f(A)$, it is clear we also have $E+y=y+E$ for all $y\in B$. Therefore, E is normal in B.

REMARK 2.11. Note that if f is an anti-additive function of X into Y in the sense that $f(x+y)=f(y)+f(x)$ for all $x,y\in X$, then f is an additive function of X into the dual group $Y(\oplus)$, where $z\oplus w=w+z$ for all $z,w\in Y$.

Moreover, if E is an additive (normal) nonnegativity domain of a subset B of $Y(\oplus)$, then E is also an additive (normal) nonnegativity domain of B as a subset of Y. Therefore, as a useful consequence of Theorem 2.10, we can also state

COROLLARY 2.12. *If D is a nonnegativity domain of a subset A of a group X, then $E=-D$ is also a nonnegativity domain of A. Moreover, if D is additive (normal in A), then E is also additive (normal in A).*

EXAMPLE 2.13. By Example 2.7 and Corollary 2.12, it is clear that $-\mathbb{Z}_{\oplus}$, $-\mathbb{Q}_{\oplus}$ and $-\mathbb{R}_{\oplus}$ are also additive nonnegativity domains of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively.

THEOREM 2.14. *If D and E are nonnegativity domains of a subset A of a group X, with $0\in A$, such that $D\subset E$, then $D=E$.*

Proof. If $x\in E$, then since $E\subset A=D\cup(-D)$ we have either $x\in D$ or $x\in -D$. If $x\in -D$, then since $D\subset E$ we also have $x\in -E$. Hence, it follows that $x\in E\cap(-E)\subset\{0\}$, and thus $x=0$. Therefore, by Theorem 2.4, we also have $x\in D$. Consequently, the inclusion $E\subset D$ is also true.

EXAMPLE 2.15. If D is an additive nonnegativity domain of the additive group $\mathbb{Z}$ of all integers, then either $D=\mathbb{Z}_{\oplus}$ or $D=-\mathbb{Z}_{\oplus}$.

Namely, by Theorem 2.4, we have $0\in D$. Moreover, since $\mathbb{Z}=D\cup(-D)$, have either $1\in D$ or $-1\in D$. If $1\in D$, then since $D+D\subset D$ we also have $\mathbb{N}\subset D$. Therefore, $\mathbb{Z}_{\oplus}=\{0\}\cup\mathbb{N}\subset D$. Hence, by Theorem 2.14, it is clear that $D=\mathbb{Z}_{\oplus}$. While, if $-1\in D$, then we can similarly see that $D=-\mathbb{Z}_{\oplus}$.

3. The existence of nonnegativity domains

DEFINITION 3.1. A subset A of a group X is called n-cancellable for some $n\in\mathbb{N}$ if $nx=0$ implies $x=0$ for all $x\in A$.

REMARK 3.2. By using the notation $X_n=\{x\in X: nx=0\}$, the above condition can be briefly expressed by writing that $A\cap X_n\subset\{0\}$.

Hence, it is clear that X is n-cancellable if and only if $X_n=\{0\}$. Besides, it can be easily seen that $\{0\}\cup(X\setminus X_n)$ is the largest n-cancellable subset of X.

The following theorem shows that an arbitrary group need not have a nonnegativity domain.

THEOREM 3.3. *If a subset A of a group X has a nonnegativity domain, then A is 2-cancellable.*

Proof. Let D be a nonnegativity domain of A, and assume that $x \in A$ such that $2x = 0$. Then, since $2x = x + x$, we also have $x = -x$. Moreover, since $A = D \cup (-D)$ we have either $x \in D$ or $x \in (-D)$.

If $x \in D$, then since $-x = x$, we also have $-x \in D$, and hence $x \in -D$. Quite similarly, if $x \in -D$, then since $-x = x$, we also have $-x \in -D$, and hence $x \in D$. Consequently, we have $x \in D \cap (-D) \subset \{0\}$, and hence $x = 0$. Therefore, the required assertion is true.

EXAMPLE 3.4. If A is a subset of the multiplicative group of all nonzero complex numbers such that $-1 \in A$, then A has no nonnegativity domain.

To check this, by the multiplicative form of Theorem 3.3, it is enough to note only that $(-1)^2 = 1$, but $-1 \neq 1$. Thus, A is not 2-cancellable.

A standard application of Zorn's lemma gives the existence of a nonnegativity domain of a 2-cancellable group. More generally, we can prove the following

THEOREM 3.5. *If A is a symmetric and 2-cancellable subset of a group X and B is an antisymmetric subset of A, then there exists a nonnegativity domain D of A such that $B \subset D$.*

Proof. Let $\mathcal{D}$ be the family of all antisymmetric subsets D of A such that $B \subset D$. Then, $B \in \mathcal{D}$, and thus $\mathcal{D} \neq \emptyset$. Moreover, since the union of a directed family of antisymmetric subsets of X is also antisymmetric, it is clear that $\mathcal{D}$ is, in particular, inductive.

Therefore, by Zorn's lemma, there exists a maximal element D of $\mathcal{D}$. Now, since D is an antisymmetric subset of A such that $B \subset D$ and $D \cup (-D) \subset A \cup (-A) = A$, it remains only to show that $A \subset D \cup (-D)$. For this, assume on the contrary that there exists $x \in A$ such that $x \notin D$ and $-x \notin D$. Define $E = D \cup \{x\}$. Then, we evidently have $B \subset E \subset A$. Moreover, if $y \in E \cap (-E)$, i. e., $y \in D \cup \{x\}$ and $-y \in D \cup \{x\}$, then by examining the four possible cases and using the assumptions $x \notin D$ and $-x \notin D$, we can see that either $y \in D \cap (-D)$ or $2y = 0$ can hold. Hence, by the antisymmetry of D and the 2-cancellability of A, it follows that $y = 0$. Therefore, $E \cap (-E) \subset \{0\}$, and thus $E \in \mathcal{D}$. Hence, by using the maximality of D and the inclusion $D \subset E$, we can infer that $E = D$, which is a contradiction. Therefore, the required assertion is true.

Now, as an immediate consequence of Theorems 2.5, 3.3 and 3.5, we can also state

THEOREM 3.6. *If A is a subset of a group X, then the following assertions are equivalent:*

(1) *A has a nonnegativity domain;*

(2) *A is symmetric and 2-cancellable;*

(3) *each antisymmetric subset of A can be extended to a nonnegativity domain of A.*

Hint. To prove the implication (2)⇒(1), note that $B = \emptyset$ is an antisymmetric subset of A. Therefore, if the assertion (2) holds, then Theorem 3.5 can be applied to get the assertion (1).

4. Infinitely cancellable subsets of groups

DEFINITION 4.1. A subset A of a group X is called infinitely cancellable if it is n-cancellable for all $n \in \mathbb{N}$.

REMARK 4.2. By using the notation $X_\infty = \bigcup_{n=1}^{\infty} X_n$, the above condition can be briefly expressed by writing that $A \cap X_\infty \subset \{0\}$.

Hence, it is clear that X is infinitely cancellable if and only if $X_\infty = \{0\}$. Moreover, it can be easily seen that $\{0\} \cup (X \setminus X_\infty)$ is the largest infinitely cancellable subset of X.

Analogously to Theorem 3.3, we can also easily prove the following

THEOREM 4.3. *If a subset A of a group X has an additive nonnegativity domain, then A is infinitely cancellable.*

Proof. Let D be an additive nonnegativity domain of A, and assume on the contrary that there exist $n \in \mathbb{N}$ and $x \in A$ such that $nx = 0$, but $x \neq 0$. Then, we necessarily have $n > 1$. Moreover, since $A = D \cup (-D)$, we have either $x \in D$ or $-x \in D$. If $x \in D$, then by using the assumptions $nx = 0$ and $D + D \subset D$, we can infer that $-x = (n-1)x \in D$. While, if $-x \in D$, then by using the above assumptions, we can infer that $x = (n-1)(-x) \in D$. Consequently, we have $x \in D \cap (-D) \subset \{0\}$, and hence $x = 0$. This contradiction proves the theorem.

EXAMPLE 4.4. If A is a subset of the multiplicative group of all nonzero complex numbers such that $i \in A$, then A has no multiplicative nonnegativity domain.

To check this, by the multiplicative form of Theorem 4.3, it is enough to note only that $i^4 = 1$, but $i \neq 1$. Thus, A is not 4-cancellable.

EXAMPLE 4.5. If A is finite subset of a group X such that $A \not\subset \{0\}$, then A has no additive nonnegativity domain.

To check this, assume on the contrary that D is an additive nonnegativity domain of A. Then, since $D \cup (-D) = A \not\subset \{0\}$, there exists $x \in D$ such that $x \neq 0$. Moreover, since $D + D \subset D$, we have $nx \in D \subset A$ for all $n \in \mathbb{N}$. Hence, by using the finiteness of A, we can infer that there exist $k, l \in \mathbb{N}$, with $k < l$, such that $kx = lx$, and hence $(l-k)x = 0$. Hence, by Theorem 4.3, it follows that $x = 0$. This contradiction proves the required assertion.

By using Theorem 4.3, we can also easily prove the following

THEOREM 4.6. *If D is an additive nonnegativity domain of a subset A of a group X, then $nx = y$ implies $x \in D$ for all $n \in N$, $x \in A$ and $y \in D$.*

Proof. Assume on the contrary that there exist $n \in \mathbb{N}$, $x \in A$ and $y \in D$ such that $nx = y$, but $x \notin D$. Then, since $A = D \cup (-D)$, we necessarily have $x \in -D$, and thus $-x \in D$. Hence, by using the assumptions $nx = y$ and $D + D \subset D$, we can infer that $-y = n(-x) \in D$, and hence $y \in -D$. Therefore, $y \in D \cap (-D) \subset \{0\}$, and hence $y = 0$. Consequently, $nx = 0$, and hence, by Theorem 4.3, $x = 0 = y \in D$. This contradiction proves the theorem.

EXAMPLE 4.7. If D is an additive nonnegativity domain of $\mathbb{Q}$, then either $D = \mathbb{Q}_{\oplus}$ or $D = -\mathbb{Q}_{\oplus}$.

Namely, by Theorem 2.4, we have $0 \in D$. Moreover, since $\mathbb{Q} = D \cup (-D)$, we have either $1 \in D$ or $-1 \in D$. If $1 \in D$, then by using the equality $n(1) = 1$ and Theorem 4.6 we can see that $1 \in D$ for all $n \in \mathbb{N}$. Hence, since $D + D \subset D$, it is clear that $k(1) \in D$ for all $n, k \in \mathbb{N}$. Therefore, $\mathbb{Q}_{\oplus} \subset D$. Hence, by Theorem 2.14, it is clear that $D = \mathbb{Q}_{\oplus}$. While, if $-1 \in D$, then we can similarly see that $D = -\mathbb{Q}_{\oplus}$.

EXAMPLE 4.8. If D is a closed and additive nonnegativity domain of $\mathbb{R}$, then either $D = \mathbb{R}_{\oplus}$ or $D = -\mathbb{R}_{\oplus}$.

Namely, by Theorem 2.6, $E = D \cap \mathbb{Q}$ is an additive nonnegativity domain of $\mathbb{Q}$. Moreover, by Example 4.7, we have either $E = \mathbb{Q}_{\oplus}$ and $E = -\mathbb{Q}_{\oplus}$. If $E = \mathbb{Q}_{\oplus}$, then $\mathbb{Q}_{\oplus} \subset D$. Hence, by using the denseness of $\mathbb{Q}$ and the closedness of D in $\mathbb{R}$, we can infer that $\mathbb{R}_{\oplus} = (\mathbb{Q}_{\oplus})^{-} \subset D^{-} = D$. Therefore, by Theorem 2.14, we have $D = \mathbb{R}_{\oplus}$. While, if $E = -\mathbb{Q}_{\oplus}$, then we can similarly see that $D = -\mathbb{R}_{\oplus}$.

5. Perfectly cancellable subsets of groups

DEFINITION 5.1. For any point x and any subset A of a group X, we define

$$F_A(x) = \Big(\big(A \cup \{x\} \big) \setminus \{0\} \Big)^{\#}.$$

The usefulness of this notation is apparent from the following two theorems.

THEOREM 5.2. *If A is a subset of a group X, then the following assertions are equivalent:*

(1) *A is infinitely cancellable;*
(2) $0 \notin F_{\emptyset}(x)$ *for all* $x \in A$.

Proof. If the assertion (2) does not hold, then there exists $x \in A$ such that $0 \in F_{\emptyset}(x) = (\{x\} \setminus \{0\})^{\#}$. Therefore, $x \neq 0$ and there exists $n \in \mathbb{N}$ such that $0 = nx$. Thus, the assertion (1) does not hold.

While, if the assertion (1) does not hold, then there exist $n \in \mathbb{N}$ and $x \in A$ such that $nx = 0$, but $x \neq 0$. Therefore, $0 \in (\{x\} \setminus \{0\})^{\#} = F_{\emptyset}(x)$, and thus the assertion does not hold.

THEOREM 5.3. *If B is a subset of a subset A of a group X such that B is contained in some additive nonnegativity domain of A, then for each $x \in A$ we have either $0 \notin F_B(x)$ or $0 \notin F_B(-x)$.*

Proof. Let D be an additive nonnegativity domain of A such that $B \subset D$, and assume on the contrary that there exists $x \in A$ such that $0 \in F_B(x)$ and $0 \in F_B(-x)$. Then, since $x \in A = D \cup (-D)$, we have either $x \in D$ or $-x \in D$. Moreover, since $0 \in F_B(x) = ((B \cup \{x\}) \setminus \{0\})^{\#}$, there exist finite families $(n_i)_{i=1}^k$ in $\mathbb{N}$ and $(u_i)_{i=1}^k$ in $(B \cup \{x\}) \setminus \{0\}$ such that $0 = \sum_{i=1}^k n_i u_i$. Hence, since $\{u_i\}_{i=1}^k \subset A \setminus \{0\}$, by Theorem 4.3 it is clear that $k > 1$. Therefore, we also have $n_k u_k = -\sum_{i=1}^{k-1} n_i u_i$. Now, if $x \in D$, then by using the inclusion $\{u_i\}_{i=1}^k \subset D$ and the additivity of D, we can see that $n_k u_k \in D \cap (-D) \subset \{0\}$, and thus $n_k u_k = 0$. Hence, since $u_k \in A$, by Theorem 4.3 it follows that $u_k = 0$, which is a contradiction.

Moreover, to complete the proof, we can quite similarly see that the inclusions $-x \in D$ and $0 \in F_B(-x)$ also lead to a contradiction.

REMARK 5.4. Since the operation $\#$ is increasing and $A = \emptyset$ implies $B = \emptyset$, it is clear that in addition we also have $0 \notin F_B(0)$.

Moreover, by Theorem 5.3, we may also naturally introduce the following

DEFINITION 5.5. A subset A of a group X is called perfectly cancellable if for any additive and antisymmetric subset B of A and any $x \in A \setminus (B \cup (-B))$ we have either $0 \notin F_B(x)$ or $0 \notin F_B(-x)$.

The relationship between infinite and perfect cancellability can be cleared up by the following two theorems.

THEOREM 5.6. *If A is a perfectly cancellable subset of a group X, then A is in particular infinitely cancellable.*

Proof. If A is not infinitely cancellable, then by Theorem 5.2 there exist $x \in A$ such that $0 \in F_{\emptyset}(x)$. Thus, since $F_{\emptyset}(-x) = -F_{\emptyset}(x)$, we also have $0 \in F_{\emptyset}(-x)$. Hence, since $B = \emptyset$ is an additive and antisymmetric subset of A such that $A = A \setminus (B \cup (-B)$, it is clear A cannot be perfectly cancellable.

THEOREM 5.7. *If A is a commutative subset of a group X, then the following assertions are equivalent:*

(1) *A is perfectly cancellable;*
(2) *A is infinitely cancellable.*

Proof. By Theorem 5.6, we need only to prove the implication (2)$\Rightarrow$(1). For this, assume on the contrary that the assertion (2) holds, but the assertion (1) does not hold. Then, there exist an additive and antisymmetric subset B of A and an $x \in A$ such that $0 \in F_B(x)$ and $0 \in F_B(-x)$.

From the first inclusion, because of $F_B(x) = \big((B \cup \{x\}) \setminus \{0\}\big)^{\#}$, it follows that there exist finite families $(n_i)_{i=1}^k$ in $\mathbb{N}$ and $(u_i)_{i=1}^k$ in $\big(B \cup \{x\}\big) \setminus \{0\}$ such that $0 = \sum_{i=1}^k n_i u_i$. Hence, by the assertion (2), it is clear $u_i \in B \setminus \{0\}$ for some $i = 1, \ldots, k$. Moreover, by using a similar argument as in the proof of Theorem 5.3, we can also easily see that $u_i = x \neq 0$ for some $i = 1, \ldots, k$.

Now, by the commutativity of A and the additivity of B it is clear that there exist $n \in \mathbb{N}$ and $y \in B \setminus \{0\}$ such that $0 = nx + y$. Moreover, by the second inclusion $0 \in F_B(-x)$, it is clear that there exist $m \in \mathbb{N}$ and $z \in B \setminus \{0\}$ such that $0 = m(-x) + z$. This implies that $(nm)x = n(mx) = nz$ and $(nm)x = m(nx) = m(-y) = -my$. Hence, by the additivity and the antisymmetry of B, it follows that $(nm)x \in B \cap (-B) \subset \{0\}$, and thus $(nm)x = 0$. Therefore, by the assertion (2), we necessarily have either $n = 0$ or $m = 0$. This contradiction proves the required assertion.

6. The existence of additive nonnegativity domains

Analogously to Theorem 3.5, we can also prove the following

THEOREM 6.1. *If A is a symmetric and perfectly cancellable subset of a group X and B is an additive and antisymmetric subset of A, then there exists an additive nonnegativity domain D of X such that $B \subset D$.*

Proof. Let $\mathcal{D}$ be the family of all additive and antisymmetric subsets D of A such that $B \subset D$. Then, $B \in \mathcal{D}$, and thus $\mathcal{D} \neq \emptyset$. Moreover, since the union of a directed family of additive and antisymmetric subsets of X is also additive and antisymmetric, it is clear that $\mathcal{D}$ is, in particular, inductive.

Therefore, by Zorn's lemma, there exists a maximal element D of $\mathcal{D}$. Now, since D is an additive and antisymmetric subset of A such that $B \subset D$ and $D \cap (-D) \subset A \cap (-A) = A$, it remains only to show that $A \subset D \cup (-D)$. For this, assume on the contrary that there exists $x \in A$ such that $x \notin D$ and $-x \notin D$. Define $D_1 = \big(D \cup \{x\}\big)^{\#}$ and $D_1 = \big(D \cup \{-x\}\big)^{\#}$. Then, it is clear that, for $i = 1, 2$, D_i is an additive subset of X such that $D \subset D_i$, but $D_i \neq D$. Therefore, by the maximality of D in $\mathcal{D}$, we necessarily have $D_i \cap (-D_i) \not\subset \{0\}$.

Since $D_1 \cap (-D_1) \not\subset \{0\}$, there exists $y \in D_1$ such that $-y \in D_1$, but $y \neq 0$. Therefore, there exist finite families $(n_i)_{i=1}^{k}$ and $(m_j)_{j=1}^{l}$ in $\mathbb{N}$ and $(u_i)_{i=1}^{k}$ and $(v_j)_{j=1}^{l}$ in $D \cup \{x\}$ such that $y = \sum_{i=1}^{k} n_i u_i$ and $-y = \sum_{j=1}^{l} m_j v_i$. Since $y \neq 0$ and $-y \neq 0$, we may assume that $u_i \neq 0$ for all $i = 1, \dots, k$ and $v_j \neq 0$ for all $j = 1, \dots, l$. Thus, we have

$$0 = y + (-y) = \sum_{i=1}^{k} n_i u_i + \sum_{j=1}^{l} m_j v_i \in \Big(\big(D \cup \{x\}\big) \setminus \{0\} \Big)^{\#} = F_D(x).$$

By using $D_2 \cap (-D_2) \not\subset \{0\}$, we can quite similarly see that $0 \in F_D(-x)$. Thus, since $x \in A \setminus \big(D \cup (-D)\big)$, the set A cannot be perfectly cancellable. This contradiction proves the required assertion.

Now, as some immediate consequences of Theorems 2.5, 5.3 and 6.1, we can also state the following two theorems.

THEOREM 6.2. *If A is a subset of group X, then the following assertions are equivalent:*

(1) *A has an additive nonnegativity domain;*

(2) *A is symmetric and perfectly cancellable;*

(3) *each additive and antisymmetric subset of A can be extended to an additive nonnegativity domain of A.*

THEOREM 6.3. *If A is a symmetric and perfectly cancellable subset of a group X, then for any additive and antisymmetric subset B of A and any $x \in A$ we have either $0 \notin F_B(x)$ or $0 \notin F_B(-x)$.*

From Theorems 6.1 and 6.2, by using Theorem 5.7, we can also immediately get the following two theorems.

THEOREM 6.4. *If A is a symmetric, commutative and infinitely cancellable subset of a group X and B is an additive and antisymmetric subset of A, then there exists an additive nonnegativity domain D of X such that $B \subset D$.*

THEOREM 6.5. *If A is a commutative subset of a group X, then the following assertions are equivalent:*

(1) *A has an additive nonnegativity domain;*

(2) *A is symmetric and infinitely cancellable.*

REMARK 6.6. According to Fuchs [4, pp. 36 and 39], the $A = X$ particular cases of Theorems 6.5 and 6.4 have already been established by Levi [6] and Lorenzen [7], respectively.

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INSTITUTE OF MATHEMATICS AND INFORMATICS
UNIVERSITY OF DEBRECEN
H-4010 DEBRECEN, Pf. 12
HUNGARY
E-mails: glavosit@dragon.klte.hu, szaz@math.klte.hu

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