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CHARACTERIZATIONS BASED ON k-TH UPPER AND LOWER RECORD VALUES

Abstract. Let $\{Y_n^{(k)},\, n \geq 1\}$ and $\{Z_n^{(k)},\, n \geq 1\}$ denote respectively the sequences of k-th upper and lower record values of the sequence $\{X_n,\, n \geq 1\}$ of independent identically distributed random variables with distribution function F. Let n, k and r be given positive integers. We characterize distributions for which one of the conditional expectations $E(Y_{n+r}^{(k)} \mid Y_n^{(k)})$, $E(Y_n^{(k)} \mid Y_{n+r}^{(k)})$, $E(Z_{n+r}^{(k)} \mid Z_n^{(k)})$ or $E(Z_n^{(k)} \mid Z_{n+r}^{(k)})$, is linear. For example, distributions for which $E(Y_{n+r}^{(k)} \mid Y_n^{(k)})$ has the form $E(Y_{n+r}^{(k)} \mid Y_n^{(k)}) = aY_n^{(k)} + b$ for some $a, b \in \mathbb{R}$.

1. Introduction

Let $\{X_n,\, n \geq 1\}$ be a sequence of independent identically distributed random variables with distribution function (df) F and probability density function (pdf) f. Let $X_{1:n}, \ldots, X_{n:n}$ denote the order statistics of a sample $X_1, \ldots, X_n$. For a fixed $k \geq 1$ we define the sequence $U_k(1), U_k(2), \ldots$ of k-th upper record times of $X_1, X_2, \ldots$ as follows: $U_k(1) = 1$, and for $n = 2, 3, \ldots$,

$$U_k(n) = \min\left\{j > U_k(n-1) : X_{j:j+k-1} > X_{U_k(n-1):U_k(n-1)+k-1}\right\}.$$

Write

$$Y_n^{(k)} = X_{U_k(n):U_k(n)+k-1}, \quad n \geq 1.$$

The sequence $\{Y_n^{(k)},\, n \geq 1\}$ is called the sequence of k-th (upper) record values of the above sequence [3]. Note that for $k = 1$ we have $Y_n^{(1)} = X_{U_1(n):U_1(n)} := R_n$ — the upper record values of the sequence $\{X_n,\, n \geq 1\}$ and that $Y_1^{(k)} = X_{1:k}$.

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Similarly for a fixed $k \geq 1$ we define the sequence $L_k(1), L_k(2), \ldots$ of k-th lower record times of $X_1, X_2, \ldots$ as follows: $L_k(1) = 1$, and for $n = 2, 3, \ldots,$

$$L_k(n) = \min\left\{j > L_k(n-1) : X_{k:L_k(n-1)+k-1} > X_{k:j+k-1}\right\},$$

Write

$$Z_n^{(k)} = X_{k:L_k(n)+k-1}, \quad n \geq 1.$$

The sequence $\{Z_n^{(k)},\, n \geq 1\}$ is called the sequence of k-th (lower) record values of the above sequence [7]. Note that for $k = 1$ we have $Z_n^{(1)} = X_{1:L_1(n)} := R'_n$ — the lower record values of the sequence $\{X_n,\, n \geq 1\}$ and that $Z_1^{(k)} = X_{k:k}$.

The aim of the paper is to characterize absolutely continuous distributions by linearity of regression of k-th upper and lower record values. We show that $E(Y_{n+r}^{(k)} \mid Y_n^{(k)})$ is linear iff F is an exponential, Pareto or power distribution function with regression parameters depending on k, and $E(Y_n^{(k)} \mid Y_{n+r}^{(k)})$ is linear iff F is a negative Gumbel, negative Fréchet or Weibull distribution function with regression parameters which, in contrast with the previous case, do not depend on k. These results generalize those given by [5, 6] where the case $k = 1$ and $r = 1$ was considered and by [2] who studied the case $k = 1$.

Moreover, we prove that $E(Z_{n+r}^{(k)} \mid Z_n^{(k)})$ is linear iff F is a negative exponential, negative Pareto or inverse power distribution with regression parameters depending on k and $E(Z_n^{(k)} \mid Z_{n+r}^{(k)})$ is linear iff F is a Gumbel, Fréchet or negative Weibull distribution with regression parameters which also do not depend on k.

In Section 2 we give definitions of distribution functions for which the regressions $E(Y_{n+r}^{(k)} \mid Y_n^{(k)})$, $E(Y_n^{(k)} \mid Y_{n+r}^{(k)})$, $E(Z_{n+r}^{(k)} \mid Z_n^{(k)})$, $E(Z_n^{(k)} \mid Z_{n+r}^{(k)})$, are linear while in Section 3 we discuss linearity of regression for k-th upper and lower record values. Characterizations of distributions by linearity of regressions of the k-th upper and lower record values are presented in Section 4.

2. List of working distributions

In the paper we use the following notation of probability distribution functions:

- $\mathrm{Exp}(\lambda, \gamma)$ – the exponential distribution with

$$F(x) = 1 - \exp(-\lambda(x - \gamma)), \quad x > \gamma;\ \lambda > 0,\ \gamma \in \mathbb{R}.$$

- $\mathrm{Par}(\theta,\mu,\delta)$ – the Pareto distribution with
$$F(x) = 1 - \left(\frac{\mu+\delta}{x+\delta}\right)^{\theta}, \quad x > \mu;\ \theta > 0,\ \mu,\delta \in \mathbb{R},\ \mu+\delta > 0.$$
- $\mathrm{Pow}(\theta,\mu,\nu)$ – the power distribution with
$$F(x) = 1 - \left(\frac{\nu-x}{\nu-\mu}\right)^{\theta}, \quad \mu < x < \nu;\ \theta > 0,\ \mu,\nu \in \mathbb{R},\ \mu < \nu.$$
- $\mathrm{NG}(\beta,\gamma)$ – the negative Gumbel distribution with
$$F(x) = 1 - \exp\left(-e^{\beta(x-\gamma)}\right), \quad x \in \mathbb{R};\ \beta > 0, \gamma \in \mathbb{R}.$$
- $\mathrm{NFre}(\theta,\gamma,\alpha)$ – the negative Fréchet distribution with
$$F(x) = 1 - \exp\left(-\left(\frac{\alpha-\gamma}{\alpha-x}\right)^{\theta}\right), \quad x < \alpha;\ \theta > 0,\ \gamma,\alpha \in \mathbb{R},\ \alpha > \gamma.$$
- $\mathrm{W}(\theta,\mu,\gamma)$ – the Weibull distribution with
$$F(x) = 1 - \exp\left(-\left(\frac{x-\mu}{\gamma-\mu}\right)^{\theta}\right), \quad x > \mu;\ \theta > 0,\ \mu,\gamma \in \mathbb{R},\ \gamma > \mu.$$
- $\mathrm{NExp}(\lambda,\nu)$ – the negative exponential distribution with
$$F(x) = \exp\left(\lambda(x-\nu)\right), \quad x < \nu;\ \lambda > 0,\ \nu \in \mathbb{R}.$$
- $\mathrm{NPar}(\theta,\nu,\delta)$ – the negative Pareto distribution with
$$F(x) = \left(\frac{\delta-\nu}{\delta-x}\right)^{\theta}, \quad x < \nu;\ \theta > 0,\ \nu,\delta \in \mathbb{R},\ \nu < \delta.$$
- $\mathrm{IPow}(\theta,\alpha,\beta)$ – the inverse power distribution with
$$F(x) = \left(\frac{x-\alpha}{\beta-\alpha}\right)^{\theta}, \quad \alpha < x < \beta;\ \theta > 0,\ \alpha,\beta \in \mathbb{R},\ \alpha < \beta.$$
- $\mathrm{G}(\beta,\gamma)$ – the Gumbel distribution with
$$F(x) = \exp\left(-e^{-\beta(x-\gamma)}\right), \quad x \in \mathbb{R};\ \beta > 0,\ \gamma \in \mathbb{R}.$$
- $\mathrm{Fre}(\theta,\mu,\delta)$ – the Fréchet distribution with
$$F(x) = \exp\left(-\left(\frac{\delta-\mu}{x-\mu}\right)^{\theta}\right), \quad x > \mu;\ \theta > 0,\ \mu,\delta \in \mathbb{R},\ \mu < \delta.$$
- $\mathrm{NW}(\theta,\mu,\gamma)$ – the negative Weibull distribution with
$$F(x) = \exp\left(-\left(\frac{\mu-x}{\mu-\gamma}\right)^{\theta}\right), \quad x < \mu;\ \theta > 0,\ \mu,\gamma \in \mathbb{R},\ \mu > \gamma.$$

Moreover, for the df F we denote
$$l_F = \inf\{x : F(x) > 0\}, \quad r_F = \sup\{x : F(x) < 1\}.$$

3. Linearity of regression of k-th upper and lower record values

3.1. k-th upper record values

It is known [3] that the pdf of $Y_n^{(k)}$ is

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!}[-\log(1-F(x))]^{n-1}[1-F(x)]^{k-1}f(x)$$

and the joint pdf of the vector $(Y_m^{(k)}, Y_n^{(k)})$ is

$$f_{Y_m^{(k)}, Y_n^{(k)}}(x,y) = \frac{k^n}{(m-1)!(n-m-1)!}[-\log(1-F(x))]^{m-1}\frac{f(x)}{1-F(x)} \times \left(\log\frac{1-F(x)}{1-F(y)}\right)^{n-m-1}[1-F(y)]^{k-1}f(y)$$

for $x < y$ and 0 for $x \geq y$ [4]. Therefore, the conditional density of $Y_{n+r}^{(k)}$ given $Y_n^{(k)} = x$ is

$$f_{Y_{n+r}^{(k)}|Y_n^{(k)}}(y \mid x) = \frac{k^r}{(r-1)!}\left(\log\frac{1-F(x)}{1-F(y)}\right)^{r-1}\left(\frac{1-F(y)}{1-F(x)}\right)^{k-1}\frac{f(y)}{1-F(x)} \tag{3.1}$$

for $y \geq x$, and the conditional density of $Y_n^{(k)}$ given $Y_{n+r}^{(k)} = y$ is

$$f_{Y_n^{(k)}|Y_{n+r}^{(k)}}(x \mid y) = \frac{(n+r-1)!}{(n-1)!(r-1)!}\frac{(-\log(1-F(x)))^{n-1}}{(-\log(1-F(y)))^{n+r-1}} \times \left(\log\frac{1-F(x)}{1-F(y)}\right)^{r-1}\frac{f(x)}{1-F(x)} \tag{3.2}$$

for $x < y$.

Using (3.1) we see that for $x \in (l_F, r_F)$

$$E(Y_{n+r}^{(k)} \mid Y_n^{(k)} = x) = \int_x^{r_F} y\frac{k^r}{(r-1)!}\left(\log\frac{1-F(x)}{1-F(y)}\right)^{r-1}\left(\frac{1-F(y)}{1-F(x)}\right)^{k-1}\frac{f(y)}{1-F(x)}\,dy$$

which for the exponential, Pareto and power distributions gives

$$E(Y_{n+r}^{(k)} \mid Y_n^{(k)}) = a_{rk}Y_n^{(k)} + b_{rk},$$

where for:

(a) $\mathrm{Exp}(\lambda, \gamma)$

$$a_{rk} = 1, \quad b_{rk} = \frac{r}{k\lambda};$$

(b) $\mathrm{Par}(\theta, \mu, \delta)$, $\theta > \frac{1}{k}$,

$$a_{rk} = \left(\frac{k\theta}{k\theta-1}\right)^r, \quad b_{rk} = \delta\left\{\left(\frac{k\theta}{k\theta-1}\right)^r - 1\right\};$$

(*c*) Pow(θ, μ, ν)

$$a_{rk} = \left(\frac{k\theta}{k\theta+1}\right)^r, \quad b_{rk} = \nu\left\{1-\left(\frac{k\theta}{k\theta+1}\right)^r\right\}.$$

For instance, if $F \sim \text{Exp}(\lambda, \gamma)$, then for $x \geq \gamma$

$$E(Y_{n+r}^{(k)} \mid Y_n^{(k)} = x) = \frac{(k\lambda)^r}{(r-1)!}\int_x^\infty y\,(y-x)^{r-1}\, e^{-k\lambda(y-x)}\,dy$$

$$= \frac{(k\lambda)^r}{(r-1)!}\int_0^\infty (x+u)u^{r-1}\, e^{-k\lambda u}\,du$$

$$= x\frac{(k\lambda)^r}{(r-1)!}\frac{\Gamma(r)}{(k\lambda)^r} + \frac{(k\lambda)^r}{(r-1)!}\frac{\Gamma(r+1)}{(k\lambda)^{r+1}} = x + \frac{r}{k\lambda}.$$

Using (3.2) we see that the regression

$$(3.3)\quad E\,(Y_n^{(k)} \mid Y_{n+r}^{(k)} = y) = \frac{(n+r-1)!}{(n-1)!(r-1)!}\frac{1}{(-\log(1-F(y)))^{n+r-1}}$$
$$\times \int_{l_F}^{y} x\,(-\log(1-F(x)))^{n-1}\left(\log\frac{1-F(x)}{1-F(y)}\right)^{r-1}\frac{f(x)}{1-F(x)}\,dx$$

does not depend on k. Moreover, we see that for negative Gumbel, negative Fréchet and Weibull distributions we have

$$E(Y_n^{(k)} \mid Y_{n+r}^{(k)}) = a_{rk}Y_{n+r}^{(k)} + b_{rk},$$

where a_{rk} and b_{rk} are a_r and b_r, respectively, for all $k \geq 1$ and (c.f. [2]) for:

(*a*) NG(β, γ)

$$a_r = 1, \quad b_r = \frac{(n+r-1)!}{\beta(n-1)!}\sum_{i=0}^{r-1}\frac{(-1)^{i+1}}{i!(r-i-1)!(n+i)^2},$$

(*b*) NFre(θ, γ, α), if $\theta > 1/n$,

$$a_r = \frac{\Gamma(n+r)\Gamma(n-\frac{1}{\theta})}{\Gamma(n)\Gamma(n+r-\frac{1}{\theta})}, \quad b_r = \alpha(a-1),$$

(*c*) W(θ, μ, γ)

$$a_r = \frac{\Gamma(n+r)\Gamma(n+\frac{1}{\theta})}{\Gamma(n)\Gamma(n+r+\frac{1}{\theta})}, \quad b_r = \mu(1-a).$$

3.2. k-th lower record values

It is known that the pdf of $Z_n^{(k)}$ is

$$f_{Z_n^{(k)}}(x) = \frac{k^n}{(n-1)!}\,(-\log F(x))^{n-1}\,(F(x))^{k-1}f(x),$$

and the joint pdf of $(Z_m^{(k)}, Z_n^{(k)})$ is

$$f_{Z_m^{(k)}, Z_n^{(k)}}(x,y) = \frac{k^n}{(m-1)!(n-m-1)!}(-\log F(x))^{m-1}\frac{f(x)}{F(x)} \times \left(\log\frac{F(x)}{F(y)}\right)^{n-m-1}(F(y))^{k-1}f(y)$$

for $x > y$ and 0 for $x \le y$ [7]. Therefore, the conditional density of $Z_{n+r}^{(k)}$ given $Z_n^{(k)} = x$ is

$$f_{Z_{n+r}^{(k)}|Z_n^{(k)}}(y \mid x) = \frac{k^r}{(r-1)!}\left(\log\frac{F(x)}{F(y)}\right)^{r-1}\left(\frac{F(y)}{F(x)}\right)^{k-1}\frac{f(y)}{F(x)}, \tag{3.4}$$

for $y < x$, and the conditional density of $Z_n^{(k)}$ given $Z_{n+r}^{(k)}$ is

$$f_{Z_n^{(k)}|Z_{n+r}^{(k)}}(x \mid y) = \frac{(n+r-1)!}{(n-1)!(r-1)!}\frac{(-\log F(x))^{n-1}}{(-\log F(y))^{n+r-1}} \times \left(\log\frac{F(x)}{F(y)}\right)^{r-1}\frac{f(x)}{F(x)}, \tag{3.5}$$

for $x > y$.

Using (3.4) we obtain that for $x \in (l_F, r_F)$

$$E(Z_{n+r}^{(k)} \mid Z_n^{(k)} = x) = \int_{l_F}^{x} y\frac{k^r}{(r-1)!}\left(\log\frac{F(x)}{F(y)}\right)^{r-1}\left(\frac{F(y)}{F(x)}\right)^{k-1}\frac{f(y)}{F(x)}\,dy,$$

which for the negative Gumbel, negative Fréchet and Weibull distributions gives

$$E(Z_{n+r}^{(k)} \mid Z_n^{(k)}) = a_{rk}Z_n^{(k)} + b_{rk},$$

where for:

(a) NExp(λ, ν)

$$a_{rk} = 1, \quad b_{rk} = -\frac{r}{k\lambda};$$

(b) NPar(θ, ν, δ), $\theta > 1/k$,

$$a_{rk} = \left(\frac{k\theta}{k\theta-1}\right)^r, \quad b_{rk} = \delta\left\{1 - \left(\frac{k\theta}{k\theta-1}\right)^r\right\};$$

(c) IPow(θ, α, β)

$$a_{rk} = \left(\frac{k\theta}{k\theta+1}\right)^r, \quad b_{rk} = \alpha\left\{\left(\frac{k\theta}{k\theta+1}\right)^r - 1\right\}.$$

Using (3.5) we see that for $y \in (l_F, r_F)$

$$E(Z_n^{(k)} \mid Z_{n+r}^{(k)} = y) = \frac{(n+r-1)!}{(n-1)!(r-1)!} \frac{1}{(-\log F(y))^{n+r-1}}$$
$$\times \int_y^{r_F} x\,(-\log F(x))^{n-1} \left(\log \frac{F(x)}{F(y)}\right)^{r-1} \frac{f(x)}{F(x)}\,dx$$

which for the Gumbel, Fréchet and negative Weibull distribution gives

$$E(Z_n^{(k)} \mid Z_{n+r}^{(k)}) = a_{rk} Z_{n+r}^{(k)} + b_{rk},$$

where a_{rk} and b_{rk} are a_r and b_r, respectively, for all $k \geq 1$ and for:

(*a*) $\mathrm{G}(\beta, \gamma)$

$$a_r = 1, \quad b_r = \frac{(n-r-1)!}{\beta(n-1)!} \sum_{j=0}^{r-1} \frac{(-1)^j}{j!(r-1-j)!(n+j)^2},$$

(*b*) $\mathrm{Fre}(\theta, \mu, \delta)$, $\theta > 1/n$,

$$a_r = \frac{\Gamma(n+r)\Gamma\left(n - \frac{1}{\theta}\right)}{\Gamma(n)\Gamma\left(n + r - \frac{1}{\theta}\right)}, \quad b_r = \delta(1-a),$$

(*c*) $\mathrm{NW}(\theta, \mu, \gamma)$

$$a_r = \frac{\Gamma(n+r)\Gamma\left(n + \frac{1}{\theta}\right)}{\Gamma(n)\Gamma\left(n + r + \frac{1}{\theta}\right)}, \quad b_r = \mu(a-1).$$

4. Characterizations by linearity of regression of k-th upper and lower record values

4.1. k-th upper record values

THEOREM 1. *Let $\{X_j,\, j \geq 1\}$ be a sequence of i.i.d. random variables with df F. If for fixed positive integers $n, r, k \geq 1$*

$$E(Y_{n+r}^{(k)} \mid Y_n^{(k)}) = aY_n^{(k)} + b \tag{4.1}$$

for some $a, b \in \mathbb{R}$, then only the following three cases are possible:

(*a*) *$a = 1$ and $X_i \sim \mathrm{Exp}(\lambda, \gamma)$ where $\lambda = \frac{r}{kb}$ and $\gamma \in \mathbb{R}$;*

(*b*) *$a > 1$ and $X_i \sim \mathrm{Par}(\theta, \mu, \delta)$ where*

$$\theta = \frac{\sqrt[r]{a}}{k\,(\sqrt[r]{a} - 1)}, \quad \delta = \frac{b}{a-1},$$

and $\mu \in \mathbb{R}$ is such that $\mu + \delta > 0$;

(*c*) $0 < a < 1$ *and* $X_i \sim \text{Pow}(\theta, \mu, \nu)$ *where*

$$\theta = \frac{\sqrt[r]{a}}{k\,(1 - \sqrt[r]{a})}, \quad \nu = \frac{b}{1-a},$$

and $\mu \in \mathbb{R}$ *is such that* $\mu < \nu$.

Proof. We use Theorem 1 of [2] and the fact that for any $k \geq 1$ the sequence $\{Y_n^{(k)},\, n \geq 1\}$ of k-th upper record values from df F has the same finite-dimensional distributions as the sequence $\{\widetilde{R}_n,\, n \geq 1\}$ of record values from the df F

$$G_k(x) = 1 - (1 - F(x))^k \tag{4.2}$$

(c.f. [1]). Then (4.1) is equivalent to

$$E(\widetilde{R}_{n+r} \mid \widetilde{R}_n) = a\widetilde{R}_n + b.$$

It follows from Theorem 1 of [2] that only three cases are possible:

(*a*) $a = 1$ and

$$G_k(x) = 1 - \exp\left(-\widetilde{\lambda}(x - \gamma)\right), \quad x > 0,$$

where $\widetilde{\lambda} = \frac{r}{b}$ and $\gamma \in \mathbb{R}$;

(*b*) $a > 1$ and

$$G_k(x) = 1 - \left(\frac{\mu + \delta}{x + \delta}\right)^{\widetilde{\theta}}, \quad x > \mu,$$

where

$$\widetilde{\theta} = \frac{\sqrt[r]{a}}{\sqrt[r]{a} - 1}, \quad \delta = \frac{b}{a-1},$$

and $\mu \in \mathbb{R}$ is such that $\mu + \delta > 0$;

(*c*) $0 < a < 1$ and

$$G_k(x) = 1 - \left(\frac{\nu - x}{\nu - \mu}\right)^{\widetilde{\theta}}, \quad \mu < x < \nu,$$

where

$$\widetilde{\theta} = \frac{\sqrt[r]{a}}{1 - \sqrt[r]{a}}, \quad \nu = \frac{b}{1-a},$$

and $\mu \in \mathbb{R}$ is such that $\mu < \nu$. Using (4.2) we complete the proof. ∎

For the regression $E(Y_n^{(k)} \mid Y_{n+r}^{(k)})$ we have

THEOREM 2. *Let* $\{X_j,\, j \geq 1\}$ *be a sequence of i.i.d. random variables with df* F. *If for fixed positive integers* $n, r, k \geq 1$

$$E(Y_n^{(k)} \mid Y_{n+r}^{(k)}) = aY_{n+r}^{(k)} + b \tag{4.3}$$

for some $a, b \in \mathbb{R}$, *then only the following three cases are possible:*

(*a*) $a = 1$ *and* $X_i \sim \mathrm{NG}(\beta, \gamma)$;
(*b*) $a > 1$ *and* $X_i \sim \mathrm{NFre}(\theta, \gamma, \alpha)$;
(*c*) $0 < a < 1$ *and* $X_i \sim \mathrm{W}(\theta, \mu, \gamma)$.

Proof. Note that if (4.3) holds then by (3.3) for $y \in (l_F, r_F)$

$$E(R_n \mid R_{n+r} = y) = E(Y_n^{(k)} \mid Y_{n+r}^{(k)} = y) = ay + b. \tag{4.4}$$

Therefore (4.4) implies that

$$E(R_n \mid R_{n+r}) = aR_{n+r} + b.$$

Now the result follows from Theorem 3 of [2]. ∎

The parameters of the distributions are determined from the respective formulae for a and b of Section 3.

4.2. k-th lower record values

THEOREM 3. *Let* $\{X_j,\ j \geq 1\}$ *be a sequence of i.i.d. random variables with df* F. *If for fixed positive integers* $n, r, k \geq 1$

$$E(Z_{n+r}^{(k)} \mid Z_n^{(k)}) = aZ_n^{(k)} + b \tag{4.5}$$

for some $a, b \in \mathbb{R}$, *then only the following three cases are possible:*
(*a*) $a = 1$ *and* $X_i \sim \mathrm{NExp}(\lambda, \nu)$ *where* $\lambda = -\frac{r}{kb}$ *and* $\nu \in \mathbb{R}$;
(*b*) $a > 1$ *and* $X_i \sim \mathrm{NPar}(\theta, \nu, \delta)$ *where*

$$\theta = \frac{\sqrt[r]{a}}{k\,(\sqrt[r]{a} - 1)}, \quad \delta = \frac{b}{1 - a},$$

and $\nu \in \mathbb{R}$ *is such that* $\nu < \delta$;
(*c*) $0 < a < 1$ *and* $X_i \sim \mathrm{IPow}(\theta, \alpha, \beta)$ *where*

$$\theta = \frac{\sqrt[r]{a}}{k\,(1 - \sqrt[r]{a})}, \quad \alpha = \frac{b}{a - 1},$$

and $\beta \in \mathbb{R}$ *is such that* $\beta > \alpha$.

Proof. Note that by (3.1) and (3.4)

$$f_{Z_{n+r}^{(k)} \mid Z_n^{(k)}}(y \mid x) = f_{\widetilde{Y}_{n+r}^{(k)} \mid \widetilde{Y}_n^{(k)}}(-y \mid -x),$$

where $\{\widetilde{Y}_n^{(k)},\ n \geq 1\}$ is the sequence of k-th upper record values from the df $G(x) = 1 - F(-x)$ with the pdf $g(x) = f(-x)$. Thus, for $x \in (l_F, r_F)$

$$E(Z_{n+r}^{(k)} \mid Z_n^{(k)} = x) = -E(\widetilde{Y}_{n+r}^{(k)} \mid \widetilde{Y}_n^{(k)} = -x),$$

which implies that if (4.5) holds true, then

$$E(\widetilde{Y}_{n+r}^{(k)} \mid \widetilde{Y}_n^{(k)} = x) = ax - b,$$

for $x \in (l_G, r_G)$, where $l_G = -r_F$, $r_G = -l_F$. Therefore, by Theorem 1 only the following three cases are possible:

(*a*) $a = 1$ and

$$G(x) = 1 - \exp\left(-\lambda(x-\gamma)\right), \quad x > \gamma,$$

where

$$\lambda = -\frac{r}{kb}, \quad \gamma \in \mathbb{R};$$

(*b*) $a > 1$ and

$$G(x) = 1 - \left(\frac{\mu+\delta}{x+\delta}\right)^{\theta}, \quad x > \mu,$$

where

$$\theta = \frac{\sqrt[r]{a}}{k\left(\sqrt[r]{a}-1\right)}, \quad \delta = \frac{b}{1-a},$$

and $\mu \in \mathbb{R}$ is such that $\mu + \delta > 0$;

(*c*) $0 < a < 1$ and

$$G(x) = 1 - \left(\frac{\nu - x}{\nu - \mu}\right)^{\theta}, \quad \mu < x < \nu.$$

$$\theta = \frac{\sqrt[r]{a}}{k\left(1-\sqrt[r]{a}\right)}, \quad \nu = \frac{b}{a-1},$$

and $\mu \in \mathbb{R}$ is such that $\mu < \nu$.

Taking into account that $G(x) = 1 - F(-x)$ we get the statement of Theorem 3. ∎

In the same way, we can prove the following result, which characterizes the Gumbel, Fréchet and negative Weibull distributions.

THEOREM 4. *Let* $\{X_j,\ j \geq 1\}$ *be a sequence of i.i.d. random variables with df* F. *If for fixed positive integers* $n, r, k \geq 1$

$$E(Z_n^{(k)} \mid Z_{n+r}^{(k)}) = aZ_{n+r}^{(k)} + b$$

for some $a, b \in \mathbb{R}$, *then only the following three cases are possible:*

(*a*) $a = 1$ *and* $X_i \sim \mathrm{G}(\beta, \gamma)$;

(*b*) $a > 1$ *and* $X_i \sim \mathrm{Fre}(\theta, \mu, \delta)$;

(*c*) $0 < a < 1$ *and* $X_i \sim \mathrm{NW}(\theta, \mu, \gamma)$.

The parameters of the distributions are determined from the respective formulae for a and b of Section 3.

References

[1] B. C. Arnold, N. Balakrishnan, H. N. Nagaraja, *Records*, John Wiley & Sons, New York, 1998.

[2] A. Dembińska, J. Wesołowski, *Linearity of regression for non-adjacent record values*, J. Statist. Plann. Inference, 90 (2000), 195–205.

[3] W. Dziubdziela, B. Kopociński, *Limiting properties of the k-th record values*, Zastos. Mat., 15 2 (1976), 187–190.

[4] Z. Grudzień, *Charakteryzacja rozkładów w terminach statystyk rekordowych oraz rozkłady i momenty statystyk porządkowych i rekordowych z prób o losowej liczebności*, (in Polish) Ph.D. Thesis, UMCS, Lublin, 1982.

[5] H. N. Nagaraja, *On a characterization based on record values*, Austral. J. Statist., 19 (1977), 70–73.

[6] H. N. Nagaraja, *Some characterizations of continuous distributions based on regressions of order statistics and record values*, Sankhyā, Ser. A, 50 (1988), 70–73.

[7] P. Pawlas, D. Szynal, *Relations for single and product moments of k–th record values from exponential and Gumbel distributions*, J. Appl. Stat. Sci, 7 (1998), 53–62.

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