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LIFTINGS OF PROJECTABLE PROJECTABLE VECTOR FIELDS TO 1-FORMS ON HIGHER ORDER COTANGENT BUNDLES OVER FIBERED FIBERED MANIFOLDS

Abstract. For an (m_1, m_2, n_1, n_2)-fibered fibered manifold Y and a projectable projectable vector field Z on Y we give a construction of a 1-form $A(Z)$ on the $(r_1, \dots, r_8)$-cotangent bundle $T^{r_1,\dots,r_8*}Y$. It is reflected in the concept of a natural operator $\mathcal{A}_Y : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}}Y \to T^*T^{r_1,\dots,r_8*}Y$ lifting projectable projectable vector fields on a Y to 1-forms on $T^{r_1,\dots,r_8*}Y$. We determine all natural operators of this kind and prove that they form a free $C^\infty(\mathbb{R}^{r_1+r_4+r_6+r_8})$-module of dimension $2(r_1+r_4+r_6+r_8)$. We construct explicitely a basis of such module.

0. Introduction

Roughly speaking, we generalize the construction of a 1-form $A(Z)$ on a higher order cotangent bundle from a vector field Z or a projectable vector field Z (in case of the category of fibered manifolds) to the category of (m_1, m_2, n_1, n_2)-fibered fibered manifolds. It is reflected in the concept of a natural operator over such objects, the classification result on which will be also presented.

We start with the notation of categories over manifolds to be used and the survey of already achieved results. We consider the category $\mathcal{M}f_m$ of m-dimensional manifolds with embeddings, the category $\mathcal{FM}$ of fibered manifolds and fibered maps, the category $\mathcal{FM}_{m,n}$ of fibered manifolds with m-dimensional bases, n-dimensional fibers and fibered embeddings. Further, we consider the category $\mathcal{F}^2\mathcal{M}$ of fibered fibered manifolds with fibered fibered maps, [16] and the category $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ of (m_1, m_2, n_1, n_2)-

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dimensional fibered fibered manifolds with fibered fibered embeddings and finally the category $\mathcal{VB}$ of vector bundles with vector bundle morphisms. We follow the basic notation of bundle functors and natural operators from the fundamental monograph [4]. The concept of a fibered fibered manifold was introduced in [16].

Fibered fibered manifolds are surjective fibered submersions between fibered manifolds. They naturally appear in differential geometry if we consider transverse natural bundles in the sense of Wolak, [18]. In [17], we have classified all product preserving bundle functors on the category $\mathcal{F}^2\mathcal{M}$. In the present paper, we consider a bundle functor $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2} \to \mathcal{VB}$ which does not preserve products, namely the cotangent bundle functor of order $(r_1,\dots,r_8)$, [8].

We are going to generalize the following results concerning the category $\mathcal{M}f_m$ together with an r-th order cotangent bundle $T^{r*}M = J^r(M,\mathbb{R})_0$ and the category $\mathcal{FM}_{m,n}$ together with an (r,s,q)-th order cotangent bundle $T^{r,s,q*}Y = J^{r,s,q}(Y,\mathbb{R}^{1,1})_0$, see [14] and [15].

In [14], the first author studied the problem how a vector field Z on an m-dimensional manifold M induces a 1-form $A(Z)$ on $T^{r*}M$. Such a concept is reflected in the concept of a natural operator $\mathcal{A} : T_{\mathcal{M}f_m} \to T^*T^{r*}$. He proved that for natural numbers $m \geq 2$ and all r, all natural operators $\mathcal{A} : T_{\mathcal{M}f_m} \to T^*T^{r*}$ form a free $2r$-dimensional module over $C^\infty(\mathbb{R}^r)$. Further, the basis of such a module was constructed.

In [15], the similar problem was studied, i.e. how a projectable projectable vector field Z on an (m,n)-dimensional fibered manifold Y induces a 1-form $A(Z)$ on the (r,s,q)-cotangent bundle $T^{r,s,q*}Y$. The problem is reflected in the concept of a natural operator $\mathcal{A} : T_{proj|\mathcal{FM}_{m,n}} \to T^*T^{r,s,q*}$. It is proved that for all natural numbers m,n,r,s,q satisfying $m \geq 2$ ans $s \geq r \leq q$ all natural operators $\mathcal{A} : T_{proj|\mathcal{FM}_{m,n}} \to T^*T^{r,s,q*}$ form a free $2(q+r)$-dimensional module over $C^\infty(\mathbb{R}^{q+r})$ and its basis is constructed.

In the present paper, we are going to extend the above mentioned results to fibered fibered manifolds, i.e. we study the problem how an $\mathcal{F}^2\mathcal{M}$-projectable (i.e. projectable projectable) vector field Z on $Y \in \mathcal{O}b(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$ induces a 1-form $A(Z)$ on on the $(r_1,\dots,r_8)$-th order cotangent bundle $T^{r_1,\dots,r_8*}Y = J^{r_1,\dots,r_8}(Y,\mathbb{R}^{1,1,1,1})_0$. This problem is reflected in the concept of a natural operator $\mathcal{A} : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\dots,r_8*}$. We prove that for all natural numbers $m_1,m_2,n_1,n_2,r_1,r_2,\dots,r_8$ satisfying $m_1 \geq 2$, $r_8 \geq r_4 \leq r_5 \geq r_3$, $r_8 \geq r_6 \leq r_7 \geq r_2$ and $r_1 \leq r_i$ for $i = 2,\dots,8$ all natural operators $\mathcal{A} : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\dots,r_8*}$ form a free $C^\infty(\mathbb{R}^{r_1+r_4+r_6+r_8})$-module of dimension $2(r_1+r_4+r_6+r_8)$ and we explicitely construct the basis of this module.

As a matter of fact, natural operators lifting functions, vector fields and 1-forms to some bundle functors played an important role in all papers devoted to prolongations of geometric structures, e.g. [19]. That is why such operators are studied, [1]–[17], [19]–[21], etc. Operators like this concerning higher-order cotangent bundle functors were studied in [1]–[4], [6]–[11], [14], [15], [20], [21] e.t.c.

From now on the usual coordinates on $\mathbb{R}^{m_1,m_2,n_1,n_2} = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ are denoted by $x^1,\dots,x^{m_1},y^1,\dots,y^{m_2},w^1,\dots,w^{n_1},v^1,\dots,v^{n_2}$. All manifolds and maps are assumed to be smooth, i.e. of class C^∞.

1. Fibered fibered manifold

The concept of a fibered fibered manifold was introduced in [16]. A fibered fibered manifold is a fibered surjective submersion $\pi : Y \to X$ between fibered manifolds, i.e. a surjective submersion which sends fibers into fibers such that the restricted and corestricted maps are submersions. We will write Y instead π if π is clear. If $\pi' : Y' \to X'$ is another fibered fibered manifold, a morphism $\pi \to \pi'$ is a fibered map $f : Y \to Y'$ such that there is a fibered map $f_0 : X \to X'$ satisfying $\pi' \circ f = f_0 \circ \pi$. Thus all fibered fibered manifolds form a category which we will denote by $\mathcal{F}^2\mathcal{M}$. The category is over manifolds, local and admissible in the sense of [4].

A fibered fibered manifold $\pi : Y \to X$ is said to be of dimension (m_1, m_2, n_1, n_2) if the fibered manifold Y is of dimension (m_1+n_1, m_2+n_2) and the fibered manifold X of dimension (m_1, m_2). All fibered fibered manifolds of dimension (m_1, m_2, n_1, n_2) and their local $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2} \subset \mathcal{F}^2\mathcal{M}$. Every $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object is locally isomorphic to $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ where the fibered manifolds forming the total space and the basis in this object are considered over $\mathbb{R}^{m_1} \times \mathbb{R}^{n_1}$ and $\mathbb{R}^{m_1}$' respectively. Let us denote such an object by $\mathbb{R}^{m_1,m_2,n_1,n_2}$.

In the end of the section, we define the concept of a projectable projectable vector field as follows. Let $\pi : Y \to X$ be a fibered fibered manifold (an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object). A projectable vector field Z on Y is said to be projectable projectable if there is a π-related projectable vector field Z_0 on X, [16]. Further, the flow of a projectable projectable vector field is formed by local $\mathcal{F}^2\mathcal{M}$-isomorphisms.

2. $(r_1,\dots,r_8)$-cotangent bundles of fibered fibered manifolds

Let $r_1,\dots,r_8$ be natural numbers such that $r_8 \ge r_4 \le r_5 \ge r_3$, $r_8 \ge r_6 \le r_7 \ge r_2$ and $r_1 \le r_i$ for $i = 2,\dots,8$. The concept of the r-jet and of the (r,s,q)-jet can be generalized as follows, see e.g. [8]. Let $\pi : Y \to X$ be a fibered fibered manifold being a surjective fibered submersion between fibered manifolds $p^Y : Y \to \underline{Y}$ and $p^X : X \to \underline{X}$. Further, let $\pi' : Y' \to X'$

be another fibered fibered manifold being a surjective fibered submersion between fibered manifolds $p^{Y'} : Y' \to \underline{Y}'$ and $p^{X'} : X' \to \underline{X}'$. Let $y \in Y$ be a point and $\underline{y} = p^Y(y) \in \underline{Y}$, $x = \pi(y) \in X$ and $\underline{x} = p^X(x) \in \underline{X}$ be its underlying points. Further, let $f, g : Y \to Y'$ be two fibered fibered maps with underlying maps $\underline{f}, \underline{g} : \underline{Y} \to \underline{Y}'$, $f_0, g_0 : X \to X'$ and $\underline{f}_0, \underline{g}_0 : X \to \underline{X}$. We say that f, g determine the same $(r_1, \ldots, r_8)$-jet at $y \in Y$, i.e. $j_y^{r_1,\ldots,r_8} f = j_y^{r_1,\ldots,r_8} g$ if the equalities $j_y^{r_1} f = j_y^{r_1} g$, $j_y^{r_2}(f|Y_x) = j_y^{r_2}(g|Y_x)$, $j_y^{r_3}(f|Y_{\underline{y}}) = j_y^{r_3}(g|Y_{\underline{y}})$, $j_x^{r_4} f_0 = j_x^{r_4} g_0$, $j_x^{r_5}(f_0|X_{\underline{x}}) = j_x^{r_5}(g_0|X_{\underline{x}})$, $j_{\underline{y}}^{r_6}(\underline{f}) = j_{\underline{y}}^{r_6}(\underline{g})$ $j_{\underline{y}}^{r_7}(\underline{f}|\underline{Y}_{\underline{x}}) = j_{\underline{y}}^{r_7}(\underline{g}|\underline{Y}_{\underline{x}})$ and $j_{\underline{x}}^{r_8}(\underline{f}_0) = j_{\underline{x}}^{r_8}(\underline{g}_0)$ hold. The space of all $(r_1, \ldots, r_8)$-jets of fibered fibered maps between fibered fibered manifolds is denoted $J^{(r_1,\ldots,r_8)}(Y, Y')$. The composition of fibered fibered maps induces the composition of $(r_1, \ldots, r_8)$-jets.

Let $r_1, \ldots, r_8$ be as above. Let m_1, m_2, n_1, n_2 be natural numbers. The r-cotangent bundle functor $T^{r*} = J^r(\cdot, \mathbb{R})_0 : \mathcal{M}f_m \to \mathcal{VB}$ ([4]) and the (r, s, q)-cotangent bundle functor $T^{r,s,q*} = J^{(r,s,q)}(\cdot, \mathbb{R}^{1,1})_0 : \mathcal{FM}_{m,n} \to \mathcal{VB}$ ([5], [13], [15]) can be generalized as follows. The space $T^{r_1,\ldots,r_8*}Y = J^{(r_1,\ldots,r_8)}(Y, \mathbb{R}^{1,1,1,1})_0$, $0 \in \mathbb{R}^4$ has an induced structure of a vector bundle over Y. Every $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-map $f : Y \to Y'$ induces a vector bundle map $T^{r_1,\ldots,r_8*}f : T^{r_1,\ldots,r_8*}Y \to T^{r_1,\ldots,r_8*}Y'$ covering f defined by

$$T^{r_1,\ldots,r_8*}f(j_y^{(r_1,\ldots,r_8)}\gamma) = j_{f(y)}^{(r_1,\ldots,r_8)}(\gamma \circ f^{-1})$$

for any fibered fibered map $\gamma : Y \to \mathbb{R}^{1,1,1,1}$ satisfying $\gamma(y) = 0$. The correspondence $T^{r_1,\ldots,r_8*} : \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2} \to \mathcal{VB}$ determines a vector bundle functor in the sense of [4]. In what follows, a bundle functor of this kind is said to be the $(r_1, \ldots, r_8)$-cotangent bundle.

3. Examples of natural operators $T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\ldots,r_8*}$

Let $m_1, m_2, n_1, n_2, r_1, \ldots, r_8$ be natural numbers as in Section 1. We are going to study how a $\mathcal{F}^2\mathcal{M}$-projectable vector field on an (m_1, m_2, n_1, n_2)-dimensional fibered fibered manifold Y induces canonically a 1-form $A(Z)$ on $T^{r_1,\ldots,r_8*}Y$. This problem is reflected in the concept of a natural operator $T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\ldots,r_8*}$ in the sense of [4].

EXAMPLE 1. Let Z be an $\mathcal{F}^2\mathcal{M}$-projectable vector field on an (m_1, m_2, n_1, n_2)-dimensional manifold. For every $k_1 = 1, \ldots, r_8$ we have a map $\overset{(k_1)_1}{Z} : T^*T^{r_1,\ldots,r_8*}Y \to \mathbb{R}$ defined by

$$\overset{(k_1)_1}{Z}(j_y^{(r_1,\ldots,r_8)}\gamma) = Z^{k_1}\gamma_1(y) \quad \text{where} \quad \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) : Y \to \mathbb{R}^{1,1,1,1}$$

$y \in Y, \gamma(y) = 0$ and $\overset{(k_1)_1}{Z} = Z \circ \ldots \circ Z$ (k_1-times). The map $\overset{(k_1)_1}{Z}$ is well-defined since $j_y^{(r_1,\ldots,r_8)}\gamma = j_y^{(r_1,\ldots,r_8)}\tilde{\gamma}$ follows $j_y^{k_1}\gamma_1 = j_y^{k_1}\tilde{\gamma}_1$ for all $k_1 = 1, \ldots, r_8$. Then for every $k_1 = 1, \ldots, r_8$ we have a 1-form $d\overset{(k_1)_1}{Z}$ on $T^{r_1,\ldots,r_8*}Y$. The correspondence $\overset{(k_1)_1}{A} : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\ldots,r_8*}$ defined by $Z \mapsto d\overset{(k_1)_1}{Z}$ determines a natural operator.

EXAMPLE 2. Let Z be as in Example 1. For every $k_2 = 1, \ldots, r_4$ we have a map $\overset{(k_2)_2}{Z} : T^*T^{r_1,\ldots,r_8*}Y \to \mathbb{R}$ defined by

$$\overset{(k_2)_2}{Z}(j_y^{(r_1,\ldots,r_8)}\gamma) = Z^{k_2}\gamma_2(y) \quad \text{where} \quad \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) : Y \to \mathbb{R}^{1,1,1,1}$$

and $y \in Y$, $\gamma(y) = 0$. Then for every $k_2 = 1, \ldots, r_4$ we have a 1-form $d\overset{(k_2)_2}{Z}$ on $T^{r_1,\ldots,r_8*}Y$. The correspondence $\overset{(k_2)_2}{A} : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\ldots,r_8*}$ defined by $Z \mapsto d\overset{(k_2)_2}{Z}$ determines a natural operator.

EXAMPLE 3. Let Z be as in Example 1. For every $k_3 = 1, \ldots, r_6$ we have a map $\overset{(k_3)_3}{Z} : T^*T^{r_1,\ldots,r_8*}Y \to \mathbb{R}$ defined by

$$\overset{(k_3)_3}{Z}(j_y^{(r_1,\ldots,r_8)}\gamma) = Z^{k_3}\gamma_3(y) \quad \text{where} \quad \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) : Y \to \mathbb{R}^{1,1,1,1}$$

and $y \in Y$, $\gamma(y) = 0$. Then for every $k_3 = 1, \ldots, r_6$ we have a 1-form $d\overset{(k_3)_3}{Z}$ on $T^{r_1,\ldots,r_8*}Y$. The correspondence $\overset{(k_3)_3}{A} : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\ldots,r_8*}$ defined by $Z \mapsto d\overset{(k_3)_3}{Z}$ determines a natural operator.

EXAMPLE 4. Let Z be as in Example 1. For every $k_4 = 1, \ldots, r_1$ we have a map $\overset{(k_4)_4}{Z} : T^*T^{r_1,\ldots,r_8*}Y \to \mathbb{R}$ defined by

$$\overset{(k_4)_4}{Z}(j_y^{(r_1,\ldots,r_8)}\gamma) = Z^{k_4}\gamma_4(y) \quad \text{where} \quad \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) : Y \to \mathbb{R}^{1,1,1,1}$$

and $y \in Y$, $\gamma(y) = 0$. Then for every $k_4 = 1, \ldots, r_1$ we have a 1-form $d\overset{(k_4)_4}{Z}$ on $T^{r_1,\ldots,r_8*}Y$. The correspondence $\overset{(k_4)_4}{A} : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\ldots,r_8*}$ defined by $Z \mapsto d\overset{(k_4)_4}{Z}$ determines a natural operator.

EXAMPLE 5. Let Z be as in Example 1. For every $k_1 = 1, \dots, r_8$ we have a 1-form $\overset{\langle k_1 \rangle_1}{Z} : TT^{r_1,\dots,r_8*}Y \to \mathbb{R}$ on $T^*T^{r_1,\dots,r_8*}Y$ defined by

$$\overset{\langle k_1 \rangle_1}{Z}(v) = \langle d(Z^{k_1-1}\gamma_1)(y), T\tilde{\pi}_Y(v)\rangle$$

for $v \in (TT^{r_1,\dots,r_8*})_y Y$, $y \in Y$, $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) : Y \to \mathbb{R}^{1,1,1,1}$, $\gamma(y) = 0$ and $p_Y^T(v) = j_y^{(r_1,\dots,r_8)}\gamma$ where $p_Y^T : TT^{r_1,\dots,r_8*}Y \to T^{r_1,\dots,r_8*}Y$ is the tangent bundle projection. Further, $\tilde{\pi} : T^{r_1,\dots,r_8*}Y \to Y$ is the base projection. The correspondence $\overset{\langle k_1 \rangle_1}{A} : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\dots,r_8*}$ defined by $Z \mapsto \overset{\langle k_1 \rangle_1}{Z}$ determines a natural operator.

EXAMPLE 6. Let Z be as in Example 1. For every $k_2 = 1, \dots, r_4$ we have a 1-form $\overset{\langle k_2 \rangle_2}{Z} : TT^{r_1,\dots,r_8*}Y \to \mathbb{R}$ on $T^*T^{r_1,\dots,r_8*}Y$ defined by

$$\overset{\langle k_2 \rangle_2}{Z}(v) = \langle d(Z^{k_2-1}\gamma_2)(y), T\tilde{\pi}_Y(v)\rangle$$

for $v \in (TT^{r_1,\dots,r_8*})_y Y$, $y \in Y$, $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) : Y \to \mathbb{R}^{1,1,1,1}$, $\gamma(y) = 0$ and $p_Y^T(v) = j_y^{(r_1,\dots,r_8)}\gamma$ where $p_Y^T : TT^{r_1,\dots,r_8*}Y \to T^{r_1,\dots,r_8*}Y$ is the tangent bundle projection. Further, $\tilde{\pi} : T^{r_1,\dots,r_8*}Y \to Y$ is the base projection. The correspondence $\overset{\langle k_2 \rangle_2}{A} : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\dots,r_8*}$ defined by $Z \mapsto \overset{\langle k_2 \rangle_2}{Z}$ determines a natural operator.

EXAMPLE 7. Let Z be as in Example 1. For every $k_3 = 1, \dots, r_6$ we have a 1-form $\overset{\langle k_3 \rangle_3}{Z} : TT^{r_1,\dots,r_8*}Y \to \mathbb{R}$ on $T^*T^{r_1,\dots,r_8*}Y$ defined by

$$\overset{\langle k_3 \rangle_3}{Z}(v) = \langle d(Z^{k_3-1}\gamma_3)(y), T\tilde{\pi}_Y(v)\rangle$$

for $v \in (TT^{r_1,\dots,r_8*})_y Y$, $y \in Y$, $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) : Y \to \mathbb{R}^{1,1,1,1}$, $\gamma(y) = 0$ and $p_Y^T(v) = j_y^{(r_1,\dots,r_8)}\gamma$ where $p_Y^T : TT^{r_1,\dots,r_8*}Y \to T^{r_1,\dots,r_8*}Y$ is the tangent bundle projection. Further, $\tilde{\pi} : T^{r_1,\dots,r_8*}Y \to Y$ is the base projection. The correspondence $\overset{\langle k_3 \rangle_3}{A} : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\dots,r_8*}$ defined by $Z \mapsto \overset{\langle k_3 \rangle_3}{Z}$ determines a natural operator.

EXAMPLE 8. Let Z be as in Example 1. For every $k_4 = 1, \dots, r_1$ we have a 1-form $\overset{\langle k_4 \rangle_4}{Z} : TT^{r_1,\dots,r_8*}Y \to \mathbb{R}$ on $T^*T^{r_1,\dots,r_8*}Y$ defined by

$$\overset{\langle k_4 \rangle_4}{Z}(v) = \langle d(Z^{k_4-1}\gamma_4)(y), T\tilde{\pi}_Y(v)\rangle$$

for $v \in (TT^{r_1,\dots,r_8*})_y Y$, $y \in Y$, $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) : Y \to \mathbb{R}^{1,1,1,1}$, $\gamma(y) = 0$ and $p_Y^T(v) = j_y^{(r_1,\dots,r_8)}\gamma$ where $p_Y^T : TT^{r_1,\dots,r_8*}Y \to T^{r_1,\dots,r_8*}Y$ is the tangent bundle projection. Further, $\tilde{\pi} : T^{r_1,\dots,r_8*}Y \to Y$ is the base projection. The correspondence $\overset{\langle k_4\rangle_4}{A} : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\dots,r_8*}$ defined by $Z \mapsto \overset{\langle k_4\rangle_4}{Z}$ determines a natural operator.

4. The classification theorem

In this section, we are going to formulate the main result giving the classification of all natural operators $A_Y : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}}Y \to T^*T^{r_1,\dots,r_8*}Y$ transforming $\mathcal{F}^2\mathcal{M}$-projectable vector fields on a fibered fibered manifold $Y \in \mathcal{O}b(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$ to 1-forms on the $(r_1,\dots,r_8)$-cotangent bundle of Y.

The set of all natural operators $T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\dots,r_8*}$ is a module over $C^\infty(\mathbb{R}^{r_1+r_4+r_6+r_8})$. Indeed, for any $f \in C^\infty(\mathbb{R}^{r_1+r_4+r_6+r_8})$ and a natural operator $A : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\dots,r_8*}$, we have a natural operator $fA : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\dots,r_8*}$ defined as follows

$$(fA)(Z) = f(\overset{(k_1)_1}{Z}, \overset{(k_2)_2}{Z}, \overset{(k_3)_3}{Z}, \overset{(k_4)_4}{Z})A(Z)$$

where Z is an $\mathcal{F}^2\mathcal{M}$ projectable vector field and $1 \le k_1 \le r_8$, $1 \le k_2 \le r_4$, $1 \le k_3 \le r_6$ and $1 \le k_4 \le r_1$.

The main classification theorem of this paper reads

THEOREM 1. *Let $m_1, m_2, n_1, n_2, r_1, \dots, r_8$ be natural numbers satisfying $m_1 \ge 2$, $r_8 \ge r_4 \le r_5 \ge r_3$, $r_8 \ge r_6 \le r_7 \ge r_2$ and $r_1 \le r_i$ for $i = 2, \dots, 8$. Then the $C^\infty(\mathbb{R}^{r_1+r_4+r_6+r_8})$-module of all natural operators $T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\dots,r_8*}$ is free and $2(r_1 + r_4 + r_6 + r_8)$-dimensional. Moreover, for $k_1 = 1, \dots r_8$, $k_2 = 1, \dots, r_4$, $k_3 = 1, \dots, r_6$ and $k_4 = 1, \dots, r_1$, the natural operators $\overset{(k_1)_1}{A}, \overset{(k_2)_2}{A}, \overset{(k_3)_3}{A}, \overset{(k_4)_4}{A}, \overset{\langle k_1\rangle_1}{A}, \overset{\langle k_2\rangle_2}{A}, \overset{\langle k_3\rangle_3}{A}$ and $\overset{\langle k_4\rangle_4}{A}$ form the basis of the module.*

The proof of Theorem 1 will occupy the rest of the paper. It is a complicated adaptation of the proofs of Theorem 1 in [14] and [15].

5. Some preparations to the proof of Theorem 1

Let us consider a natural operator $A : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*T^{r_1,\dots,r_8*}$. The operators $\overset{(k_1)_1}{A}, \overset{(k_2)_2}{A}, \overset{(k_3)_3}{A}, \overset{(k_4)_4}{A}, \overset{\langle k_1\rangle_1}{A}, \overset{\langle k_2\rangle_2}{A}, \overset{\langle k_3\rangle_3}{A}$ and $\overset{\langle k_4\rangle_4}{A}$

for $k_1 = 1,\dots r_8$, $k_2 = 1,\dots,r_4$, $k_3 = 1,\dots,r_6$ and $k_4 = 1,\dots,r_1$ are $C^\infty(\mathbb{R}^{r_1+r_4+r_6+r_8})$-linearly independent. Thus we only prove that A is a linear combination of $\overset{(k_1)_1}{A}$, $\overset{(k_2)_2}{A}$, $\overset{(k_3)_3}{A}$, $\overset{(k_4)_4}{A}$, $\overset{\langle k_1\rangle_1}{A}$, $\overset{\langle k_2\rangle_2}{A}$, $\overset{\langle k_3\rangle_3}{A}$ and $\overset{\langle k_4\rangle_4}{A}$ for k_1, k_2, k_3, k_4 as above with $C^\infty(\mathbb{R}^{r_1+r_4+r_6+r_8})$-coefficients.

In the following lemma we show that A is uniquely determined by the restriction $A(\frac{\partial}{\partial x^1})|(TT^{r_1,\dots,r_8*}\mathbb{R}^{m_1,m_2,n_1,n_2})_0$.

Lemma 1. *If* $A(\frac{\partial}{\partial x^1})|(TT^{r_1,\dots,r_8*}\mathbb{R}^{m_1,m_2,n_1,n_2})_0 = 0$ *then* $A = 0$.

Proof. The proof is standard. We use the naturality of A and the fact that any $\mathcal{F}^2\mathcal{M}$-projectable vector field Z having an underlying projectable vector field with an underlyng non-vanishing vector field is locally $\frac{\partial}{\partial x^1}$ in some fibered fibered manifold coordinates. ∎

It follows from Lemma 1 that our investigations of natural operators in question can be reduced to their restriction to the canonical vector field $\frac{\partial}{\partial x^1}$, i.e. we are going to study all $A(\frac{\partial}{\partial x^1})|(TT^{r_1,\dots,r_8*}\mathbb{R}^{m_1,m_2,n_1,n_2})_0$.

Lemma 2. *There are functions* $f_{(1)_1},\dots,f_{(r_8)_1}$, $f_{(1)_2},\dots,f_{(r_4)_2}$, $f_{(1)_3},\dots,f_{(r_6)_3}$, $f_{(1)_4},\dots,f_{(r_1)_4} \in C^\infty(\mathbb{R}^{r_1+r_4+r_6+r_8})$ *such that*

$$(A - \sum_{k_1=1}^{r_8} f_{(k_1)_1}\overset{(k_1)_1}{A} - \sum_{k_2=1}^{r_4} f_{(k_2)_2}\overset{(k_2)_2}{A} - \sum_{k_3=1}^{r_6} f_{(k_3)_3}\overset{(k_3)_3}{A} - \sum_{k_4=1}^{r_1} f_{(k_4)_4}\overset{(k_4)_4}{A})(v) = 0$$

for any $v \in (VT^{r_1,\dots,r_8*})_0\mathbb{R}^{m_1,m_2,n_1,n_2}$, *the vertical subbundle with respect to the projection* $\widetilde{\pi} : T^{r_1,\dots,r_8*}\mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2}$.

Proof. We have the identification $(VT^{r_1,\dots,r_8*})_0\mathbb{R}^{m_1,m_2,n_1,n_2} \simeq T_0^{r_1,\dots,r_8*}\mathbb{R}^{m_1,m_2,n_1,n_2} \times T_0^{r_1,\dots,r_8*}\mathbb{R}^{m_1,m_2,n_1,n_2}$ by $\frac{d}{dt}_{|t=0}(u+tw) \simeq (u,w)$ for $u, w \in T_0^{r_1,\dots,r_8*}\mathbb{R}^{m_1,m_2,n_1,n_2}$.

For $k_1 = 1,\dots,r_8$ we define $f_{(k_1)_1} : \mathbb{R}^{r_1+r_4+r_6+r_8} \to \mathbb{R}$ by

$$\begin{aligned} &f_{(k_1)_1}(a,b,c,e) = \\ &= A\Big(\frac{\partial}{\partial x^1}\Big)\Big(j_0^{(r_1,\dots,r_8)}\Big(\sum_{\widetilde{k}_1=1}^{r_8}\frac{1}{\widetilde{k}_1!}a_{\widetilde{k}_1}(x^1)^{\widetilde{k}_1}, \sum_{\widetilde{k}_2=1}^{r_4}\frac{1}{\widetilde{k}_2!}b_{\widetilde{k}_2}(x^1)^{\widetilde{k}_2}, \\ &\qquad \sum_{\widetilde{k}_3=1}^{r_6}\frac{1}{\widetilde{k}_3!}c_{\widetilde{k}_3}(x^1)^{\widetilde{k}_3}, \sum_{\widetilde{k}_4=1}^{r_1}\frac{1}{\widetilde{k}_4!}e_{\widetilde{k}_4}(x^1)^{\widetilde{k}_4}\Big), j_0^{(r_1,\dots,r_8)}\Big(\frac{1}{k_1!}(x^1)^{k_1},0,0,0\Big)\Big), \end{aligned}$$

where $a = (a_1,\dots,a_{r_8}) \in \mathbb{R}^{r_8}$, $b = (b_1,\dots,b_{r_4}) \in \mathbb{R}^{r_4}$, $c = (c_1,\dots,c_{r_6}) \in$

$\mathbb{R}^{r_6}$, $e = (e_1, \dots, e_{r_1}) \in \mathbb{R}^{r_1}$. For $k_2 = 1, \dots, r_4$ we define $f_{(k_2)_2} : \mathbb{R}^{r_1+r_4+r_6+r_8} \to \mathbb{R}$ by

$$f_{(k_2)_2}(a,b,c,e) =$$
$$= A\left(\frac{\partial}{\partial x^1}\right)\Bigg(j_0^{(r_1,\dots,r_8)}\Bigg(\sum_{\widetilde{k}_1=1}^{r_8}\frac{1}{\widetilde{k}_1!}a_{\widetilde{k}_1}(x^1)^{\widetilde{k}_1}, \sum_{\widetilde{k}_2=1}^{r_4}\frac{1}{\widetilde{k}_2!}b_{\widetilde{k}_2}(x^1)^{\widetilde{k}_2},$$
$$\sum_{\widetilde{k}_3=1}^{r_6}\frac{1}{\widetilde{k}_3!}c_{\widetilde{k}_3}(x^1)^{\widetilde{k}_3}, \sum_{\widetilde{k}_4=1}^{r_1}\frac{1}{\widetilde{k}_4!}e_{\widetilde{k}_4}(x^1)^{\widetilde{k}_4}\Bigg), j_0^{(r_1,\dots,r_8)}\left(0, \frac{1}{k_2!}(x^1)^{k_2}, 0, 0\right)\Bigg),$$

where a, b, c, e are as above.

For $k_3 = 1, \dots, r_6$ we define $f_{(k_3)_3} : \mathbb{R}^{r_1+r_4+r_6+r_8} \to \mathbb{R}$ by

$$f_{(k_3)_3}(a,b,c,e) =$$
$$= A\left(\frac{\partial}{\partial x^1}\right)\Bigg(j_0^{(r_1,\dots,r_8)}\Bigg(\sum_{\widetilde{k}_1=1}^{r_8}\frac{1}{\widetilde{k}_1!}a_{\widetilde{k}_1}(x^1)^{\widetilde{k}_1}, \sum_{\widetilde{k}_2=1}^{r_4}\frac{1}{\widetilde{k}_2!}b_{\widetilde{k}_2}(x^1)^{\widetilde{k}_2},$$
$$\sum_{\widetilde{k}_3=1}^{r_6}\frac{1}{\widetilde{k}_3!}c_{\widetilde{k}_3}(x^1)^{\widetilde{k}_3}, \sum_{\widetilde{k}_4=1}^{r_1}\frac{1}{\widetilde{k}_4!}e_{\widetilde{k}_4}(x^1)^{\widetilde{k}_4}\Bigg), j_0^{(r_1,\dots,r_8)}\left(0, 0, \frac{1}{k_3!}(x^1)^{k_3}, 0\right)\Bigg),$$

where a, b, c, e are as above.

For $k_4 = 1, \dots, r_1$ we define $f_{(k_4)_4} : \mathbb{R}^{r_1+r_4+r_6+r_8} \to \mathbb{R}$ by

$$f_{(k_4)_4}(a,b,c,e) =$$
$$= A\left(\frac{\partial}{\partial x^1}\right)\Bigg(j_0^{(r_1,\dots,r_8)}\Bigg(\sum_{\widetilde{k}_1=1}^{r_8}\frac{1}{\widetilde{k}_1!}a_{\widetilde{k}_1}(x^1)^{\widetilde{k}_1}, \sum_{\widetilde{k}_2=1}^{r_4}\frac{1}{\widetilde{k}_2!}b_{\widetilde{k}_2}(x^1)^{\widetilde{k}_2},$$
$$\sum_{\widetilde{k}_3=1}^{r_6}\frac{1}{\widetilde{k}_3!}c_{\widetilde{k}_3}(x^1)^{\widetilde{k}_3}, \sum_{\widetilde{k}_4=1}^{r_1}\frac{1}{\widetilde{k}_4!}e_{\widetilde{k}_4}(x^1)^{\widetilde{k}_4}\Bigg), j_0^{(r_1,\dots,r_8)}\left(0, 0, 0, \frac{1}{k_4!}(x^1)^{k_4}\right)\Bigg),$$

where a, b, c, e are as above.

In order to simplify the notation, put $\widetilde{A} = A - \sum_{k_1=1}^{r_8} f_{(k_1)_1} \overset{(k_1)_1}{A} - \sum_{k_2=1}^{r_4} f_{(k_2)_2} \overset{(k_2)_2}{A} - \sum_{k_3=1}^{r_6} f_{(k_3)_3} \overset{(k_3)_3}{A} - \sum_{k_4=1}^{r_1} f_{(k_4)_4} \overset{(k_4)_4}{A}$.

Consider $\mathcal{F}^2\mathcal{M}$-morphisms γ, η : $\mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{1,1,1,1}$ satisfying $\gamma(0) = 0$, $\eta(0) = 0$, $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ and $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$.

Define $a = (a_1, \dots, a_{r_8}) \in \mathbb{R}^{r_8}$, $b = (b_1, \dots, b_{r_4}) \in \mathbb{R}^{r_4}$, $c = (c_1, \dots, c_{r_6})$

$\in \mathbb{R}^{r_6}$, $e = (e_1, \dots, e_{r_1}) \in \mathbb{R}^{r_1}$ by

$$j_0^{(r_1,\dots,r_8)}(\gamma_1(x^1,0,\dots,0),0,0,0) = j_0^{(r_1,\dots,r_8)}\Big(\sum_{k_1=1}^{r_8} \frac{1}{k_1!} a_{k_1}(x^1)^{k_1},0,0,0\Big)$$

$$j_0^{(r_1,\dots,r_8)}(0,\gamma_2(x^1,0,\dots,0),0,0) = j_0^{(r_1,\dots,r_8)}\Big(0,\sum_{k_2=1}^{r_4} \frac{1}{k_2!} b_{k_2}(x^1)^{k_2},0,0\Big)$$

$$j_0^{(r_1,\dots,r_8)}(0,0,\gamma_3(x^1,0,\dots,0),0) = j_0^{(r_1,\dots,r_8)}\Big(0,0,\sum_{k_3=1}^{r_6} \frac{1}{k_3!} c_{k_3}(x^1)^{k_3},0\Big)$$

$$j_0^{(r_1,\dots,r_8)}(0,0,0,\gamma_4(x^1,0,\dots,0)) = j_0^{(r_1,\dots,r_8)}\Big(0,0,0,\sum_{k_4=1}^{r_1} \frac{1}{k_4!} e_{k_4}(x^1)^{k_4}\Big).$$

Further, define $\widetilde{a} = (\widetilde{a}_1, \dots, \widetilde{a}_{r_8}) \in \mathbb{R}^{r_8}$, $\widetilde{b} = (\widetilde{b}_1, \dots, \widetilde{b}_{r_4}) \in \mathbb{R}^{r_4}$, $\widetilde{c} = (\widetilde{c}_1, \dots, \widetilde{c}_{r_6}) \in \mathbb{R}^{r_6}$, $\widetilde{e} = (\widetilde{e}_1, \dots, \widetilde{e}_{r_1}) \in \mathbb{R}^{r_1}$ by

$$j_0^{(r_1,\dots,r_8)}(\eta_1(x^1,0,\dots,0),0,0,0) = j_0^{(r_1,\dots,r_8)}\Big(\sum_{k_1=1}^{r_8} \frac{1}{k_1!} \widetilde{a}_{k_1}(x^1)^{k_1},0,0,0\Big)$$

$$j_0^{(r_1,\dots,r_8)}(0,\eta_2(x^1,0,\dots,0),0,0) = j_0^{(r_1,\dots,r_8)}\Big(0,\sum_{k_2=1}^{r_4} \frac{1}{k_2!} \widetilde{b}_{k_2}(x^1)^{k_2},0,0\Big)$$

$$j_0^{(r_1,\dots,r_8)}(0,0,\eta_3(x^1,0,\dots,0),0) = j_0^{(r_1,\dots,r_8)}\Big(0,0,\sum_{k_3=1}^{r_6} \frac{1}{k_3!} \widetilde{c}_{k_3}(x^1)^{k_3},0\Big)$$

$$j_0^{(r_1,\dots,r_8)}(0,0,0,\eta_4(x^1,0,\dots,0)) = j_0^{(r_1,\dots,r_8)}\Big(0,0,0,\sum_{k_4=1}^{r_1} \frac{1}{k_4!} \widetilde{e}_{k_4}(x^1)^{k_4}\Big).$$

Taking into account the naturality of $\widetilde{A}$ with respect to $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-morphisms given by $(x^1, tx^2, \dots, tx^{m_1}, ty^1, \dots, ty^{m_2}, tu_1, \dots, tu^{n_1}, tv_1, \dots, tv^{n_2}) : \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ for $t \neq 0$ and putting $t \to 0$, we obtain

$$\widetilde{A}\Big(\frac{\partial}{\partial x^1}\Big)(j_0^{(r_1,\dots,r_8)}\gamma, j_0^{(r_1,\dots,r_8)}\eta) = \widetilde{A}\Big(\frac{\partial}{\partial x^1}\Big)(j_0^{(r_1,\dots,r_8)}(\gamma(x^1,0,\dots 0)), j_0^{(r_1,\dots,r_8)}(\eta(x^1,0,\dots 0))).$$

Then we have

$$\widetilde{A}\Big(\frac{\partial}{\partial x^1}\Big)(j_0^{(r_1,\dots,r_8)}\gamma, j_0^{(r_1,\dots,r_8)}\eta) =$$

$$= \sum_{k_1=1}^{r_8} f_{(k_1)_1}(a,b,c,e)\widetilde{a}_{k_1} + \sum_{k_2=1}^{r_4} f_{(k_2)_2}(a,b,c,e)\widetilde{b}_{k_2}$$

$$+ \sum_{k_3=1}^{r_6} f_{(k_3)_3}(a,b,c,e)\widetilde{c}_{k_3} + \sum_{k_4=1}^{r_1} f_{(k_4)_4}(a,b,c,e)\widetilde{e}_{k_4}$$

$$- \sum_{k_1=1}^{r_8} f_{(k_1)_1}(a,b,c,e)\widetilde{a}_{k_1} - \sum_{k_2=1}^{r_4} f_{(k_2)_2}(a,b,c,e)\widetilde{b}_{k_2}$$

$$- \sum_{k_3=1}^{r_6} f_{(k_3)_3}(a,b,c,e)\widetilde{c}_{k_3} - \sum_{k_4=1}^{r_1} f_{(k_4)_4}(a,b,c,e)\widetilde{e}_{k_4} = 0.$$

The proof of Lemma 2 is complete. ∎

6. Proof of Theorem 1

Consider the functions $f_{(k_1)_1}, f_{(k_2)_2}, f_{(k_3)_3}, f_{(k_4)_4}$ from Lemma 2 and replace A by $A - \sum_{k_1=1}^{r_8} f_{(k_1)_1} \overset{(k_1)_1}{A} - \sum_{k_2=1}^{r_4} f_{(k_2)_2} \overset{(k_2)_2}{A} - \sum_{k_3=1}^{r_6} f_{(k_3)_3} \overset{(k_3)_3}{A} - \sum_{k_4=1}^{r_1} f_{(k_4)_4} \overset{(k_4)_4}{A}$ as in Lemma 2. Then any natural operator in question can be assumed to satisfy

$$A\left(\frac{\partial}{\partial x^1}\right)|(VT^{r_1,\dots,r_8*})_0\mathbb{R}^{m_1,m_2,n_1,n_2} = 0.$$

It remains to show the existence of functions $g_{\langle k_1\rangle_1}, g_{\langle k_2\rangle_2}, g_{\langle k_3\rangle_3}, g_{\langle k_4\rangle_4} \in C^\infty(\mathbb{R}^{r_1+r_4+r_6+r_8})$ so that $A = \sum_{k_1=1}^{r_8} g_{\langle k_1\rangle_1} \overset{\langle k_1\rangle_1}{A} + \sum_{k_2=1}^{r_4} g_{\langle k_2\rangle_2} \overset{\langle k_2\rangle_2}{A} + \sum_{k_3=1}^{r_6} g_{\langle k_3\rangle_3} \overset{\langle k_3\rangle_3}{A} + \sum_{k_4=1}^{r_1} g_{\langle k_4\rangle_4} \overset{\langle k_4\rangle_4}{A}$. For $k_1 = 1,\dots,r_8$ define $g_{\langle k_1\rangle_1} : \mathbb{R}^{r_1+r_4+r_6+r_8} \to \mathbb{R}$ by

$$g_{\langle k_1\rangle_1}(a,b,c,e) =$$

$$A\left(\frac{\partial}{\partial x^2}\right)\left(T^{r_1,\dots,r_8*}\frac{\partial}{\partial x^2}\left(j_0^{(r_1,\dots,r_8)}\left(\sum_{\widetilde{k}_1=1}^{r_8}\frac{1}{\widetilde{k}_1!}a_{\widetilde{k}_1}(x^1)^{\widetilde{k}_1} + \frac{1}{(\widetilde{k}_1-1)!}(x^1)^{\widetilde{k}_1-1}x^2,\right.\right.\right.$$

$$\left.\left.\left.\sum_{\widetilde{k}_2=1}^{r_4}\frac{1}{\widetilde{k}_2!}b_{\widetilde{k}_2}(x^1)^{\widetilde{k}_2}, \sum_{\widetilde{k}_3=1}^{r_6}\frac{1}{\widetilde{k}_3!}c_{\widetilde{k}_3}(x^1)^{\widetilde{k}_3}, \sum_{\widetilde{k}_4=1}^{r_1}\frac{1}{\widetilde{k}_4!}e_{\widetilde{k}_4}(x^1)^{\widetilde{k}_4}\right)\right)\right),$$

where $a = (a_1,\dots,a_{r_8}) \in \mathbb{R}^{r_8}$, $b = (b_1,\dots,b_{r_4}) \in \mathbb{R}^{r_4}$, $c = (c_1,\dots,c_{r_6}) \in \mathbb{R}^{r_6}$, and $e = (e_1,\dots,e_{r_1}) \in \mathbb{R}^{r_1}$. Here $T^{r_1,\dots,r_8*}Z$ denotes the flow lifting of the $\mathcal{F}^2\mathcal{M}$-projectable vector field Z on Y to $T^{r_1,\dots,r_8*}$.

Analogously, for $k_2 = 1, \dots, r_4$ define $g_{\langle k_2\rangle_2} : \mathbb{R}^{r_1+r_4+r_6+r_8} \to \mathbb{R}$ by

$$g_{\langle k_2\rangle_2}(a,b,c,e) = A\Big(\frac{\partial}{\partial x^2}\Big)\Big(T^{r_1,\dots,r_8*}\frac{\partial}{\partial x^2}\Big(j_0^{(r_1,\dots,r_8)}\Big(\sum_{\widetilde{k}_1=1}^{r_8}\frac{1}{\widetilde{k}_1!}a_{\widetilde{k}_1}(x^1)^{\widetilde{k}_1}, \sum_{\widetilde{k}_2=1}^{r_4}\frac{1}{\widetilde{k}_2!}b_{\widetilde{k}_2}(x^1)^{\widetilde{k}_2}+\frac{1}{(\widetilde{k}_2-1)!}(x^1)^{\widetilde{k}_2-1}x^2, \sum_{\widetilde{k}_3=1}^{r_6}\frac{1}{\widetilde{k}_3!}c_{\widetilde{k}_3}(x^1)^{\widetilde{k}_3}, \sum_{\widetilde{k}_4=1}^{r_1}\frac{1}{\widetilde{k}_4!}e_{\widetilde{k}_4}(x^1)^{\widetilde{k}_4}\Big)\Big)\Big),$$

Further, for $k_3 = 1, \dots, r_6$ define $g_{\langle k_3\rangle_3} : \mathbb{R}^{r_1+r_4+r_6+r_8} \to \mathbb{R}$ by

$$g_{\langle k_3\rangle_3}(a,b,c,e) = A\Big(\frac{\partial}{\partial x^1}\Big)\Big(T^{r_1,\dots,r_8*}\frac{\partial}{\partial x^2}\Big(j_0^{(r_1,\dots,r_8)}\Big(\sum_{\widetilde{k}_1=1}^{r_8}\frac{1}{\widetilde{k}_1!}a_{\widetilde{k}_1}(x^1)^{\widetilde{k}_1}, \sum_{\widetilde{k}_2=1}^{r_4}\frac{1}{\widetilde{k}_2!}b_{\widetilde{k}_2}(x^1)^{\widetilde{k}_2}, \sum_{\widetilde{k}_3=1}^{r_6}\frac{1}{\widetilde{k}_3!}c_{\widetilde{k}_3}(x^1)^{\widetilde{k}_3} + \frac{1}{(\widetilde{k}_3-1)!}(x^1)^{\widetilde{k}_3-1}x^2, \sum_{\widetilde{k}_4=1}^{r_1}\frac{1}{\widetilde{k}_4!}e_{\widetilde{k}_4}(x^1)^{\widetilde{k}_4}\Big)\Big)\Big),$$

where a, b, c, e are as above.

Finally for $k_4 = 1, \dots, r_1$ define $g_{\langle k_4\rangle_4} : \mathbb{R}^{r_1+r_4+r_6+r_8} \to \mathbb{R}$ by

$$g_{\langle k_4\rangle_4}(a,b,c,e) = A\Big(\frac{\partial}{\partial x^1}\Big)\Big(T^{r_1,\dots,r_8*}\frac{\partial}{\partial x^2}\Big(j_0^{(r_1,\dots,r_8)}\Big(\sum_{\widetilde{k}_1=1}^{r_8}\frac{1}{\widetilde{k}_1!}a_{\widetilde{k}_1}(x^1)^{\widetilde{k}_1}, \sum_{\widetilde{k}_2=1}^{r_4}\frac{1}{\widetilde{k}_2!}b_{\widetilde{k}_2}(x^1)^{\widetilde{k}_2}, \sum_{\widetilde{k}_3=1}^{r_6}\frac{1}{\widetilde{k}_3!}c_{\widetilde{k}_3}(x^1)^{\widetilde{k}_3}, \sum_{\widetilde{k}_4=1}^{r_1}\frac{1}{\widetilde{k}_4!}e_{\widetilde{k}_4}(x^1)^{\widetilde{k}_4} + \frac{1}{(\widetilde{k}_4-1)!}(x^1)^{\widetilde{k}_4-1}x^2\Big)\Big)\Big).$$

It follows from Lemma 1 and the assumption

$$A(\tfrac{\partial}{\partial x^1})|(VT^{r_1,\dots,r_8*})_0\mathbb{R}^{m_1,m_2,n_1,n_2} = 0$$

that it suffices to show that

$$A\Big(\frac{\partial}{\partial x^1}\Big)(T^{r_1,\dots,r_8*}(\partial)(j_0^{(r_1,\dots,r_8)}\gamma)) = \Big(\sum_{k_1=1}^{r_8} g_{\langle k_1\rangle_1} \overset{\langle k_1\rangle_1}{A} + \sum_{k_2=1}^{r_4} g_{\langle k_2\rangle_2} \overset{\langle k_2\rangle_2}{A}$$

$$+ \sum_{k_3=1}^{r_6} g_{\langle k_3\rangle_3} \overset{\langle k_3\rangle_3}{A} + \sum_{k_4=1}^{r_1} g_{\langle k_4\rangle_4} \overset{\langle k_4\rangle_4}{A}\Big)\Big(\frac{\partial}{\partial x^1}\Big)(T^{r_1,\dots,r_8*}(\partial)(j_0^{(r_1,\dots,r_8)}\gamma))$$

for any $\mathcal{F}^2\mathcal{M}$-morphism $\gamma : \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{1,1,1,1}$, $\gamma(0) = 0$ and any constant vector field ∂ on $\mathbb{R}^{m_1,m_2,n_1,n_2}$ and linearly independent with $\frac{\partial}{\partial x^1}$. Taking into account the naturality of A and $\widetilde{A} = \sum_{k_1=1}^{r_8} g_{\langle k_1\rangle_1} \overset{\langle k_1\rangle_1}{A} + \sum_{k_2=1}^{r_4} g_{\langle k_2\rangle_2} \overset{\langle k_2\rangle_2}{A} + \sum_{k_3=1}^{r_6} g_{\langle k_3\rangle_3} \overset{\langle k_3\rangle_3}{A} + \sum_{k_4=1}^{r_1} g_{\langle k_4\rangle_4} \overset{\langle k_4\rangle_4}{A}$ with respect to linear $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-morphisms $\mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ preserving $\frac{\partial}{\partial x^1}$ we can assume $\partial = \frac{\partial}{\partial x^2}$.

Consider an $\mathcal{F}^2\mathcal{M}$-morphism $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) : \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{1,1,1,1}$ satisfying $\gamma(0) = 0$. Define $a = (a_1, \dots, a_{r_8}) \in \mathbb{R}^{r_8}$, $b = (b_1, \dots, b_{r_4}) \in \mathbb{R}^{r_4}$, $c = (c_1, \dots, c_{r_6}) \in \mathbb{R}^{r_6}$, $e = (e_1, \dots, e_{r_6}) \in \mathbb{R}^{r_1}$ by

$$a_{k_1} = \frac{\partial^{k_1}}{\partial(x^1)^{k_1}}\gamma_1(0), \qquad b_{k_2} = \frac{\partial^{k_2}}{\partial(x^1)^{k_2}}\gamma_2(0),$$

$$c_{k_3} = \frac{\partial^{k_3}}{\partial(x^1)^{k_3}}\gamma_3(0), \qquad e_{k_4} = \frac{\partial^{k_4}}{\partial(x^1)^{k_4}}\gamma_4(0),$$

for $k_1 = 1, \dots, r_8$, $k_2 = 1, \dots, r_4$, $k_3 = 1, \dots, r_6$ and $k_4 = 1, \dots, r_1$. Further, define $\widetilde{a} = (\widetilde{a}_1, \dots, \widetilde{a}_{r_8}) \in \mathbb{R}^{r_8}$, $\widetilde{b} = (\widetilde{b}_1, \dots, \widetilde{b}_{r_4}) \in \mathbb{R}^{r_4}$, $\widetilde{c} = (\widetilde{c}_1, \dots, \widetilde{c}_{r_6}) \in \mathbb{R}^{r_6}$, $\widetilde{e} = (\widetilde{e}_1, \dots, \widetilde{e}_{r_6}) \in \mathbb{R}^{r_1}$ by

$$\widetilde{a}_{k_1} = \frac{\partial}{\partial x^2}\frac{\partial^{k_1-1}}{\partial(x^1)^{k_1-1}}\gamma_1(0), \qquad \widetilde{b}_{k_2} = \frac{\partial}{\partial x^2}\frac{\partial^{k_2-1}}{\partial(x^1)^{k_2-1}}\gamma_2(0),$$

$$\widetilde{c}_{k_3} = \frac{\partial}{\partial x^2}\frac{\partial^{k_3-1}}{\partial(x^1)^{k_3-1}}\gamma_3(0), \qquad \widetilde{e}_{k_4} = \frac{\partial}{\partial x^2}\frac{\partial^{k_4-1}}{\partial(x^1)^{k_4-1}}\gamma_4(0).$$

Using the naturality of A with respect to the $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-morphisms $(x^1, tx^2, \tau x^3, \dots, \tau x^{m_1}, \tau y^1, \dots, \tau y^{m_2}, \tau w^1, \dots, \tau w^{n_1}, \tau v^1, \dots, \tau v^{n_2}) :$ $\mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ for $t, \tau \neq 0$ we obtain the following homogenity condition

$$tA\Big(\frac{\partial}{\partial x^1}\Big)\Big(T^{r_1,\dots,r_8*}\Big(\frac{\partial}{\partial x^2}\Big)(j_0^{(r_1,\dots,r_8)}\gamma)\Big)$$

$$= A\Big(\frac{\partial}{\partial x^1}\Big)\Big(T^{r_1,\dots,r_8*}\Big(\frac{\partial}{\partial x^2}\Big)(j_0^{(r_1,\dots,r_8)}\gamma(x^1, tx^2,$$

$$\tau x^3, \dots, \tau x^{m_1}, \tau y^1, \dots, \tau y^{m_2}, \tau w^1, \dots, \tau w^{n_1}, \tau v^1, \dots, \tau v^{n_2}))\Big).$$

This kind of homogenity yields

$$A\Big(\frac{\partial}{\partial x^1}\Big)\Big(T^{r_1,\dots,r_8*}\Big(\frac{\partial}{\partial x^2}\Big)(j_0^{(r_1,\dots,r_8)}\gamma)\Big)$$
$$= \sum_{k_1=1}^{r_8} g_{\langle k_1\rangle_1}(a,b,c,e)\widetilde{a}_{k_1} + \sum_{k_2=1}^{r_4} g_{\langle k_2\rangle_2}(a,b,c,e)\widetilde{b}_{k_2}$$
$$+ \sum_{k_3=1}^{r_6} g_{\langle k_3\rangle_3}(a,b,c,e)\widetilde{c}_{k_3} + \sum_{k_4=1}^{r_1} g_{\langle k_4\rangle_4}(a,b,c,e)\widetilde{d}_{k_4}$$

which follows from the homogenous function theorem. On the other hand

$$\widetilde{A}\Big(\frac{\partial}{\partial x^1}\Big)\Big(T^{r_1,\dots,r_8*}(\frac{\partial}{\partial x^2})(j_0^{(r_1,\dots,r_8)}\gamma)\Big)$$
$$= \sum_{k_1=1}^{r_8} g_{\langle k_1\rangle_1}(a,b,c,e)\widetilde{a}_{k_1} + \sum_{k_2=1}^{r_4} g_{\langle k_2\rangle_2}(a,b,c,e)\widetilde{b}_{k_2}$$
$$+ \sum_{k_3=1}^{r_6} g_{\langle k_3\rangle_3}(a,b,c,e)\widetilde{c}_{k_3} + \sum_{k_4=1}^{r_1} g_{\langle k_4\rangle_4}(a,b,c,e)\widetilde{d}_{k_4}$$

which is easy to observe. It completes the proof of Theorem 1. ∎

7. A corollary

In this section, we deduce a corollary of Theorem 1 giving the classification of canonical 1-forms λ on $T^*T^{r_1,\dots,r_8*}$ for fibered fibered manifolds Y of dimension m_1, m_2, n_1, n_2. They correspond to constant (i.e. independent on vector fields) natural operators $A_Y : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}}Y \to T^*T^{r_1,\dots,r_8*}Y$. Such a correspondence can be equivalently formulated as follows.

Any canonical 1-form λ on $T^{r_1,\dots,r_8*}Y$ is of the form $A_Y(0)$ for a natural operator $A_Y : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}}Y \to T^*T^{r_1,\dots,r_8*}Y$, where Y are $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$- objects. Thus we have the following corollary

COROLLARY 1 ([8]). *Let m_1, m_2, n_1, n_2 and $r_1, \dots, r_8$ be natural numbers such that $m_1 \geq 2$, $r_8 \geq r_4 \leq r_5 \geq r_3$, $r_8 \geq r_6 \leq r_7 \geq r_2$ and $r_1 \leq r_i$ for $i = 2, \dots, 8$. Then the vector space over $\mathbb{R}$ of all canonical 1-forms λ on $T^*T^{r_1,\dots,r_8*}Y$ for $Y \in Ob(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$ is 4-dimensional and the 1-forms $\overset{\langle k_1\rangle_1}{A}$, $\overset{\langle k_2\rangle_2}{A}$, $\overset{\langle k_3\rangle_3}{A}$ and $\overset{\langle k_4\rangle_4}{A}$ form its basis.*

Proof. The proof is clear.

REMARK. We have four canonical maps $\pi_i : T^{r_1,\dots,r_8 *}Y \to T^*Y$ defined by formulas $\pi_i(j_y^{(r_1,\dots,r_8)}\gamma) = d_y\gamma_i$, $i = 1,\dots,4$, $\gamma = (\gamma_1,\gamma_2,\gamma_3,\gamma_4) : Y \to \mathbb{R}^{1,1,1,1}$, $\gamma(y) = 0$, $y \in Y \in \mathrm{Obj}(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$. Then it is easy to verify that $\overset{\langle k_i \rangle_i}{A} = \pi_i^*\lambda$, the pull-back of the well-known canonical (Liouville) 1-form on T^*Y.

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