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ON THE SEQUENTIAL DENSITY POINTS

Abstract. This paper contains some results about density with respect to a sequence and an extension of the Lebesgue measure. There are some properties of topologies associated with such density point.

Throughout the paper $\mathbb{N}$ will denote the set of all positive intrgers, $\mathbb{Q}$—the set of all rational numbers, l—the standard Lebesgue measure on the real line $\mathbb{R}$ and $\mathcal{L}$—the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$. By $\mathcal{T}_d$ we shall denote the density topology on $\mathbb{R}$ and by S the family of all unbounded and nondecreasing sequences of positive numbers. We shall denote a sequence $\{s_n\}_{n\in\mathbb{N}} \in S$ by $\langle s\rangle$.

Let μ be any complete extension of the Lebesgue measure on $\mathbb{R}$. Let $\mathcal{S}_\mu$ be the σ-algebra of μ-measurable sets and $\mathcal{I}_\mu$ the σ-ideal of μ-null sets. We shall denote by μ^* the outer measure induced by μ and by μ_* the inner measure induced by μ. We shall also write $A \sim B$ if and only if $\mu(A\triangle B) = 0$ for sets $A, B \subset \mathbb{R}$ where $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Definition 1. (see [FFH]). *We shall say that $x \in \mathbb{R}$ is a density point of a set $A \in \mathcal{L}$ with respect to a sequence $\{s_n\}_{n\in N} \in S$ (in abr. $\langle s\rangle$-density point) if*

$$\lim_{n\to\infty} \frac{l(A\cap[x-\frac{1}{s_n}, x+\frac{1}{s_n}])}{\frac{2}{s_n}} = 1.$$

Let $\langle s\rangle \in S$ and $A \in \mathcal{L}$. Putting

$$\Phi^{\langle s\rangle}(A) = \{x \in \mathbb{R} : x \text{ is a } \langle \text{s}\rangle\text{-density point of A}\},$$

and

$$\mathcal{T}^{\langle s\rangle} = \{A \in \mathcal{L} : A \subset \Phi^{\langle s\rangle}(A)\}$$

we have the following result

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THEOREM 1. (see [FFH]). *Let $\langle s\rangle \in S$. The family $\mathcal{T}^{\langle s\rangle}$ is a topology on the real line.*

Let us introduce the following notation:
$S_0 = \{\langle s\rangle \in S : \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} = 0\}$ and
$S_1 = \{\langle s\rangle \in S : \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} > 0\}$.

THEOREM 2. (cf. [FFH] and [W]). *Let $\langle s\rangle \in S$. Then $\mathcal{T}^{\langle s\rangle} = \mathcal{T}_d$ if and only if $\langle s\rangle \in S_1$.*

Let μ be an arbitrary complete extension of the Lebesgue measure on $\mathbb{R}$.

DEFINITION 2. *We say that $x \in \mathbb{R}$ is a density point of a set $A \in \mathcal{S}_\mu$ with respect to the measure μ if*

$$\lim_{h\to 0^+} \frac{\mu(A\cap[x-h,x+h])}{2h} = 1.$$

Let

$\Phi_\mu(A) = \{x \in \mathbb{R} : x$ is a density point of A with respect to the measure $\mu\}$

for $A \in \mathcal{S}_\mu$ and

$$\mathcal{T}_\mu = \{A \in \mathcal{S}_\mu : A \subset \Phi_\mu(A)\}.$$

In paper [H] it is proved that family $\mathcal{T}_\mu$ forms a topology on $\mathbb{R}$ such that it has the form

$$\mathcal{T}_\mu = \mathcal{T}_d \ominus \mathcal{I}_\mu$$

(where $\mathcal{T}_d \ominus \mathcal{I}_\mu = \{A \setminus B : A \in \mathcal{T}_d, B \in \mathcal{I}_\mu\}$).

Similarly as above we can define $\langle s\rangle$-density point of a set $A \in \mathcal{S}_\mu$ with respect to measure μ.

DEFINITION 3. *Let $\langle s\rangle \in S$. We say that $x \in \mathbb{R}$ is a $\langle s\rangle$-density point of a set $A \in \mathcal{S}_\mu$ with respect to the measure μ if*

$$\lim_{n\to\infty} \frac{\mu(A\cap[x-\frac{1}{s_n},x+\frac{1}{s_n}])}{\frac{2}{s_n}} = 1.$$

Let $\langle s\rangle \in S$ and $A \in \mathcal{S}_\mu$. Putting

$\Phi_\mu^{\langle s\rangle}(A) =$
$= \{x \in \mathbb{R} : x$ is a $\langle \mathrm{s}\rangle$ − density point of A with respect to the measure $\mu\}$

we have the following result:

PROPERTY 1. *For each $A, B \in \mathcal{S}_\mu$ and $\langle s\rangle \in S$*

1. $\Phi_\mu^{\langle s\rangle}(\emptyset) = \emptyset$, $\Phi_\mu^{\langle s\rangle}(\mathbb{R}) = \mathbb{R}$,
2. *if* $A \subset B$ *then* $\Phi_\mu^{\langle s\rangle}(A) \subset \Phi_\mu^{\langle s\rangle}(B)$,
3. $\Phi_\mu^{\langle s\rangle}(A\cap B) = \Phi_\mu^{\langle s\rangle}(A) \cap \Phi_\mu^{\langle s\rangle}(B)$,
4. *if* $A \sim B$ *then* $\Phi_\mu^{\langle s\rangle}(A) = \Phi_\mu^{\langle s\rangle}(B)$.

Moreover, it is easy to see that

PROPERTY 2. *Let* $\langle s\rangle \in S_1$ *and* $X \in \mathcal{S}_\mu$. *A point* $x \in \mathbb{R}$ *is an* $\langle s\rangle$*-density point of* X *with respect to the measure* μ *if and only if a point* x *is a density point of* X *with respect to the measure* μ.

Proof. Let $\langle s\rangle \in S_1$ and $X \in \mathcal{S}_\mu$. The only difficulty in this proof is to show that if $x \in \mathbb{R}$ is a $\langle s\rangle$-density point of the set X with respect to the measure μ then this point is a density point of X with respect to the measure μ.

We assume that $x \in \mathbb{R}$ is a $\langle s\rangle$-density point of the set X with respect to the measure μ and fix $\varepsilon > 0$. There exists subsequence $\{s_{n_k}\}_{k\in\mathbb{N}}$ of the sequence $\{s_n\}_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty} \frac{s_{n_k}}{s_{n_{k+1}}} = c > 0$. So, there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ we have

$$\frac{c}{2} < \frac{s_{n_k}}{s_{n_{k+1}}} < \frac{3}{2}c.$$

Moreover, it is easy to see that x is a $\langle s\rangle$-density point of X with respect to the measure μ if

$$\lim_{n\to\infty} \frac{\mu(X' \cap [x - \frac{1}{s_n}, x + \frac{1}{s_n}])}{\frac{2}{s_n}} = 0,$$

where X' denotes $\mathbb{R} \setminus X$.

Therefore, there exists $k_1 > k_0$ such that for any $k \geq k_1$ we have

$$\frac{\mu(X' \cap [x - \frac{1}{s_{n_k}}, x + \frac{1}{s_{n_k}}])}{\frac{2}{s_{n_k}}} < \frac{c\cdot\varepsilon}{2}.$$

Put $\delta = \frac{1}{s_{n_{k_1}}}$ and fix $0 < h < \delta$. Of course, there exists $k > k_1$ such that $\frac{1}{s_{n_{k+1}}} \leq h < \frac{1}{s_{n_k}}$. Hence we have

$$\frac{\mu(X' \cap [x-h, x+h])}{2h} \leq \frac{\mu(X' \cap [x - \frac{1}{s_{n_k}}, x + \frac{1}{s_{n_k}}])}{\frac{2}{s_{n_{k+1}}}} \leq$$

$$\leq \frac{\mu(X' \cap [x - \frac{1}{s_{n_k}}, x + \frac{1}{s_{n_k}}])}{\frac{2}{s_{n_k}}} \cdot \frac{s_{n_{k+1}}}{s_{n_k}} < \frac{c\cdot\varepsilon}{2}\cdot\frac{2}{c} = \varepsilon.$$

It means that x is a density point of X with respect to the measure μ. □

Let $\langle s\rangle \in S$ and $A \in \mathcal{S}_\mu$. Let us define

$$\mathcal{T}_\mu^{\langle s\rangle} = \{A \in \mathcal{S}_\mu : A \subset \Phi_\mu^{\langle s\rangle}(A)\}.$$

From Property 2 it follows that $\Phi_\mu(A) = \Phi_\mu^{\langle s\rangle}(A)$ for each $A \in \mathcal{S}_\mu$ and $\langle s\rangle \in S_1$. Then we have $\mathcal{T}_\mu = \mathcal{T}_\mu^{\langle s\rangle}$ for each $\langle s\rangle \in S_1$. It implies, by Theorems 2 and results in [H], that

COROLLARY 1. *For every $\langle s\rangle \in S_1$ we have that the family $\mathcal{T}_\mu^{\langle s\rangle}$ is a topology on $\mathbb{R}$ and $\mathcal{T}_\mu^{\langle s\rangle} = \mathcal{T}^{\langle s\rangle} \ominus \mathcal{I}_\mu$.*

In our future considerations we will show that the previous result holds for every sequence $\langle s\rangle \in S$. Regarding the Property 2 we should limit our considerations to sequences belonging to S_0, because only then we can obtain new results. Let us fix $\langle s\rangle \in S_0$.

LEMMA 1. (cf.[H]). *Let $\{X_w\}_{w\in W} \in \mathcal{T}_\mu^{\langle s\rangle}$ and $X = \bigcup_{w\in W} X_w$. For any $\varepsilon > 0$ there exists a set $C_\varepsilon \in \mathcal{L}$ such that $X \subset C_\varepsilon$ and $\mu^*(C_\varepsilon \setminus X) < \varepsilon$.*

Proof. Let $\{X_w\}_{w\in W} \in \mathcal{T}_\mu^{\langle s\rangle}$ and $X = \bigcup_{w\in W} X_w$. At first, we suppose that X is bounded. Let P be an open interval such that $X \subset P$. Fix $\varepsilon \in (0, l(P))$ and put

$$\mathcal{P} = \left\{K \subset \mathbb{R} : \mu_*(K \cap X) > \left(1 - \frac{\varepsilon}{l(P)}\right) \cdot \mu(K)\right\},$$

where K denotes an interval. Family $\mathcal{P}$ is not empty. Indeed, for each $w \in W$ and every $x \in X_w$ we have that $X_w \subset X$ and

$$\lim_{n\to\infty} \frac{\mu(X_w \cap [x - \frac{1}{s_n}, x + \frac{1}{s_n}])}{\frac{2}{s_n}} = 1.$$

So, there exist $w_0 \in W, x_0 \in X_{w_0}$ and $n_0 \in \mathbb{N}$ such that for any $n > n_0$

$$\left|\frac{\mu(X_{w_0} \cap [x_0 - \frac{1}{s_n}, x_0 + \frac{1}{s_n}])}{\frac{2}{s_n}} - 1\right| < \frac{\varepsilon}{l(P)}.$$

Hence $\mu(X_{w_0} \cap [x_0 - \frac{1}{s_n}, x_0 + \frac{1}{s_n}]) > (1 - \frac{\varepsilon}{l(P)}) \cdot \frac{2}{s_n}$ for any $n > n_0$. Since $X_{w_0} \subset P$ and $\langle s\rangle \in S_0$, so there exists $N > n_0$ such that $[x_0 - \frac{1}{s_N}, x_0 + \frac{1}{s_N}] \subset P$ and

$$\mu_*\left(X \cap \left[x_0 - \frac{1}{s_N}, x_0 + \frac{1}{s_N}\right]\right) > \left(1 - \frac{\varepsilon}{l(P)}\right) \cdot \frac{2}{s_n}.$$

It means that $[x_0 - \frac{1}{s_N}, x_0 + \frac{1}{s_N}] \in \mathcal{P}$. Moreover, the family $\mathcal{P}$ forms a Vitali covering of the set X. So, there exists sequence $\{K_n\}_{n\in\mathbb{N}} \subset \mathcal{P}$ of pairwise disjoint intervals such that $l(X \setminus \bigcup_{n=1}^\infty K_n) = 0$. Simultaneously $\mu^*(\bigcup_{n=1}^\infty (K_n \setminus X)) < \varepsilon$. Indeed, for any $n \in \mathbb{N}$ there exists a μ-measurable set B_n such that

$$B_n \subset K_n \cap X \text{ and } \mu(B_n) > \left(1 - \frac{\varepsilon}{l(P)}\right) \cdot \mu(K_n).$$

Therefore

$$\mu^*\Big(\bigcup_{n=1}^{\infty} K_n \setminus X\Big) \leq \sum_{n=1}^{\infty} \mu^*(K_n \setminus X) \leq \sum_{n=1}^{\infty} \mu^*(K_n \setminus B_n) = \sum_{n=1}^{\infty} \mu(K_n \setminus B_n) =$$
$$= \sum_{n=1}^{\infty}(\mu(K_n) - \mu(B_n)) < \sum_{n=1}^{\infty} \left(\frac{\varepsilon}{l(P)}\mu(K_n)\right) =$$
$$= \frac{\varepsilon}{l(P)} \sum_{n=1}^{\infty} \mu(K_n) = \frac{\varepsilon}{l(P)}\mu\Big(\bigcup_{n=1}^{\infty} K_n\Big) \leq \varepsilon.$$

Putting $C_\varepsilon = (X \setminus \bigcup_{n=1}^{\infty} K_n) \cup (\bigcup_{n=1}^{\infty} K_n)$ we have that $C_\varepsilon \in \mathcal{L}$ and $X \subset C_\varepsilon$. Moreover, $\mu^*(C_\varepsilon \setminus X) = \mu^*(\bigcup_{n=1}^{\infty} K_n \setminus X) < \varepsilon$.

If X is unbounded the proof is obvious because we can present X as a countable union of bounded sets. □

THEOREM 3. *The family $\mathcal{T}_\mu^{\langle s\rangle}$ forms a topology on $\mathbb{R}$.*

Proof. By Property 1 it is clear that $\emptyset, \mathbb{R}$ belong to $\mathcal{T}_\mu^{\langle s\rangle}$ and for any $A, B \in \mathcal{T}_\mu^{\langle s\rangle}$ we have that $A \cap B \in \mathcal{T}_\mu^{\langle s\rangle}$. The only difficulty is to show that the union of an arbitrary subfamily of $\mathcal{T}_\mu^{\langle s\rangle}$ is a μ - measurable set because by the monotonicity property of operator $\Phi_\mu^{\langle s\rangle}$ such union is a member of $\mathcal{T}_\mu^{\langle s\rangle}$.

Let $\{X_w\}_{w \in W} \in \mathcal{T}_\mu^{\langle s\rangle}$. From Lemma 1 it follows that there exists a sequence $\{C_n\}_{n \in \mathbb{N}}$ of μ - measurable sets such that $\bigcup_{w \in W} X_w = X \subset C_n$ and $\mu^*(C_n \setminus X) < \frac{1}{n}$ for every $n \in \mathbb{N}$. Let $C = \bigcap_{n=1}^{\infty} C_n$. Evidently C is a Lebesgue measurable set, $X \subset C$ and

$$\forall_{n \in \mathbb{N}}\ \mu^*(C \setminus X) \leq \mu^*(C_n \setminus X) < \frac{1}{n}.$$

From arbitrary $n \in \mathbb{N}$ we have that $\mu(C \setminus X) = 0$. It implies that $X = C \setminus (C \setminus X) \in \mathcal{S}_\mu$. Obviously $X \subset \Phi_\mu^{\langle s\rangle}(X)$, so $\bigcup_{w \in W} X_w = X \in \mathcal{T}_\mu^{\langle s\rangle}$. □

THEOREM 4. *For every $\langle s\rangle \in S_0$ we have $\mathcal{T}_\mu^{\langle s\rangle} = \mathcal{T}^{\langle s\rangle} \ominus \mathcal{I}_\mu$.*

Proof. Let $\langle s\rangle \in S$. Firstly, we shall prove that $\mathcal{T}_\mu^{\langle s\rangle} \subset \mathcal{T}^{\langle s\rangle} \ominus \mathcal{I}_\mu$. Let $X \in \mathcal{T}_\mu^{\langle s\rangle}$. It implies that $X \in \mathcal{S}_\mu$ and $X \subset \Phi_\mu^{\langle s\rangle}(X)$. According to the proof of Theorem 3 there exists a Lebesgue measurable set $C \supset X$ such that $\mu(C \setminus X) = 0$. Since $X = C \setminus (C \setminus X)$, then $X \in \mathcal{L} \ominus \mathcal{I}_\mu$. So, $X = A \setminus B$ where $A \in \mathcal{L}$ and $\mu(B) = 0$. We obtain immediately, by Property 1, that

$$A \setminus B = X \subset \Phi_\mu^{\langle s\rangle}(X) = \Phi_\mu^{\langle s\rangle}(A \setminus B) = \Phi_\mu^{\langle s\rangle}(A) = \Phi^{\langle s\rangle}(A).$$

Since $A \cap \Phi^{\langle s \rangle}(A) \in \mathcal{T}^{\langle s \rangle}$ and

$$X = X \cap A = (A \cap \Phi^{\langle s \rangle}(A)) \setminus ((A \cap \Phi^{\langle s \rangle}(A)) \setminus (X \cap A)) = \\ = (A \cap \Phi^{\langle s \rangle}(A)) \setminus (\Phi^{\langle s \rangle}(A) \cap A \cap B)$$

then $X \in \mathcal{T}^{\langle s \rangle} \ominus \mathcal{I}_\mu$.

Now we show that $\mathcal{T}^{\langle s \rangle} \ominus \mathcal{I}_\mu \subset \mathcal{T}_\mu^{\langle s \rangle}$. Let $X \in \mathcal{T}^{\langle s \rangle} \ominus \mathcal{I}_\mu$, so $X = A \setminus B$ where $A \in \mathcal{T}^{\langle s \rangle}$ and $B \in \mathcal{I}_\mu$. Of course $X \in \mathcal{S}_\mu$ and

$$\Phi_\mu^{\langle s \rangle}(X) = \Phi_\mu^{\langle s \rangle}(A) = \Phi^{\langle s \rangle}(A) \supset A \supset A \setminus B = X.$$

Hence $X \in \mathcal{T}_\mu^{\langle s \rangle}$. □

From the above and Corollary 1 we obtain

THEOREM 5. *For every $\langle s \rangle \in S$ we have that the family $\mathcal{T}_\mu^{\langle s \rangle}$ is a topology on $\mathbb{R}$ and $\mathcal{T}_\mu^{\langle s \rangle} = \mathcal{T}^{\langle s \rangle} \ominus \mathcal{I}_\mu$.*

PROPERTY 3. *Let $\langle s \rangle, \langle t \rangle \in S$. We have that $\mathcal{T}_\mu^{\langle s \rangle} = \mathcal{T}_\mu^{\langle t \rangle}$ if and only if $\mathcal{T}^{\langle s \rangle} = \mathcal{T}^{\langle t \rangle}$.*

Proof. Let $\langle s \rangle, \langle t \rangle \in S$. The only difficulty in this proof is to show that if $\mathcal{T}_\mu^{\langle s \rangle} = \mathcal{T}_\mu^{\langle t \rangle}$ then $\mathcal{T}^{\langle s \rangle} = \mathcal{T}^{\langle t \rangle}$. So, let $\mathcal{T}_\mu^{\langle s \rangle} = \mathcal{T}_\mu^{\langle t \rangle}$ and suppose that, there exists a set A such that $A \in \mathcal{T}^{\langle s \rangle} \setminus \mathcal{T}^{\langle t \rangle}$. Since $A \in \mathcal{T}_\mu^{\langle s \rangle} = \mathcal{T}_\mu^{\langle t \rangle}$, so, by Theorem 5, there exist sets $B \in \mathcal{T}^{\langle t \rangle}$ and $C \in \mathcal{I}_\mu$ such that $A = B \setminus C$. We can assume that $C \subset B$. Because $A \in \mathcal{L}$ and $B \in \mathcal{L}$, so $l(C) = 0$. And we have that $\Phi^{\langle t \rangle}(A) = \Phi^{\langle t \rangle}(B)$. From the above we have that

$$A \subset B \subset \Phi^{\langle t \rangle}(B) = \Phi^{\langle t \rangle}(A),$$

which means that $A \in \mathcal{T}^{\langle t \rangle}$. It contradicts with fact that $A \notin \mathcal{T}^{\langle t \rangle}$. □

Let $C((X,\tau),(Y,\varrho))$ denotes the family of all continuous functions acting from topological space (X,τ) to the topological space (Y,ϱ). In the future considerations, the application of the following theorem of Martin will be useful.

THEOREM 6. (see [M]). *Let $(X,\mathcal{T})$ be any arbitrary topological space, (Y,τ)— a regular topological space and $\mathcal{I}$—a σ-ideal of subsets X free from nonempty $\mathcal{T}$—open sets and such that the family $\mathcal{T} \ominus \mathcal{I}$ forms a topology. Then*

$$C((X,\mathcal{T}),(Y,\tau)) = C((X,\mathcal{T} \ominus \mathcal{I}),(Y,\tau)).$$

At this moment we also recall the following result presented in [FFH].

THEOREM 7. *For every sequences $\langle s \rangle, \langle t \rangle \in S$ we have that $C((\mathbb{R},\mathcal{T}^{\langle s \rangle}),(\mathbb{R},\tau_0)) = C(\mathbb{R},\mathcal{T}^{\langle t \rangle}),(\mathbb{R},\tau_0))$ if and only if $\mathcal{T}^{\langle s \rangle} = \mathcal{T}^{\langle t \rangle}$, where τ_0 is the natural topology.*

Combining Theorems 4, 7 and 6 we have that

THEOREM 8. *For every sequences* $\langle s\rangle, \langle t\rangle \in S$ *we have that* $C((\mathbb{R}, \mathcal{T}_\mu^{\langle s\rangle}), (\mathbb{R}, \tau_0)) = C(\mathbb{R}, \mathcal{T}_\mu^{\langle t\rangle}), (\mathbb{R}, \tau_0))$ *if and only if* $\mathcal{T}_\mu^{\langle s\rangle} = \mathcal{T}_\mu^{\langle t\rangle}$, *where* τ_0 *is the natural topology.*

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