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STRONG CONVERGENCE OF APPROXIMANTS TO FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES WITHOUT UNIFORM CONVEXITY

Abstract. Let X be a real reflexive Banach space whose norm is uniformly Gâteaux differentiable, D be a closed convex subset of X and T be an asymptotically nonexpansive mapping from D into itself with $F(T) \neq \emptyset$. Suppose that every closed convex bounded subset of D enjoys the fixed point property for T. Then for an element $x \in D$ and a sequence $\{a_n\}$ in $[0,1]$, there exist the sequence $\{x_n\}$ defined by the equation $x_n = a_n x + (1-a_n)\frac{1}{n}\sum_{i=1}^{n} T^i x_n, \quad n \in \mathbb{N}$ and the sunny nonexpansive retraction P form D onto $F(T)$, the set of fixed point of T such that $\{x_n\}$ converges strongly to Px. No uniform convexity assumption is made on X. Our results improve the results [Shimizu and W. Takahashi, Nonlinear Anal. 26 (1996), 265–272] and [N. Shioji and W. Takahashi, J. Approx. Theory, 97 (1999), 53–64].

1. Introduction

Let X be a real Banach space and D be a nonempty subset of X. A mapping $T : D \to D$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[1,\infty)$ with $\lim_n k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x-y\|$$

for all $x, y \in D$. The class of asymptotically nonexpansive mappings is essentially wider than the class nonexpansive mappings. If D is assumed to be a closed convex bounded subset of a uniformly convex Banach space, then

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every asymptotically nonexpansive mapping $T: D \to D$ has a fixed point in D.

Using an idea of Browder [1], Shimizu and Takahashi [10] studied strong convergence of the sequence $\{x_n\}$ defined by

$$(1.1) \qquad x_n = a_n x + (1 - a_n)\frac{1}{n}\sum_{i=1}^{n} T^i x_n, \quad n \in \mathbb{N}, \quad x \in D,$$

where T is asymptotically nonexpansive mapping from a nonempty closed convex bounded subset of a Hilbert space into itself and $\{a_n\}$ is a real sequence in $(0, 1)$ such that $a_n \to 0$ as $n \to \infty$.

Recently, Shioji and Takahashi [11] extended the result of Shimizu and Takahashi [10] for asymptotically nonexpansive mappings in the framework of a uniformly convex Banach space with a uniformly Gâteaux differentiable norm as below:

THEOREM ST. *Let D be a closed convex subset of a uniformly convex Banach space X whose norm is uniformly Gâteaux differentiable, $T: D \to D$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $F(T) \neq \phi$ and let P be the sunny nonexpansive retraction from D onto $F(T)$. Let $\{a_n\}$ be a real sequence in $(0, 1]$ such that $\lim_n a_n = 0$ and $\limsup_n \frac{b_n - 1}{a_n} < 1$, where $b_n = \frac{1}{n}\sum_{i=1}^{n} k_j, n \in \mathbb{N}$. Let x be an element of D and x_n be the unique point in D such that*

$$(1.1) \qquad x_n = a_n x + (1 - a_n)\frac{1}{n}\sum_{i=1}^{n} T^i x_n, \qquad n \geq N_0,$$

where N_0 is sufficiently large natural number. Then $\{x_n\}$ converges strongly to Px.

We remark that it is easy to find examples of Banach spaces which have fixed point property for asymptotically nonexpansive mappings without uniformly convexity. To see this, consider X to be uniformly smooth Banach space then X is not uniformly convex and enjoys the fixed point property for asymptotically nonexpansive mappings (see, Xu [6]).

From Theorem ST, we have the following natural question:

QUESTION 1. *Is it possible to remove uniform convexity of Banach space of Theorem ST?*

The purpose of this paper is to answer the question and develop general results concerning the strong convergence of the sequence $\{x_n\}$ defined by the equation (1.1) in Banach spaces without uniform convexity. First, we prove that the sequence $\{x_n\}$ also converges strongly to a fixed point T, where domain of D is non-convex in a reflexive Banach space. Next, we show that if X is a reflexive Banach space whose norm is uniformly Gâteaux differentiable, T is asymptotically nonexpansive mapping from a nonempty closed convex subset D of X into itself and D has *hereditary fixed property* for T, then the sequence $\{x_n\}$ converges strongly to Px, whenever P is the sunny nonexpansive retraction from D onto $F(T)$, the set of fixed points of T. Our results extend further the recent results of Jung, Sahu and Thakur [5], Shimizu and Takahashi [10] and Shioji and Takahashi [11] and others (see [1], [9]) to more general type Banach spaces.

2. Preliminaries

Throughout this paper, we assume X being a Banach space. When $\{x_n\}$ is a sequence in X, then $x_n \to x$ (resp. $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp. weak, weak*) convergence of the sequence $\{x_n\}$ to x.

Recall that a Banach space X is said to be *smooth* provided the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in S, where $S = \{x \in X : \|x\| = 1\}$. In this case, the norm of X is said to be *Gâteaux differentiable.* It is said to be *uniformly Gâteaux differentiable* if for each $y \in S$, this limit is attained uniformly for $x \in S$. The norm is said to be *Fréchet differentiable* if for each $x \in S$, this limit is attained uniformly for $y \in S$. Finally, the norm is said to be *uniformly Fréchet differentiable* if the limit is attained uniformly for $(x, y) \in S \times S$. In this case, X is said to be *uniformly smooth.* Since the dual X^* of X is uniformly convex if and only if the norm of X is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm. The reverse is false.

Let X^* be the dual space of X. The value of $y \in X^*$ at $x \in X$ will be denoted by $\langle x, y\rangle$. We also denoted by J, the duality mapping from X into 2^{X^*}, that is,

$$J(x) = \{y \in X^* : \langle x, y\rangle = \|x\|^2 = \|y\|^2\} \quad \text{for each} \quad x \in X.$$

Suppose that J is single valued. Then J is said to be *weakly sequentially continuous* if for each $\{x_n\} \in X$ with $x_n \rightharpoonup x$, $J(x_n) \overset{*}{\rightharpoonup} J(x)$.

Let C be a convex subset of X, K be a nonempty subset of C, and P be a retraction from C onto K, that is, $Px = x$ for each $x \in K$. A retraction P is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. If the sunny retraction P is also nonexpansive, then K is said to be a *sunny, nonexpansive retract* of C.

The following lemma is well-known [2, 8]:

LEMMA 1. *Let C be a convex subset of a smooth Banach space, D a nonempty subset of C and P a retraction from C onto D. Then P is the sunny and nonexpansive if and only if*

$$\langle x - Px, J(z - Px)\rangle \leq 0$$

for all $x \in C$ and $z \in D$.

Let LIM be a Banach limit. We sometimes use $LIM_n a_n$ instead of $LIM(a)$ for any $a = \{a_n\} \in \ell^\infty$. Let $\{x_n\}$ be a bounded sequence in X. Then we can define a real-valued continuous convex function on X by

$$f(z) = LIM_n\|x_n - z\|^2$$

for all $z \in X$. The following lemma is given in [4, 9, 12].

LEMMA 2. *Let C be a nonempty closed convex subset of a Banach space whose norm is uniformly Gâteaux differentiable. Let $\{x_n\}$ be a bounded sequence in C, $u \in C$ and LIM be a Banach limit. Then*

$$f(u) = \min_{z \in C} f(z)$$

if and only if

$$LIM_n\langle z - u, J(x_n - u)\rangle \leq 0$$

for all $z \in C$.

Finally, let D be a nonempty subset of C. D is said to satisfy the property (P) (cf. [6]) if the following holds:

$$(P) \qquad x \in D \Rightarrow \omega_\omega(x) \subset D,$$

where $\omega_\omega(x)$ is weak ω-limit set of T at x, that is, the set $\{y \in C : y = \text{weak} - \lim_j T^{n_j}x$ for some $n_j \to \infty\}$.

3. Main results

PROPOSITION 1. *Let D be a nonempty convex subset of a Banach space X and $T : D \to D$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$ of Lipschitz constants. Suppose x is an element of D, $\{a_n\}$ is a real sequence in $(0, 1]$ such that $\lim_n a_n = 0$ and $b_n = \frac{1}{n}\sum_{i=1}^{n} k_i$, $n \in \mathbb{N}$. Then there exists a natural number N_0 such that for every $n \geq N_0$ there is exactly one element x_n in D satisfying*

$$x_n = a_n x + (1 - a_n)\frac{1}{n}\sum_{i=1}^{n} T^i x_n.$$

Proof. For each $n \in \mathbb{N}$, the mapping T_n defined by

$$T_n u = a_n x + (1 - a_n)\frac{1}{n}\sum_{i=1}^{n} T^i u, \quad u \in D$$

is contraction with Lipschitz constant $(1 - a_n)b_n$. Since $\lim_{n\to\infty} b_n = 1$, it follows that $(1 - a_n)b_n < 1$ for all $n \geq N_0$, where N_0 is sufficiently large natural number. Hence for each $n \geq N_0$, T_n has exactly one fixed point x_n in D. ■

In the rest of this section, we set $x_n = x$ for all $n = 1, 2, \cdots, N_0 - 1$.

PROPOSITION 2. *Let D, X, T, a_n, k_n, b_n and x be as in Proposition 1. Then:*

(i) *If T has a fixed point in D, then $\{x_n\}$ is bounded.*

(ii) *For $v \in F(T)$, there exists $j \in J(x_n - v)$ such that*

$$\langle x_n - x, j\rangle \leq \frac{b_n - 1}{a_n}\|x_n - v\|^2, \quad n \in \mathbb{N}.$$

(iii) *If $d = \max\{\sup_{n,m\in\mathbb{N}} \|\sum_{i=1}^{n} T^i x_m\|, \sup_{m\in\mathbb{N}} \|\sum_{i=1}^{m} T^i x_m\|, \varsigma\} < \infty$, where $\varsigma = \sup_{n,m\in\mathbb{N}} \|x_n - x_m\|$, then there exists $j \in J(x_n - x_m)$ such that*

$$\left\langle \frac{a_n}{1-a_n}x_n - \frac{a_m}{1-a_m}x_m, j\right\rangle \leq \left(\frac{a_n}{1-a_n} - \frac{a_m}{1-a_m}\right)\langle x, j\rangle + \left(b_n - 1 + \frac{1}{n} + \frac{1}{m}\right)d^2$$

for all $n, m \in \mathbb{N}$.

Proof. (i) Let v be a fixed point of T. Then we have

$$\|x_n - v\| \leq a_n\|x - v\| + (1 - a_n)\left\|\frac{1}{n}\sum_{i=1}^{n} T^i x_n - v\right\|$$

and thus

$$\|x_n - v\| \leq \frac{a_n}{1 - (1 - a_n)b_n}\|x - v\|.$$

Since $(1 - a_n)b_n < 1$ for all $n \geq N_0$, $\{x_n\}$ is bounded.

(ii) Let v be a fixed point T. Then for each $n \in \mathbb{N}$, there exists $j \in J(x_n - v)$ such that

$$\begin{aligned}\langle x_n - x, j\rangle &= \frac{1 - a_n}{a_n}\left\langle \frac{1}{n}\sum_{i=1}^{n} T^i x_n - x_n, j\right\rangle \\ &= \frac{1 - a_n}{a_n}\left\langle \frac{1}{n}\sum_{i=1}^{n}(T^i x_n - T^i v) - (x_n - v), j\right\rangle \\ &\leq \frac{b_n - 1}{a_n}\|x_n - v\|^2.\end{aligned}$$

(iii) Set $\lambda_n = \frac{a_n}{1-a_n}$. For $n, m \in \mathbb{N}$, there exists $j \in J(x_n - x_m)$ such that

$$\begin{aligned}\langle \lambda_n x_n - \lambda_m x_m, j\rangle &\le (\lambda_n - \lambda_m)\langle x, j\rangle + \left\langle \frac{1}{n}\sum_{i=1}^{n} T^i x_n - \frac{1}{m}\sum_{i=1}^{m} T^i x_m, j\right\rangle \\ &\quad - \|x_n - x_m\|^2 \\ &= (\lambda_n - \lambda_m)\langle x, j\rangle + \left\langle \frac{1}{n}\sum_{i=1}^{n} T^i x_m - \frac{1}{m}\sum_{i=1}^{m} T^i x_m, j\right\rangle \\ &\quad + \left\langle \frac{1}{n}\sum_{i=1}^{n}(T^i x_n - T^i x_m), j\right\rangle - \|x_n - x_m\|^2 \\ &\le (\lambda_n - \lambda_m)\langle x, j\rangle + \left(\frac{1}{n}\left\|\sum_{i=1}^{n} T^i x_m\right\| + \frac{1}{m}\left\|\sum_{i=1}^{m} T^i x_m\right\|\right) d \\ &\quad + (b_n - 1)d^2 \\ &\le (\lambda_n - \lambda_m)\langle x, j\rangle + \left(\frac{1}{n} + \frac{1}{m} + b_n - 1\right) d^2. \ \blacksquare\end{aligned}$$

First, we obtain a convergence of the sequence $\{x_n\}$ defined by equation (1.1) for asymptotically nonexpansive mappings with non-convex domain in reflexive Banach spaces.

THEOREM 1. *Let X be a reflexive Banach space which possesses a weakly sequentially continuous duality mapping $J : X \to X^*$. Let D be a nonempty closed subset of X, $T : D \to D$ an asymptotically nonexpansive mapping with Lipschitz constant k_n such that $F(T) \neq \phi$ and let $\{a_n\}$ be a real sequence in* $(0, 1]$ *such that* $\lim_n a_n = 0$, $\lim_n \frac{1}{a_n \cdot n} = 0$ *and* $\lim_n \frac{b_n - 1}{a_n} = 0$. *Let x be an element of D and let x_n be the unique point in D defined by* (1.1). *Suppose that $\{x_n\}$ satisfying the condition:*

$$d = \max\left\{ \sup_{n,m\in\mathbb{N}} \left\|\sum_{i=1}^{n} T^i x_m\right\|, \sup_{m\in\mathbb{N}} \left\|\sum_{i=1}^{m} T^i x_m\right\|, \varsigma \right\} < \infty,$$

where $\varsigma = \sup_{n,m\in\mathbb{N}} \|x_n - x_m\|$. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $x_n - Tx_n \to 0$ as $n \to \infty$.

Proof. We first observe that due to Lemma 1 of [3], in this case the duality mapping J is single-valued. Now, let $\{x_n\}$ converge to p $(= Tp)$. Then $\lim_n \|x_n - Tx_n\| = 0$. Conversely, suppose $\lim_n \|x_n - Tx_n\| = 0$. Since $\{x_n\}$ is bounded and X is reflexive, we find some subsequence $\{x_{\varphi(n)}\}$ of $\{x_n\}$ which converges weakly to some $w \in D$. Set $\lambda_{\varphi(n)} = \frac{a_{\varphi(n)}}{1-a_{\varphi(n)}}$. From Proposition 2 (iii), we obtain that

$$\langle \lambda_{\varphi(n)} x_{\varphi(n)} - \lambda_{\varphi(m)} x_m, J(x_{\varphi(n)} - x_{\varphi(m)}) \rangle$$
$$\leq (\lambda_{\varphi(n)} - \lambda_{\varphi(m)}) \langle x, J(x_{\varphi(n)} - x_{\varphi(m)}) \rangle + \left(\frac{1}{\varphi(n)} + \frac{1}{\varphi(m)} + b_{\varphi(n)} - 1 \right) d^2$$

for some $d > 0$. Letting $m \to \infty$, we obtain

$$\frac{a_{\varphi(n)}}{1 - a_{\varphi(n)}} \langle x_{\varphi(n)} - x, J(x_{\varphi(n)} - w) \rangle \leq \left(\frac{1}{\varphi(n)} + b_{\varphi(n)} - 1 \right) d^2$$

and thus

$$\|x_{\varphi(n)} - w\|^2 \leq \langle x - w, J(x_{\varphi(n)} - w) \rangle + \left(\frac{1 - a_{\varphi}(n)}{a_{\varphi(n)} \varphi(n)} + \frac{b_{\varphi(n)} - 1}{a_{\varphi(n)}} \right) d^2.$$

It follows from conditions $\lim_n \frac{b_n - 1}{a_n} = 0$, $\lim_n \frac{1}{a_n . n} = 0$ and weak sequential continuity of J that $x_{\varphi(n)} \to w$ as $n \to \infty$. From continuity of T and $\lim_n \|x_n - Tx_n\| = 0$, we have $Tw = w$. To complete the proof, let $\{x_{\phi(n)}\}$ be another subsequence of $\{x_n\}$ such the $x_{\phi(n)} \longrightarrow z$ as $n \to \infty$. Hence $Tz = z \in D$. Proposition 2 (ii) implies that

$$\langle w - x, J(w - z) \rangle \leq 0$$

and

$$\langle z - x, J(z - w) \rangle \leq 0.$$

Hence $z = w$. This proves the strong convergence of $\{x_n\}$ to $z \in F(T)$. ∎

Following Opial's [7], we remark that none of $L_p, 1 < p \neq 2$ spaces possess weakly sequentially continuous duality mapping. At this stage, there arises a natural question:

QUESTION 2. *What conditions are required for convergence of approximants $\{x_n\}$ defined by equation* (1.1) *in such Banach spaces?*

Our next theorem gives an answer to the above question.

THEOREM 2. *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable, $T : D \to D$ be an asymptotically nonexpansive mapping with Lipschitz constant k_n such that $F(T) \neq \emptyset$. Let $\{a_n\}$ be a real sequence in* $(0, 1]$ *such that* $\lim_n a_n = 0$ *and* $\limsup_n \frac{b_n - 1}{a_n} = \alpha < 1$. *Suppose that D has hereditary fixed point property for T, that is, every closed convex bounded subset of D enjoys the fixed point property for T. Let x be an element of D and let x_n be the unique point in D defined by* (1.1) *such that $x_n - Tx_n \to 0$ as $n \to \infty$. Then:*

(a) *If $\alpha = 0$, then there exists the sunny nonexpansive retraction P from D onto $F(T)$,*

(b) *If P is the sunny nonexpansive retraction from D onto $F(T)$, then $\{x_n\}$ converges strongly to Px.*

Proof. (a) Since $\{x_n\}$ is bounded by Proposition 2 (i), we can define $f : D \to [0, \infty)$ by $f(z) = LIM_n\|x_n - z\|^2$ for all $z \in D$. Since f is continuous and convex, $f(z) \to \infty$ as $\|z\| \to \infty$, and X is reflexive, f attains it infimum over D. Let $z_0 \in D$ such that $f(z_0) = \min_{z \in D} f(z)$ and let $M = \{x \in D : f(x) = \min_{z \in D} f(z)\}$. Then M is nonempty because $z_0 \in M$. Clearly that M in nonempty closed convex and bounded set. Therefore, due to the assumption on D, T has a fixed point in M. Denote such a fixed point by v. It follows from Lemma 2 that

$$LIM_n\langle x - v, J(x_n - v) \leq 0. \tag{3.1}$$

From Proposition 2 (ii), we obtain

$$\langle x_n - x, J(x_n - v)\rangle \leq \frac{b_n - 1}{a_n}\|x_n - v\|^2. \tag{3.2}$$

which yields

$$LIM_n\langle x_n - x, J(x_n - v)\rangle \leq \alpha LIM_n\|x_n - v\|^2. \tag{3.3}$$

Combining (3.1) and (3.3), we have

$$\begin{aligned} LIM_n\|x_n - v\|^2 &= LIM_n\langle x_n - v, J(x_n - v)\rangle \\ &= LIM_n\langle x_n - x, J(x_n - v)\rangle + LIM_n\langle x - v, J(x_n - v)\rangle \\ &\leq \alpha LIM_n\|x_n - v\|^2 \end{aligned}$$

which implies

$$(1 - \alpha)LIM_n\|x_n - v\|^2 \leq 0.$$

Therefore, there is a subsequence $\{x_{n_i}\}$ which converges strongly to v. We now follow the proof of Theorem 1, we conclude that $\{x_n\}$ converges strongly to an element of $F(T)$. Then there exists a mapping P from D onto $F(T)$ defined by $Px = \lim_n x_n$, since x is an arbitrary element of D. By inequality (3.2), we have

$$\langle x - Px, J(z - Px)\rangle \leq \alpha\|z - Px\|^2$$

for all $x \in D$ and $z \in F(T)$. Since $\alpha = 0$, it follows from Lemma 1 that P is the sunny nonexpansive retraction.

(b) Let $\{x_n\}$ converges strongly to an element of $y \in F(T)$. We shall show that $y = Px$. From (3.2), we have

$$\langle y - x, J(y - Px)\rangle \leq \alpha\|y - Px\|^2.$$

Since P is the sunny nonexpansive retraction from D onto $F(T)$, we have

$$\langle x - Px, J(y - Px)\rangle \leq 0.$$

Hence we obtain

$$\|y - Px\|^2 = \langle y - Px, J(y - Px)\rangle$$
$$= \langle y - x, J(y - Px)\rangle + \langle x - Px, J(y - Px)\rangle \le \alpha\|y - Px\|^2.$$

From $\alpha < 1$, we have $y = Px$. This prove the strong convergence of $\{x_n\}$ to $Px \in F(T)$. ■

We remark that Theorem 2 is more general than several recent results of this nature. Particularly, it extends Theorem 2 of Shioji and Takahashi [11], where it is assumed that the Banach space is uniformly convex. Theorem 2 also extends Theorem 1 of Jung, Sahu and Thakur [5], where it is assumed that Banach space is both uniformly convex and uniformly smooth.

On the one hand, it is easy to find examples of Banach spaces which satisfy the F.P.P. for asymptotically nonexpansive mappings (see [6, Corollary 1]), which are not uniformly convex. Therefore, as a consequence of Theorem 2, we can derive the following result which happens to be an extension of Theorem 2 of Lim and Xu [6].

COROLLARY 1. *Let X be uniformly smooth Banach space, D be a nonempty closed convex bounded subset of X, and $T : D \to D$ be an asymptotically nonexpansive mapping with Lipschitz constant k_n such that $F(T) \neq \emptyset$. Let $\{a_n\}$ be a real sequence in $(0,1]$ such that $\lim_n a_n = 0$ and $\limsup_n \frac{b_n - 1}{a_n} = 0$. Let x be an element of D and x_n be the unique point in D defined by (1.1) such that $x_n - Tx_n \to 0$ as $n \to \infty$. Then:*

(a) *There exists the sunny nonexpansive retraction P from D onto $F(T)$.*

(b) *$\{x_n\}$ converges strongly to Px.*

Proof. Since $\{x_n\}$ is bounded, as in proof of Theorem 2, we define $f : D \to [0,\infty)$ by $f(z) = LIM_n\|x_n - z\|^2$ for any $z \in D$ and let $M = \{u \in D : f(u) = \min_{z\in D} f(z)\}$. Then M is also nonempty closed convex and bounded. To use the argument of the proof of Theorem 2, we just need to show that the set M contains a fixed point of T. Let $u \in M$ and $y = \text{weak} - \lim_j T^{m_j}u$ belong to the weak ω-limit set $\omega_\omega(u)$ of T at u. Then from the weak lower semicontinuity of f and $\lim_n \|x_n - Tx_n\|$=0, we have that

$$\begin{aligned} f(y) &\le \liminf_j f(T^{m_j}u) \\ &\le \limsup_m f(T^m u) \\ &\le \limsup_m (LIM_n\|x_n - T^m u\|^2) \\ &\le (\limsup_m k_m) LIM_n\|x_n - u\|^2 \\ &\le \min_{z\in D} f(z). \end{aligned}$$

This shows that $y \in M$ and hence M satisfies the property (P). It follows from Corollary 1 of [5] that T has a fixed point v in M. We now follow the proof of Theorem 2. ∎

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