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ON SOME APPROXIMATION PROBLEMS IN MUSIELAK–ORLICZ SPACES OF MULTIFUNCTIONS

Abstract. We introduce the Musielak–Orlicz spaces of multifunctions $X_{m,\varphi}$ and $X_{c,m,\varphi}$. We prove that these spaces are complete. Also, we get some convergence and approximation theorems in these spaces.

1. Introduction

Modular approximation by a filtered family of linear operators in Musielak–Orlicz spaces was studied in [15]. The results of [15] were extended to the case of Hammerstein operators in [7]. The results of [7] and [15] were extended to the case of some spaces of multifunctions in [8]–[13] (the set images are subsets of $2^{\overline{R}}$ or the set images are very specific subsets of real Hilbert space Y).

In this paper we introduce the spaces of multifunctions $X_{c,m,\varphi}$ and $X_{m,\varphi}$. We study the structure of these spaces and we extend the results of [11]–[13] to the case of the spaces $X_{c,m,\varphi}$ and $X_{m,\varphi}$. Theorems 2 and 3 which we obtain are closely connected with superposition operators and Niemytskii operators which were studied for example in [1], [18] and [19]. We apply the results of [2]–[6] and [16]. All definitions and theorems connected with the idea of the Musielak–Orlicz space can be found in [16].

Let (Ω, Σ, μ) be a measure space with a nonnegative, nontrivial σ-finite and complete measure μ. Let φ be a φ-function i.e., $\varphi : \Omega \times R \to R_+, \varphi(t,u)$ is an even, continuous function of u, equal to zero iff $u = 0$ and nondecreasing for $u \geq 0$ for every $t \in \Omega$, is a measurable function of $t \in \Omega$ for every $u \in R$ and $\lim_{u\to\infty} \varphi(t,u) = \infty$ for μ-a.e. $t \in \Omega$. Moreover, if $\varphi(t,\cdot)$ is a convex function for every $t \in \Omega$, $\lim_{u\to 0} \frac{\varphi(t,u)}{u} = 0$ and $\lim_{u\to\infty} \frac{\varphi(t,u)}{u} = \infty$ for every

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$t \in \Omega$, then we shall say that the function φ is an N-function. Let $L^{\varphi}(\Omega, \Sigma, \mu)$ be the Musielak–Orlicz function space generated by the modular

$$\rho(x) = \int_{\Omega} \varphi(t, x(t)) d\mu.$$

Let $\|\cdot\|_{\varphi}^{O}$ denote Orlicz norm and $\|\cdot\|_{\varphi}^{L}$ denote Luxemburg norm in $L^{\varphi}(\Omega, \Sigma, \mu)$ if φ is an N-function. Let Y be a real separable Banach space with the norm $\| \odot \|_Y$. Let Θ denote the zero element of Y. If $A, B \subset Y$ are nonempty then we denote

$$H(A, B) = \max(\sup_{x \in A} \inf_{y \in B} \|x - y\|_Y, \sup_{y \in B} \inf_{x \in A} \|x - y\|_Y).$$

Denote by $C(Y)$ the set of all nonempty and compact subsets of Y, by $B(Y)$ the set of all nonempty bounded and closed subset of Y and by $E(Y)$ the set of all nonempty and closed subsets of Y.

Denote:

$$X = \{F : \Omega \to 2^Y : F(t) \in E(Y) \text{ for every } t \in \Omega\},$$
$$X_c = \{F \in X : F(t) \in C(Y) \text{ for } \mu \text{ -a.e. } t \in \Omega\},$$
$$X_b = \{F \in X : F(t) \in B(Y) \text{ for } \mu \text{ -a.e. } t \in \Omega\}.$$

Two multifunctions $F, G \in X$ such that $F(t) = G(t)$ for μ-a.e. $t \in \Omega$ will be treated as the same element of X.

Now we introduce the function $\mathbf{d}(F, G)$ by the formula:

$$\mathbf{d}(F, G)(t) = H(F(t), G(t)) \text{ for all } F, G \in X \text{ and } t \in \Omega.$$

Let $\mathbf{N}$ be the set of all positive integers. Let $\mathbf{0} \in X_c$ be such that $\mathbf{0}(t) = \{\Theta\}$ for every $t \in \Omega$. Denote $|F| = \mathbf{d}(F, \mathbf{0})$ for every $F \in X$. If $F \in X_c$ then by $r(F)$ we denote a function from Ω to Y such that $r(F)(t) \in F(t)$ and $\|r(F)(t)\|_Y = H(F(t), \{\Theta\})$ for μ-a.e. $t \in \Omega$.

2. On the spaces $X_{m,\varphi}$ and $X_{c,m,\varphi}$

Definition 1. We say that $F \in X_b$ is a step multifunction if

$$F(t) = \sum_{k=1}^{n} \chi_{A_k}(t) B_k \text{ for every } t \in \Omega$$

where χ_A is a characteristic function of the set A, $B_k \in B(Y)$ for $k = 1, \ldots, n$, $\Omega = \cup_{k=1}^{n} A_k$, $A_k \in \Sigma$ for $k = 1, \ldots, n$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

Definition 2. We say that $F \in X_b$ is measurable if there exists the sequence of step multifunctions $F_n \in X_b$ for every $n \in \mathbf{N}$ such that $\lim_{n \to \infty} \mathbf{d}(F, F_n)(t) = 0$ for μ-a.e. $t \in \Omega$.

DEFINITION 3. We say that $F \in X_c$ is c-measurable if there exists the sequence of a step multifunctions $F_n \in X_c$ for every $n \in \mathbf{N}$ such that $\lim_{n\to\infty} \mathbf{d}(F, F_n)(t) = 0$ for μ-a.e. $t \in \Omega$.

Denote:

$$X_m = \{F \in X_b : F \text{ is measurable }\},$$
$$X_{c,m} = \{F \in X_c : F \text{ is c-measurable }\},$$
$$X_{s,m} = \{F \in X_b : F \text{ is a step multifunction }\},$$
$$X_{i,s,m} = \{F \in X_{s,m} : |F| \in L^1(\Omega, \Sigma, \mu)\},$$
$$X_{m,\varphi} = \{F \in X_m : |F| \in L^\varphi(\Omega, \Sigma, \mu)\},$$
$$X_{c,m,\varphi} = \{F \in X_{c,m} : |F| \in L^\varphi(\Omega, \Sigma, \mu)\}.$$

It is easy to see that $\mathbf{d}(F, G) \in L^\varphi(\Omega, \Sigma, \mu)$ if $F, G \in X_{m,\varphi}$.

By [6], Chapter 2 Theorem 1.35 if $F \in X_m$, then F is measurable and "graph measurable" in the sense of [6], Chapter 2, Definition 1.1.

The spaces $X_{c,m,\varphi}$ and $X_{m,\varphi}$ will be called Musielak–Orlicz spaces of multifunctions.

REMARK 1. If $F \in X_m$, then there is the sequence $\{G_n\}$ such that $G_n \in X_{s,m}$ for every $n \in \mathbf{N}$ and $\mathbf{d}(F, G_n)(t) \to 0$ as $n \to \infty$ for μ a.e. $t \in \Omega$ and $\mathbf{d}(G_n, \mathbf{0})(t) \le \mathbf{d}(F, \mathbf{0})(t)$ for μ a.e. $t \in \Omega$.

Proof. Let $F_n \in X_{s,m}$ for every $n \in \mathbf{N}$ and $\mathbf{d}(F, F_n)(t) = 0$ for μ-a.e. $t \in \Omega$. Let $\{r_n\}$ be the sequence of simple functions such that $r_n \nearrow |F|$. We define

$$G_n(t) = F_n(t) \cap \overline{K}(\Theta, r_n(t)) \text{ for every } t \in \Omega. \ \Box$$

REMARK 2. If $F_n \in X_m$, $F \in X_b$ for $n \in \mathbf{N}$ and $\mathbf{d}(F, F_n)(t) \to 0$ as $n \to \infty$ for μ-a.e. $t \in \Omega$, then $F \in X_m$.

REMARK 3. If $F_n \in X_{c,m}$, $F \in X_c$ for $n \in \mathbf{N}$ and $\mathbf{d}(F, F_n)(t) \to 0$ as $n \to \infty$ for μ-a.e. $t \in \Omega$, then $F \in X_{c,m}$.

REMARK 4. Let $B \in X_m$, $f : \Omega \to R$ be Σ-measurable, $F = fB$, then $F \in X_m$.

REMARK 5. Let $F, G \in X_{c,m}$, then $F + G \in X_{c,m}$.

LEMMA 1. *Let $F_n \in X_m$ for every $n \in \mathbf{N}$. If for all $\epsilon, \delta > 0$ there is $K > 0$ such that*

$$\mu(\{t \in \Omega : \mathbf{d}(F_n, F_m)(t) \ge \epsilon\}) < \delta \textit{ for all } m, n > K,$$

then there exist a subsequence $\{F_{n_k}\}$ of the sequence $\{F_n\}$ and $F \in X_m$ such that $\mathbf{d}(F_{n_k}, F) \to 0$ μ-a.e. and $\mathbf{d}(F_n, F)$ are Σ-measurable.

Proof. The proof is very similar to that of Lemma 1 from [11] so we give only a skech of it. By the assumptions there is a subsequence $\{F_{n_k}\}$ of the sequence $\{F_n\}$ such that for μ-a.e. $t \in \Omega$ and for every $\epsilon > 0$ there is $K > 0$ such that $\mathbf{d}(F_{n_k}, F_{n_l})(t) < \epsilon$ for all $k, l > K$. Hence there is $F \in X_b$ such that $\mathbf{d}(F_{n_k}, F) \to 0$ μ-a.e., because the space $B(Y)$ with metric H is complete. So $F \in X_m$ and $\mathbf{d}(F_n, F)$ is Σ-measurable because $\mathbf{d}(F_n, F)(t) = \lim_{k\to\infty} \mathbf{d}(F_{n_k}, F_n)(t)$ for μ–a.e. $t \in \Omega$. □

THEOREM 1. *Let $F_n \in X_{m,\varphi}$ ($F_n \in X_{c,m,\varphi}$) for every $n \in \mathbf{N}$. If for every $\epsilon > 0$ and every $a > 0$ there exists $K > 0$ such that*

$$\int_\Omega \varphi(t, a\mathbf{d}(F_m, F_n)(t))d\mu < \epsilon$$

for all $m, n > K$, then there exists $F \in X_{m,\varphi}$ ($F \in X_{c,m,\varphi}$) such that

$$\int_\Omega \varphi(t, a\mathbf{d}(F_n, F)(t))d\mu \to 0 \text{ as } n \to \infty \text{ for every } a > 0.$$

P r o o f. The proof is very similar to that of Theorem 7.7 from [16] so we give only a skech of it. By the assumptions and by Lemma 1 there is $F \in X_m$ such that

$$\int_\Omega \varphi(t, a\mathbf{d}(F_n, F)(t))d\mu \le \epsilon \text{ for } n > K.$$

We have $|F|(t) \le |F_n|(t) + \mathbf{d}(F_n, F)(t)$ for μ-a.e. $t \in \Omega$, so $F \in X_{m,\varphi}$. The space $C(Y)$ with metric H is complete, so if $F_n \in X_{c,m,\varphi}$ for every $n \in \mathbf{N}$, then $F \in X_{c,m,\varphi}$. □

COROLLARY 1. *Let the function φ be an N-function. Then the function*

$$D_\varphi(F, G) = \|\mathbf{d}(F, G)\|_\varphi^L$$

for all $F, G \in X_{m,\varphi}$ is a metric in $X_{m,\varphi}$, and $< X_{m,\varphi}, D_\varphi >$ is a complete metric space.

Theorem 1 is a generalization of Proposition 5.2 Chapter 5 from [6] and Theorem 0 from [12].

DEFINITION 4. The φ-function φ will be called locally integrable, if $\int_A \varphi(t, u)d\mu < \infty$ for every $u > 0$ and $A \in \Sigma$ with $\mu(A) < \infty$.

Applying the proof of Proposition 3.3, Proposition 2.17 and Remark 3.4 Chapter 2 from [6] we easy obtain the following:

LEMMA 2. *Let the φ-function φ be locally integrable. Then for every $F \in X_{m,\varphi}$ there exists the sequence $\{f_n\} \subset L^\varphi(\Omega, \Sigma, \mu)$ such that $F(t) = \overline{\{f_n\}}(t)$ for μ-a.e. $t \in \Omega$.*

Let $Q : \Omega \times Y \to Y$. We define the operators $\tilde{\mathbf{H}}$ and $\mathbf{H}_1$ by the formulas:

$$\tilde{\mathbf{H}}(F)(t) = \begin{cases} \{Q(t,x) : x \in F(t)\}, & \text{if } F(t) \text{ is compact} \\ \{\Theta\}, & \text{if } F(t) \text{ is noncompact,} \end{cases}$$

$$\mathbf{H}_1(F)(t) = \begin{cases} \overline{\{Q(t,x) : x \in F(t)\}}, & \text{if } F(t) \text{ is bounded} \\ \{\Theta\}, & \text{if } F(t) \text{ is unbounded} \end{cases}$$

for every $F \in X_m$ and every $t \in \Omega$.

THEOREM 2. *Let the φ-function φ be locally integrable. If the function Q fulfils the following conditions:*

a) there are $L > 0$ and $g \in L^{\varphi}(\Omega, \Sigma, \mu)$ such that $\|Q(t,u)\|_Y \leq L\|u\|_Y + g(t)$ for all $u \in Y$ and $t \in \Omega$,

b) $Q(\cdot, x)$ is strongly measurable for every $x \in Y$,

c) $Q(t, \cdot)$ is continuous for every $t \in \Omega$,

then $\tilde{\mathbf{H}} : X_{c,m,\varphi} \to X_{c,m,\varphi}$.

Proof. Let $F \in X_{c,m,\varphi}$. First, let us observe that $\tilde{\mathbf{H}}(F)(t)$ is a compact and nonempty set for every $t \in \Omega$, because $Q(t,u)$ is a continuous function as a function of u for every $t \in \Omega$.

Second, let us remark that from a) we obtain:

$$|\tilde{\mathbf{H}}(F)|(t) \leq L\|r(F)(t)\|_Y + g(t)$$

for every $F \in X_{c,m,\varphi}$ and every $t \in \Omega$. By Lemma 2 we easy obtain that $|\tilde{\mathbf{H}}(F)|$ is Σ-measurable, so $|\tilde{\mathbf{H}}(F)| \in L^{\varphi}(\Omega, \Sigma, \mu)$ for every $F \in X_{c,m,\varphi}$.

Third, let $B \in C(Y)$ and let $\{x_1, x_2, \ldots, x_n\}$ be a δ-net for B. Let

$$G(t) = \{Q(t, x_i) : i = 1, \ldots, n\},$$
$$G_B(t) = \{Q(t,x) : x \in B\}$$

for every $t \in \Omega$. We have $G \in X_{c,m}$ and by c) for every $\epsilon > 0$ there is $\delta > 0$ such that $H(G(t), G_B(t)) \leq \epsilon$ for every $t \in \Omega$, so $G_B \in X_{c,m}$. Let now $A_k \in \Sigma$, $B_k \in C(Y)$ for $k = 1, \ldots, n$, $\cup_{k=1}^{n} A_k = \Omega$, $A_i \neq A_j$ if $i \neq j$. Let $F_k = \chi_{A_k} B_k$ for $k = 1, \ldots, n$. Let $F' = \sum_{k=1}^{n} F_k$. It is easy to see that $\tilde{\mathbf{H}}(F') = \sum_{k=1}^{n} \tilde{\mathbf{H}}(F_k)$ and $\tilde{\mathbf{H}}(F_k)$ is c-measurable for $k = 1, \ldots, n$ so $\tilde{\mathbf{H}}(F')$ is c-measurable.

Fourth, let $t \in \Omega$ and let $F(t)$ and $F_n(t)$ be compact. Then $F(t) \cup F_n(t)$ is compact, so from every $\epsilon > 0$ there is $\delta > 0$ such that if $x \in F(t)$, $x_n \in F_n(t)$ and $\|x - x_n\|_Y < \delta$, then $\|Q(t,x) - Q(t,x_n)\|_Y < \epsilon$.

Let $K > 0$ be such that $H(F(t), F_n(t)) < \delta$ for every $n > K$. Let $y \in \tilde{\mathbf{H}}(F)(t)$, so there is $x \in F(t)$ such that $y = Q(t,x)$. Also there are $x_n \in F_n(t)$ such that $\|x - x_n\|_Y < \delta$ for $n > K$. Let $y_n = Q(t, x_n)$, hence $\|y - y_n\|_Y < \epsilon$ for all $n > K$.

So it is easy to see that if $H(F(t), F_n(t)) \to 0$ as $n \to \infty$, then

$$H(\tilde{\mathbf{H}}(F)(t), \tilde{\mathbf{H}}(F_n)(t)) \to 0 \text{ as } n \to \infty.$$

Hence we obtain the assertion by Remark 3. □

Analogously, by Proposition 7.9 Chapter 2 from [6] we obtain

THEOREM 3. *Let the φ-function φ be locally integrable. If the function Q fulfils the following conditions:*

a) there is $L > 0$ such that $\|Q(t,u) - Q(t,v)\|_Y \le L\|u - v\|_Y$ for all $u, v \in Y$ and μ-a.e. $t \in \Omega$,

b) the multifunction $\overline{\{Q(\cdot, x) : x \in B\}}$ is measurable for every $B \in B(Y)$,

c) $\|Q(\cdot, \Theta)\|_Y \in L^{\varphi}(\Omega, \Sigma, \mu)$,

then $\mathbf{H}_1 : X_{m,\varphi} \to X_{m,\varphi}$.

3. Density and approximation

THEOREM 4. *Let μ be atomless, the φ-function φ be locally integrable and fulfils the Δ_2 condition. Then $X_{i,s,m} \subset X_{m,\varphi}$ and for every $F \in X_{m,\varphi}$ there exist a sequence $\{F_n\}$ such that $F_n \in X_{i,s,m}$ for every $n \in \mathbf{N}$ and $\rho(a\mathbf{d}(F_n, F)) \longrightarrow 0$ as $n \to \infty$ for every $a > 0$.*

Proof. First, it is easy to see that by the assumptions $X_{i,s,m} \subset X_{m,\varphi}$.

Second, let $F \in X_{m,\varphi}$. By Remark 1 there is the sequence $\{G_n\}$ such that $G_n \in X_{i,s,m}$ for every $n \in \mathbf{N}$ and $\mathbf{d}(F, G_n)(t) \to 0$ as $n \to \infty$ μ-a.e. and

$$\mathbf{d}(G_n, \mathbf{0})(t) \le \mathbf{d}(F, \mathbf{0})(t) \quad \mu\text{-a.e. .}$$

So $H(F(t), G_n(t))(t) \le 2H(F(t), \{\Theta\})$ μ-a.e. for every $n \in \mathbf{N}$. Also

$$\varphi(t, aH(F(t), G_n(t))) \to 0$$

as $n \to \infty$ μ-a.e. for every $a > 0$. So we obtain the assertion by the Lebesgue dominated convergence theorem. □

Theorem 4 is a generalization of Theorem 7.6 from [16] and Proposition 2 from [4] and Theorem 2 from [11]. If we omit the Δ_2 condition in the assumptions of Theorem 4, then we only obtain that $\rho(a\mathbf{d}(F_n, F)) \to 0$ as $n \to \infty$ for some $a > 0$.

Let now (Ω, Σ, μ) be a Lebesgue measure space. Let $K_n, K : \Omega \times \Omega \to R_+$ for every $n \in \mathbf{N}$. We introduce the family of operators $(U_n)_{n \in \mathbf{N}}$ and the operator U by the formulas:

$$U_n(F)(s) = \Big\{ \int_\Omega K_n(t,s) f(t)dt : f(t) \in F(t) \text{ for every } t \in \Omega \text{ and the integral exists} \Big\},$$

$$U(F)(s) = \Big\{ \int_\Omega K(t,s)f(t)dt : f(t) \in F(t) \text{ for every } t \in \Omega \text{ and the integral exists} \Big\},$$

for every $n \in \mathbf{N}$ every $F \in X_{m,\varphi}$ and every $s \in \Omega$.

LEMMA 3. *Let $Y = R^n$. Let φ fulfil the assumptions of Theorem 4. Let moreover φ and ψ be the complementary N-functions in the sense of Young (see [16], Definition 13.4), $K_n(\cdot, s) \in L^\psi(\Omega, \Sigma, \mu)$ for every $n \in \mathbf{N}$ and every $s \in \Omega$. Let $g_n(s) = \|K_n(\cdot, s)\|_\psi^O$ for every $n \in \mathbf{N}$ and every $s \in \Omega$ and let $g_n \in L^\varphi(\Omega, \Sigma, \mu)$ for every $n \in \mathbf{N}$, then $U_n(F) \in X_{m,\varphi}$ for all $F \in X_{m,\varphi}$ and $n \in \mathbf{N}$.*

Proof. Let $s \in \Omega$, $n \in \mathbf{N}$. By Theorem 13.13 from [16] we have

$$\int_\Omega K_n(t,s)|F|(t)dt < \infty \text{ for every } F \in X_{m,\varphi}.$$

So by Proposition 8.6.2, Theorems 8.6.3 and 8.7.2 from [3], Theorem 5.14 and Proposition 5.20 from [6] and Theorem D1.10 and Corollary D1.10.1 from [14] we obtain $U_n(F+G)(s) = U_n(F)(s) + U_n(G)(s)$ and $U_n(F)(s) \in B(Y)$ for all $F, G \in X_{m,\varphi}$.

Let $B \in C(Y)$, $C \in \Sigma$, $\mu(C) < \infty$. Let $F(t) = \chi_C(t)B$ for every $t \in \Omega$. It is easy to see that $F \in X_{m,\varphi}$ and, by Remark 4, $U_n(F)$ is measurable.

Let now $F \in X_{m,\varphi}$ be arbitrary. By Theorem 4 there is the sequence of step multifunctions $\{F_m\}$ such that $F_m \in X_{m,\varphi}$ and $\|\mathbf{d}(F_m, F)\|_\varphi^L \to 0$ as $n \to \infty$. It is easy to see that $U_n(F_m)$ is mesurable and we have

$$\begin{aligned} H(U_n(F)(s), U_n(F_m)(s)) &= H\Big(\int_\Omega K_n(t,s)F(t)dt, \int_\Omega K_n(t,s)F_m(t)dt\Big) \\ &\le \int_\Omega K_n(t,s)H(F_m(t), F(t))dt \\ &\le \|K_n\|_\psi^O \|\mathbf{d}(F_m, F)\|_\varphi^L \to 0 \text{ as } m \to \infty. \end{aligned}$$

So $U_n(F)$ is measurable.

To end the proof we must prove that $|U_n(F)| \in L^\varphi(\Omega, \Sigma, \mu)$. We have for $a > 0$

$$\begin{aligned} &\int_\Omega \varphi\Big(s, aH\Big(\int_\Omega K_n(t,s)F(t)dt, \{\Theta\}\Big)\Big)ds \\ &\qquad \le \int_\Omega \varphi\Big(s, a\Big(\int_\Omega K_n(t,s)H(F(t), \{\Theta\})dt\Big)\Big)ds \\ &\qquad \le \int_\Omega \varphi(s, a\|K_n(\cdot, s)\|_\psi^O \|\mathbf{d}(F, \mathbf{0})\|_\varphi^L)ds < \infty \end{aligned}$$

so $|U_n(F)| \in L^\varphi(\Omega, \Sigma, \mu)$. □

Analogously we obtain the following:

LEMMA 4. *Let $Y = R^n$. Let φ fulfil the assumptions of Theorem 4. Let moreover φ and ψ be the complementary N-functions in the sense of Young, $K(\cdot, s) \in L^{\psi}(\Omega, \Sigma, \mu)$ for every $s \in \Omega$. Let $g(s) = \|K(\cdot, s)\|_{\psi}^{O}$ for every $s \in \Omega$ and let $g \in L^{\varphi}(\Omega, \Sigma, \mu)$. Then $U(F) \in X_{m,\varphi}$ for all $F \in X_{m,\varphi}$.*

THEOREM 5. *Let the assumptions of Lemmas 3 and 4 hold. Denote $h_n(s) = \|K_n(\cdot, s) - K(\cdot, s)\|_{\psi}^{O}$ for every $s \in \Omega$. If $\|h_n\|_{\varphi}^{L} \to 0$ as $n \to \infty$, then*

$$D_{\varphi}(U_n(F), U(F)) \longrightarrow 0 \text{ as } n \to \infty \text{ for every } F \in X_{m,\varphi}.$$

Proof. Let $a > 0$, $n \in \mathbf{N}$, $F \in X_{m,\varphi}$. We have

$$\begin{aligned}
&\int_{\Omega} \varphi\Big(s, aH\Big(\int_{\Omega} K_n(t,s)F(t)dt, \int_{\Omega} K(t,s)F(t)dt\Big)\Big)ds \\
&\qquad \le \int_{\Omega} \varphi\Big(s, a\Big(\int_{\Omega} H(K_n(t,s)F(t), K(t,s)F(t))dt\Big)\Big)ds \\
&\qquad \le \int_{\Omega} \varphi\Big(s, a\Big(\int_{\Omega} |K_n(t,s) - K(t,s)||F|(t)dt\Big)\Big)ds \\
&\qquad \le \int_{\Omega} \varphi(s, a\|K_n(\cdot,s) - K(\cdot,s)\|_{\psi}^{O}\|\mathbf{d}(F, \mathbf{0})\|_{\varphi}^{L})ds.
\end{aligned}$$

So $D_{\varphi}(U_n(F), U(F)) \to 0$ as $n \to \infty$. □

COROLLARY 2. *If the assumptions of Theorems 2 and 5 hold, then*

$$D_{\varphi}(U_n(\tilde{H}(F)), U(\tilde{H}(F))) \to 0 \text{ as } n \to \infty \text{ for every } F \in X_{m,\varphi}.$$

4. On the convolution operators

In this section we will apply by the notation used in [12] and [16]. Let $\mathbf{V}$ be an abstract set of indices and let $\mathcal{V}$ be a filter of subsets of $\mathbf{V}$.

DEFINITION 5. A function $g : \mathbf{V} \to R$ tends to zero with respect to $\mathcal{V}$, written $g(v) \xrightarrow{\mathcal{V}} 0$, if for every $\epsilon > 0$ there is $V \in \mathcal{V}$ such that $| g(v) | < \epsilon$ for all $v \in V$.

DEFINITION 6. Let $F_v \in X_{m,\varphi}$ for every $v \in \mathbf{V}$ and let $F \in X_{m,\varphi}$. We write $F_v \xrightarrow{d,\varphi,\mathcal{V}} F$, if for every $\epsilon > 0$ and every $a > 0$ there is a set $V \in \mathcal{V}$ such that $\rho(a\mathbf{d}(F_v, F)) < \epsilon$ for every $v \in V$.

REMARK 6. Let $F_v \in X_{m,\varphi}$ for every $v \in \mathbf{V}$ and let $F, G \in X_{m,\varphi}$. If

$$F_v \xrightarrow{d,\varphi,\mathcal{V}} F \text{ and } F_v \xrightarrow{d,\varphi,\mathcal{V}} G, \text{ then } F = G.$$

DEFINITION 7. The family $T = (T_v)_{v \in \mathbf{V}}$ of operators $T_v : X_{m,\varphi} \to X_{m,\varphi}$ for every $v \in \mathbf{V}$ will be called $(\mathbf{d}, \mathcal{V})$-bounded, if there exist positive constants k_1, k_2 and a function $g : \mathbf{V} \to R_+$ such that $g(v) \xrightarrow{\mathcal{V}} 0$, and for all $F, G \in X_{m,\varphi}$ there exists a set $V_{F,G} \in \mathcal{V}$ such that $\rho(a\mathbf{d}(T_v(F), T_v(G))) \leq k_1\rho(ak_2\mathbf{d}(F,G)) + g(v)$ for every $a > 0$ and every $v \in V_{F,G}$.

Analogously as in [12] we obtain the following:

REMARK 7. Let the assumptions of Theorem 4 hold. Let the family T be $(\mathbf{d}, \mathcal{V})$-bounded. If $T_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F$ for every $F \in X_{i,s,m}$, then $T_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F$ for every $F \in X_{m,\varphi}$.

Let now and next $\Omega = [0, b)$, $0 < b < \infty$, μ = Lebesgue measure in the σ-algebra Σ of all Lebesgue measurable subsets of $[0, b)$. The translation operator $\tau_v : X_{m,\varphi} \to X$ will be defined by the equality $\tau_v(F)(t) = F(t+v)$, where F is b-periodically extended to the whole R. Also, the function φ will be periodically extended with respect to the first variable.

DEFINITION 8. We shall say that the φ-function φ is τ-bounded, if there are positive constants k_1, k_2 such that

$$\varphi(t-v, u) \leq k_1\varphi(t, k_2 u) + f(t,v) \text{ for all } u, v, t \in R,$$

where the function $f : R \times R \to R_+$ is measurable and b-periodic with respect to the first variable and such that writing $h(v) = \int_0^b f(t,v)dt$ for every $v \in R$, we have $\mathbf{M} = \sup_{v\in R} h(v) < \infty$ and $h(v) \to 0$ as $v \to 0$ or $v \to b$.

Let now $\mathbf{V} = R$ and let $\mathcal{V}$ be a filter of all neighbourhoods of zero in R.

THEOREM 6. *Let the φ-function φ fulfils the Δ_2 condition, φ is τ-bounded and $\int_0^b \varphi(t,c)dt < \infty$ for every $c > 0$. Then $\tau_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F$ for every $F \in X_{m,\varphi}$.*

Proof. Let $F, G \in X_{m,\varphi}$. Let $F_n, G_n \in X_{s,m}$ for every $n \in \mathbf{N}$ and let

$$\rho(a\mathbf{d}(F_n, F)) \to 0,\ \rho(b\mathbf{d}(G_n, G)) \to 0 \text{ as } n \to \infty \text{ for all } a, b > 0.$$

It is easy to see that $\tau_v(F_n), \tau_v(G_n) \in X_{s,m}$ for all $n \in \mathbf{N}$ and $v \in R$. Also it is easy to see that $\tau_v(F), \tau_v(G) \in X_m$ for every $v \in R$. First, we prove that $\tau_v(F), \tau_v(G) \in X_{m,\varphi}$ for every $v \in R$. For every $a > 0$ and every $v \in R$ we have

$$\begin{aligned}\int_0^b \varphi(t, a|\tau_v(F)|(t))dt &= \int_0^b \varphi(t, a\mathbf{d}(F,\mathbf{0})(t+v))dt \\ &= \int_0^b \varphi(s-v, a\mathbf{d}(F,\mathbf{0})(s))ds\end{aligned}$$

$$\leq k_1 \int_0^b \varphi(t, k_2 a\mathbf{d}(F, \mathbf{0})(t))dt + \mathbf{M} < \infty.$$

Analogously we obtain that $\tau_v(G) \in X_{m,\varphi}$ for every $v \in R$. Second, for every $a > 0$ and every $v \in R$ we have

$$\rho(a\mathbf{d}(\tau_v(F), \tau_v(G))) = \int_0^b \varphi(t, a\mathbf{d}(F, G)(t + v))dt$$
$$\leq k_1 \rho(ak_2 \mathbf{d}(F, G)) + h(v).$$

So the family $\tau = (\tau_v)_{v \in R}$ is $(\mathbf{d}, \mathcal{V})$-bounded. Third, it is easy to see that $\tau_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F$ for every $F \in X_{s,m}$. Hence we obtain the assertion from Remark 7. □

COROLLARY 3. *If the assumptions of Theorem 6 hold and moreover φ is an N-function, then $D_\varphi(\tau_v(F), F) \to 0$ as $v \to 0$.*

Now we extend the function Q b-periodically with respect to the first variable. Let $\mathbf{W}$ be an abstract nonempty set of indices and let $\mathcal{W}$ be the filter of subsets of $\mathbf{W}$. Let now $K_w : [0, b) \to R_+$ for every $w \in \mathbf{W}$ be integrable in $[0, b)$ and singular, i.e. $0 < \sigma(w) = \int_0^b K_w(t)dt \xrightarrow{\mathcal{W}} 1$, $\sigma_\delta(w) = \int_\delta^{b-\delta} K_w(t)dt \xrightarrow{\mathcal{W}} 0$ for every $0 < \delta < b/2$, $\sigma = \sup_{w \in \mathbf{W}} \sigma(w) < \infty$, and let us extend K_w b-periodically to the whole R. We introduce the family of operators $A = (A_w)_{w \in \mathbf{W}}$ by the formula:

$$A_w(F)(s) = \Big\{ \int_0^b K_w(t - s) f(t)dt : f(t) \in F(t) \text{ for every } t \in [0, b)$$
$$\text{and the integral exists} \Big\},$$

for every $w \in \mathbf{W}$ every $F \in X_{m,\varphi}$ and every $s \in [0, b)$.

LEMMA 5. *Let $Y = R^n$. Let φ and ψ be complementary N-functions in the sense of Young. If moreover φ fullfils the assumptions of Theorem 6, $K_w \in L^\psi(\Omega, \Sigma, \mu)$ for every $w \in \mathbf{W}$ and $(K_w)_{w \in \mathbf{W}}$ is singular, then $A_w(F) \in X_{m,\varphi}$ for all $F \in X_{m,\varphi}$ and $w \in \mathbf{W}$.*

Proof. Let $s \in [0, b)$, $w \in \mathbf{W}$. By Theorem 13.13 from [16] and by the proof of Theorem 6 we have

$$\int_0^b K_w(t)|F|(t + s)dt < \infty \text{ for every } F \in X_{m,\varphi}.$$

So by Proposition 8.6.2, Theorems 8.6.3 and 8.7.2 from [3], Theorem 5.14 and Proposition 5.20 from [6] and Theorem D1.10 and Corollary D1.10.1 from

[14] we obtain $A_w(F+G)(s) = A_w(F)(s) + A_w(G)(s)$ and $A_w(F)(s) \in B(Y)$ for all $F, G \in X_{m,\varphi}$.

Let $B \in C(Y)$, $C \in \Sigma$. Let $F(t) = \chi_C(t)B$ for every $t \in [0, b)$. It is easy to see that $F \in X_{m,\varphi}$ and by Remark 4 $A_w(F)$ is measurable.

Let now $F \in X_{m,\varphi}$ be arbitrary. By Theorem 4 there is the sequence of step multifunctions $\{F_n\}$ such that $F_n \in X_{m,\varphi}$ and $\|\mathbf{d}(F_n, F)\|_\varphi^L \to 0$ as $n \to \infty$. By the proof of Theorem 6 we have $\|\mathbf{d}(\tau_s(F_n), \tau_s(F))\|_\varphi^L \to 0$ as $n \to \infty$. It is easy to see that $A_w(F_n)$ is measurable and we have

$$\begin{aligned} H(A_w(F)(s), A_w(F_n)(s)) &= H\Big(\int_0^b K_w(t)F(t+s)dt, \int_0^b K_w(t)F_n(t+s)dt\Big) \\ &\le \int_0^b K_w(t)H(F_n(s+t), F(t+s))dt \\ &= \int_0^b K_w(t)H(\tau_s(F)(t), \tau_s(F_n)(t))dt \\ &\le \|K_w\|_\psi^O \|\mathbf{d}(\tau_s(F_n), \tau_s(F))\|_\varphi^L \to 0 \text{ as } n \to \infty. \end{aligned}$$

So $A_w(F)$ is measurable.

To end the proof we must prove that $|A_w(F)| \in L^\varphi(\Omega, \Sigma, \mu)$. We have (see also [16], the proof of Theorem 7.15) for $a > 0$

$$\begin{aligned} &\int_0^b \varphi\Big(s, aH\Big(\int_0^b K_w(t-s)F(t)dt, \{\Theta\}\Big)\Big)ds \\ &\qquad \le \int_0^b \varphi\Big(s, a\Big(\int_0^b K_w(t-s)H(F(t), \{\Theta\})dt\Big)\Big)ds \\ &\qquad \le \frac{1}{\sigma(w)} \int_0^b K_w(t)\Big(\int_0^b \varphi(u-t, a\sigma H(F(u), \{\Theta\})du)\Big)dt \\ &\qquad \le k_1\rho(k_2 a\sigma \mathbf{d}(F, \mathbf{0})) + \frac{1}{\sigma(w)} \int_0^b K_w(t)h(t)dt \\ &\qquad \le k_1\rho(k_2 a\sigma \mathbf{d}(F, \mathbf{0})) + M < \infty \end{aligned}$$

so $|A_w(F)| \in L^\varphi(\Omega, \Sigma, \mu)$. □

LEMMA 6. *Let $C \in \Sigma$, $B \in C(Y)$, $F(t) = \chi_C(t)B$ for every $t \in [0, b)$. If the assumptions of Lemma 5 hold, then $A_w(F) \xrightarrow{d,\varphi,\mathcal{W}} \operatorname{conv} F$.*

P r o o f. By Theorem 7.16 from [16] we have

$$\int_0^b \varphi(s, a(\int_0^b K_w(t-s)\chi_C(t)dt - \chi_C(s)))ds \xrightarrow{\mathcal{W}} 0$$

for every $a > 0$. Let $a > 0$. We have

$$\begin{aligned}
&\rho(a\mathbf{d}(A_w(F), convF))\\
&= \int_0^b \varphi\Big(s, aH\Big(\int_0^b K_w(t-s)\chi_C(t)Bdt, \chi_C(s)convB\Big)\Big)ds\\
&= \int_0^b \varphi\Big(s, aH\Big(\int_0^b K_w(t-s)\chi_C(t)dtconvB, \chi_C(s)convB\Big)\Big)ds\\
&\le \int_0^b \varphi\Big(s, aH(convB, \{\Theta\})\Big|\int_0^b K_w(t-s)\chi_C(t)dt - \chi_C(s)\Big|\Big)ds \xrightarrow{\mathcal{W}} 0. \ \square
\end{aligned}$$

LEMMA 7. *If the assumptions of Lemma 5 hold, then*

$$\rho(a\mathbf{d}(A_w(F), A_w(G))) \le k_1\rho(ak_2\sigma\mathbf{d}(F,G)) + g(w)$$

for all $F, G \in X_{m,\varphi}, w \in \mathbf{W}$ *and every* $a > 0$, *where*

$$g(w) = \frac{1}{\sigma(w)}\int_0^b K_w(t)h(t)dt \xrightarrow{\mathcal{W}} 0.$$

Proof. Let $F, G \in X_{m,\varphi}$, $a > 0$ and $w \in \mathbf{W}$. We have

$$\begin{aligned}
\rho(a\mathbf{d}(A_w(F), A_w(G))) &\le \int_0^b \varphi\Big(s, a\int_0^b H(K_w(t-s)F(t), K_w(t-s)G(t))dt\Big)ds\\
&\le \int_0^b \varphi\Big(s, a\int_0^b K_w(t-s)\mathbf{d}(F,G)(t)dt\Big)ds\\
&\le k_1\rho(ak_2\sigma\mathbf{d}(F,G)) + g(w),
\end{aligned}$$

where $g(w) \xrightarrow{\mathcal{W}} 0$ (see also [15], the proof of Proposition 2). □

By Lemmas 5–7, Remark 7 and Proposition 1.17, Chapter 1 from [6] we obtain the following theorem.

THEOREM 7. *If the assumptions of Lemma 5 hold, then* $D_\varphi(A_w(F), convF) \xrightarrow{\mathcal{W}} 0$ *for every* $F \in X_{m,\varphi}$.

Analogously, by Theorem 8.6.4 and Proposition 8.6.2 from [3] we obtain the following theorem

THEOREM 8. *Let* Y *be a real reflexive separable Banach space. If the other assumptions of Lemma 5 hold, then* $D_\varphi(A_w(\overline{conv}F), \overline{conv}F) \xrightarrow{\mathcal{W}} 0$ *for every* $F \in X_{m,\varphi}$.

Now, we define the families of Hammerstein operators $\mathbf{T} = (\mathbf{T}_w)_{w\in\mathbf{W}}$ and $\mathbf{T}^1 = (\mathbf{T}^1_w)_{w\in\mathbf{W}}$ by the formulas:

$$\mathbf{T}_w(F) = A_w(\tilde{\mathbf{H}}(F)), \quad \mathbf{T}^1_w(F) = A_w(\overline{conv}\mathbf{H}_1(F))$$

for every $w \in \mathbf{W}$ and every $F \in X_{m,\varphi}$. We easy obtain the following:

THEOREM 9. *If the assumptions of Theorems 2 and 7 hold, then*

$$D_\varphi(\mathbf{T}_w(F), conv\tilde{\mathbf{H}}(F)) \xrightarrow{\mathcal{W}} 0 \text{ for every } F \in X_{m,\varphi}.$$

THEOREM 10. *If the assumptions of Theorems 3 and 8 hold, then*

$$D_\varphi(\mathbf{T}^1_w(F), \overline{conv}\mathbf{H}_1(F)) \xrightarrow{\mathcal{W}} 0 \text{ for every } F \in X_{m,\varphi}.$$

Theorems 2 and 6 -10 are the generalization of Theorems 1, 2, 3 and 4 from [12] and Theorem 3 from [11] and [13].

5. Final remark

DEFINITION 9. Let $F_v \in X_{m,\varphi}$ for every $v \in \mathbf{V}$ and let $F \in X_{m,\varphi}$. We write $F_v \xrightarrow{d,\rho,\mathcal{V}} F$, if for every $\epsilon > 0$ there is a set $V \in \mathcal{V}$ such that $\rho(a\mathrm{d}(F_v, F)) < \epsilon$ for every $v \in V$ for some $a > 0$.

If we omit the Δ_2 condition in the assumptions of Theorems 7 and 8 then we must replace the convergence $\xrightarrow{d,\varphi,\mathcal{W}}$ by the covergence $\xrightarrow{d,\rho,\mathcal{W}}$ and moreover in Lemma 5 we must also assume that $\int_0^b \psi(t,d)dt < \infty$ for every $d > 0$ and for for every $u_0 > 0$ there exists $c > 0$ such that $\frac{\varphi(t,u)}{u} \geq c$ for $u \geq u_0$ and all $t \in [0,b)$ instead of the Δ_2 condition for φ. From the assumptions by Theorem 13.15 from [16] we obtain the assertion of Lemma 5.

References

[1] J. Appell, H. T. Nguyen, P. P. Zabrejko, *Multivalued superposition operators in ideal spaces of vector functions. I, II*, Indag. Math., N.S. 2(4) (1991) 385–395, 397–409.

[2] Z. Artstein, J. A. Burns, *Integration of compact set-valued functions*, Pacific J. Math. 58 (1975), 297–307.

[3] J.-P. Aubin, H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston/Basel/Berlin 1990.

[4] S. De Blasi, A. Lasota *Daniell's Method in the Theory of the Aumann- Hukuhara Integral of Set-Valued Functions*, Lincei-Rend. Sc. fis. mat. e nat. 45 (1968), 252–256.

[5] S. De Blasi, A. Lasota, *Characterization of the integral of set-valued functions*, Lincei-Rend. Sc. fis. mat. e nat. 46 (1969), 154–157.

[6] S. Hu, N. S. Papageorgiou, *Handbook of Multivalued Analysis Vol. 1: Theory*, Kluwer Academic Publishers, Dordrecht/Boston/London 1997

[7] A. Kasperski, *Modular approximation by a filtered family of the Hammerstein operators*, Demonstratio Math. 17 (1984), 409–427.

[8] A. Kasperski, *Modular approximation in X_φ^1 by a filtered family of "linear operators"*, Comment. Math. XXX (1991), 335–341.

[9] A. Kasperski *Modular approximation in X_φ^1 by a filtered family of the Hammarstein operators*, Comment. Math. XXXII (1992), 69–74.

[10] A. Kasperski, *Modular approximation in $\tilde{X}_\varphi$ by a filtered family of $\tilde{X}_\varphi$– linear operators*, Functiones et Approximatio XX (1992), 183–187.

[11] A. Kasperski, *Musielak–Orlicz spaces of multifunctions, convergence and approximation*, Comment. Math. 34 (1994), 99–107.

[12] A. Kasperski, *Notes on approximation in Musielak–Orlicz spaces of multifunctions*, Comment. Math. 34 (1994), 109–122.

[13] A. Kasperski, *Notes on approximation in the Musielak–Orlicz spaces of vector multifunctions*, Comment. Math. Univ. Carolinae 35 (1994), 81–93.

[14] B. Mordukchowich: *Approximation Metods in the Problems of Optimization and control*, Nauka, Moscow 1988 (in Russian).

[15] J. Musielak, *Modular approximation by a filtered family of linear operators*, "Functional Analysis and Approximation, Proc. Conf. Oberwolfach, August 9–16, 1980", Birkhäuser, Basel, 1981, 99–110.

[16] J. Musielak, *Orlicz Spaces and Modular Spaces*, "Lecture Notes in Mathematics Vol. 1034" Springer, Berlin, 1983.

[17] A. Pliś, *Remark on measurable set-valued functions*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. IX, 12, (1961), 857–859.

[18] S. Rolewicz, Wen Song, *On lower semi-continuity and metric upper-semicontinuity of Nemytskii set-valued operators*, Z. Anal. Angew. 13 (4) (1994), 739–748.

[19] S. Rolewicz, Wen Song, *On automatic boudedness of Nemytskii set-valued operators*, Studia Math. 113 (1) (1995), 67–72.

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