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ON A NEW CLASS OF SEQUENCES

Dedicated to my parents Sri Hadu Tripathy and Smt. Sasikala Tripathy

Abstract. In this article we introduce the notion of statistical bounded variation sequences. We prove some results, those include the decomposition theorem and obtain the Köthe-Toeplitz dual of the space of statistical bounded variation sequences.

1. Introduction

In order to extend the notion of convergence of sequences, statistical convergence was introduced. First it was studied by Fast [2] and Schoenberg [9] independently. Later on it was studied from sequence space point of view and linked with Summability theory by Fridy [3], Connor [1], Maddox [6], Šalàt [8], Rath and Tripathy [7], Kolk [5], Tripathy ([10], [11], [12]) and many others.

The idea depends on the natural density of the subsets of the set N of natural numbers. A subset E of N is said to have *natural density* $\delta(E)$ if $\delta(E) = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \chi_E(k)$ exists, where χ_E is the characteristic function of E. Clearly finite subsets of N have zero natural density and $\delta(E^c) = \delta(N - E) = 1 - \delta(E)$. In this article we introduce the notion of statistically bounded variation sequences, on extending the notion of bounded variation sequences. The idea is quite similar to the idea of statistical convergence of sequences.

2. Definitions and background

Throughout the paper w, c, bv, $\bar{c}$, $\bar{bv}$, ℓ_∞, ℓ_1 represent the spaces of *all*, *convergent*, *bounded variation*, *statistically convergent*, *statistically bounded*

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variation, *bounded* and *absolutely summable sequences* respectively. A sequence (x_n) is said to be of *bounded variation* type if $(\Delta x_n) \in \ell_1$, where $\Delta x_n = x_n - x_{n+1}$ for all $n \in N$.

A sequence (x_n) is said to be *statistically convergent* to L if for every $\epsilon > 0$, $\delta(\{k \in N : |x_k - L| \geq \epsilon\}) = 0$. We write $x_k \overset{stat}{\to} L$ or $stat - lim\ x_k = L$.

DEFINITION. A sequence (x_n) is said to be statistically bounded variation sequence if $(\Delta x_{n_i}) \in \ell_1$ such that $\delta(\{n_i : i \in N\}) = 1$, where $\Delta x_{n_i} = x_{n_i} - x_{n_{i+1}}$ for all $i \in N$.

To have a clear picture of the above definition, let us consider the following example.

EXAMPLE 1. Define the sequence (x_n) as:

$$x_n = \begin{cases} n, & \text{if } n = k^2,\ k \in N, \\ n^{-1}, & \text{otherwise.} \end{cases}$$

The above example shows that $\bar{bv}$ contains some unbounded sequences too. It is clear that $\bar{bv}$ and $\bar{bv} \cap \ell_\infty$ are linear spaces.

Let (x_k) and (y_k) be two sequences, then we say that $x_k = y_k$ for *almost all k (a.a.k)* if $\delta(\{k \in N : x_k \neq y_k\}) = 0$.

Let $E \subset w$, then the α-dual of the sequence space E is defined as

$$E^\alpha = \{(y_k) \in w : (x_k y_k) \in \ell_1, \quad \text{for all} \quad (x_k) \in E\}.$$

The following result will be used for establishing the results of this article.

LEMMA 1 (Šalàt [12], Lemma 1.1). *A sequence (x_n) is statistically convergent to L if and only if there exists a subset $K = \{k_i : i \in N\}$ of N such that $\delta(K) = 1$ and $\lim_{i \to \infty} x_{k_i} = L$.*

LEMMA 2. $(bv)^\alpha = \ell_1$.

LEMMA 3 (Hardy [4], Theorem 4). *If $\sum_{n=1}^{\infty} a_n u_n$ is convergent whenever $\sum_{n=1}^{\infty} u_n$ is convergent, then $(a_n) \in bv$.*

3. Main results

In this section we state and prove the results of this article. The proof of the following three results are routine works in view of the definitions and Lemma 2.

THEOREM 1. *$bv \subset \bar{bv} \subset \bar{c}$ and the inclusions are proper.*

THEOREM 2. *A necessary and sufficient condition for $(x_n) \in \bar{bv}$, where $x_n = a_n + ib_n$ for all $n \in N$ is that its real and imaginary parts (a_n) and (b_n) belong to $\bar{bv}$, where i is the imaginary unit.*

THEOREM 3. $(\bar{bv} \cap \ell_\infty)^\alpha = \ell_1$.

THEOREM 4. *Let (x_n) be a sequence of real numbers. Then a necessary and sufficient condition for $(x_n) \in \bar{bv}$ is that it contains a subsequence of the form $A_{m_k} - B_{m_k}$, where $A_{m_k} > 0$, $B_{m_k} > 0$, and decrease steadily as m_k increases over $M = \{m_i : i \in N\}$, where $\delta(M) = 1$ and $(A_{m_k}) \in bv$ and $(B_{m_k}) \in bv$.*

Proof. The sufficiency is clear in view of the following inequality and the hypothesis:

$$|\Delta x_{m_k}| \leq |\Delta A_{m_k}| + |\Delta B_{m_k}| \quad \text{for all } k \in N.$$

For the converse part, let $(x_n) \in \bar{bv}$ and $\lim_{i\to\infty} x_{m_i} = a$ for some $M = \{m_i : i \in N\}$ with $\delta(M) = 1$. Let

$$p_{m_i} = \Delta x_{m_i} \text{ (if } \Delta x_{m_i} \geq 0) \text{ and } p_{m_i} = 0 \text{ otherwise,}$$
$$q_{m_i} = |\Delta x_{m_i}| \text{ (if } \Delta x_{m_i} \leq 0) \text{ and } q_{m_i} = 0 \text{ otherwise.}$$

Let $C_{m_n} = \sum_{k=n}^{\infty} p_{m_k}$ and $D_{m_n} = \sum_{i=n}^{\infty} q_{m_i}$.

Then C_{m_n} and D_{m_n} are positive and decrease steadily as n increases and $C_{m_n} - D_{m_n} = x_{m_n} - a$ for all $n \in N$.

We can choose $A_{m_n} = C_{m_n} + \alpha$ and $B_{m_n} = D_{m_n} + \beta$ where α , β are suitably chosen constants.

This completes the proof.

THEOREM 5. *The following are equivalent:*

(i) $(a_n) \in \bar{bv}$,

(ii) *there exists sequences (x_n) and (y_n) such that $a_n = x_n + y_n$ for all $n \in N$, where $(x_n) \in bv$ and $\delta(\{k \in N : y_k \neq 0\}) = 0$.*

Proof. Let $(a_n) \in \bar{bv}$, then $\sum_{i=1}^{\infty} |\Delta a_{k_i}| < \infty$ for some $S = \{k_i : i \in N\}$ with $\delta(S) = 1$. Let us construct the sequences (x_n) and (y_n) as follows:

$x_n = a_n$ and $y_n = 0$ for $1 \leq n \leq k_1$;

$x_n = a_{k_i}$ and $y_n = 0$ if $n = k_i \in S$;

and

$x_n = a_{k_{i+1}}$ and $y_n = a_n - a_{k_{i+1}}$ if $k_1 < n < k_{i+1}$, $i \in N$.

Clearly $(x_n) \in bv$ and $\delta(\{k \in N : y_k \neq 0\}) = 0$.

Conversely let (ii) holds. Let $M = \{k \in N : y_k = 0\}$, then $\delta(M) = 1$. Let $M = \{n_i : i \in N\}$. Then we have $\sum_{i=1}^{\infty} |\Delta a_{n_i}| < \infty$ as such $(a_n) \in \bar{bv}$.

NOTE. The above result is called as the decomposition theorem for $\bar{bv}$. From the construction of the above result, one can have the following statement.

"The sequence $(x_n) \in \bar{bv}$ if and only if there exists $(y_n) \in bv$ such that $x_n = y_n$ for a.a.n".

THEOREM 6. *Let $K = \{n_i : i \in N\} \subset N$ be such that $\delta(K) = 1$. If $\sum_{i=1}^{\infty} a_{n_i} u_{n_i}$ is convergent whenever $\sum_{i=1}^{\infty} u_{n_i}$ is convergent, then $(a_n) \in \bar{bv}$.*

Proof. By Lemma 3 we have $(a_{n_i}) \in bv$ and since $\delta(K) = 1$, it follows that $(a_n) \in \bar{bv}$.

THEOREM 7. *If $(a_n) \in \bar{bv}$ and $\sum_{n \in M} |u_n| < \infty$ for some $M \subset N$ with $\delta(M) = 1$, there exists $K \subset N$ such that $\delta(K) = 1$ and $\sum_{i \in K} a_i u_i$ is convergent.*

Proof. Let $(a_n) \in \bar{bv}$, then there exists $T = \{t_i : i \in N\} \subset N$ such that $\delta(T) = 1$ and $\sum_{i=1}^{\infty} |\Delta a_{t_i}| < \infty$. Let $\lim_{i \to \infty} a_{t_i} = a$. Let $\sum_{n \in M} |u_n| < \infty$ be such that $\delta(M) = 1$. Let $K = T \cap M = \{k_i : i \in N\}$. Then clearly $\delta(K) = 1$. Let $s_{k_n} = \sum_{i=1}^{n} u_{k_i}$ for all $n \in N$ and $\lim_{n \to \infty} s_{k_n} = \alpha$ say. By Abel's summation formula we have

$$\sum_{i=1}^{n} u_{k_i} a_{k_i} = \sum_{i=1}^{n-1} s_{k_i} \Delta a_{k_i} + s_{k_n} a_{k_n} \Rightarrow \sum_{i=1}^{\infty} u_{k_i} a_{k_i} = \sum_{i=1}^{\infty} s_{k_i} \Delta a_{k_i} + \alpha a.$$

Since $(s_{k_i} \Delta a_{k_i}) \in \ell_1$, it follows that $\sum_{i=1}^{\infty} u_{k_i} a_{k_i}$ is convergent.

REMARK. Theorem 6 and Theorem 7 cannot be combined togather to get a statistical analogue of Theorem 5 of Hardy [4]. It is possible when $u_n \geq 0$ for all $n \in M$ with $\delta(M) = 1$. In this case one will have a statement "*The necessary and sufficient condition for (a_n) to be a statistical convergence factor for $\sum_{n \in M} u_n < \infty$, where $\delta(M) = 1$ and $u_n \geq 0$ for all $n \in M$ is that $(a_n) \in \bar{bv}$*".

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