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ON HYERS-ULAM STABILITY OF THE PEXIDER EQUATION

Let $(S, +)$ be a commutative semigroup and let X be a sequentially complete linear topological Hausdorff space. In the theory of functional equations the problem of the stability (in a sense) has been considered by many authors. We recall only two results concerning the stability of the Pexider equation. In [1] E. Głowacki and Z. Kominek have proved that under the above assumptions, for arbitrary functions $f, g, h : S \to X$ fulfilling the condition

$$S \times S \ni (x, y) \longrightarrow f(x + y) - g(x) - h(y) \text{ is bounded,}$$

there exists an additive function $A : S \to X$ such that the differences $f(x) - A(x),\ x \in S + S,\quad g(x) - A(x),\ x \in S,\quad h(x) - A(x),\ x \in S$ are bounded. This theorem has a qualitative character. It says nothing about numerical dependence between these bounded sets. On the other hand, in [3] K. Nikodem has proved, assuming additionally that S is a semigroup with zero, that if $f(x+y)-g(x)-h(y) \in V$ (V is a bounded, convex and symmetric with respect to zero subset of X), then there exist functions $f_1, g_1, h_1 : S \to X$ satisfying the Pexider equation $f_1(x + y) - g_1(x) - h_1(y) = 0,\ x, y \in S$, such that $f(x) - f_1(x) \in 3U$, $g(x) - g_1(x) \in 4U$, $h(x) - h_1(x) \in 4U$ where $U := seqclV$. We denote by $seqclA$ the sequential closure of A.

We start with the following lemma.

LEMMA. *Let $(S, +)$ be a commutative semigroup and let X be a sequentially complete, linear topological Hausdorff space. Assume that V is a sequentially closed, bounded, convex and symmetric with respect to zero subset of X. If*

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$f : S \to X$ fulfils the condition

$$(1) \qquad f(x+y) - \frac{f(2x)+f(2y)}{2} \in 2V, \qquad x, y \in S,$$

then there exist an additive function $A : S \to X$ and a constant $c(:= 2f(2x_0) - f(4x_0)) \in X$ such that

$$(2)\ f(2x) - A(2x) - c \in 16V \quad \text{and} \quad f(x+y) - A(x+y) - c \in 18V, \quad x, y \in S.$$

Proof. Fix an $x_0 \in S$ and put

$$(3) \qquad a(x) := f(x+2x_0) - f(2x_0),\ x \in S.$$

By virtue of (1) we have

$$\begin{aligned} a(x+y) - a(x) - a(y) & \\ &= f(x+y+2x_0) - f(x+2x_0) - f(y+2x_0) + f(2x_0) \\ &= f((x+x_0)+(y+x_0)) - \frac{f(2(x+x_0)) + f(2(y+x_0))}{2} \\ &\quad + \frac{f(2(x+x_0)) + f(2x_0)}{2} - f((x+x_0)+x_0) \\ &\quad + \frac{f(2(y+x_0)) + f(2x_0)}{2} - f((y+x_0)) + x_0) \\ &\in 2V + 2V + 2V = 6V. \end{aligned}$$

Now, by a standard way ([2], for example, where the boundedneity of V is needed) one can check that $(\frac{a(2^n x)}{2^n})_{n\in\mathbb{N}}$ is a Cauchy sequence and the limit

$$A(x) := \lim_{n\to\infty} \frac{a(2^n x)}{2^n}, \qquad x \in S$$

is additive function and, moreover,

$$(4) \qquad a(x) - A(x) \in 6V, \qquad x \in S.$$

By definition of c and (3), for every $x, y \in S$, we obtain

$$\begin{aligned} A(2x) + c - f(2x) &= A(2x) - 2a(x) + 2a(x) + 2f(2x_0) - f(4x_0) - f(2x) \\ &= 2[A(x) - a(x)] + 2\left[f(x+2x_0) - \frac{f(2x) + f(4x_0)}{2}\right] \\ &\in 12V + 4V = 16V, \end{aligned}$$

and

$$\begin{aligned}
&A(x+y)+c-f(x+y)\\
&= A(x)+A(y)+2f(2x_0)-f(4x_0)-f(x+y)\\
&\quad+\frac{f(2x)+f(2y)}{2}-\frac{f(2x)+f(2y)}{2}+a(x)+a(y)-a(x)-a(y)\\
&=[A(x)-a(x)]+[A(y)-a(y)]-\left[f(x+y)-\frac{f(2x)+f(2y)}{2}\right]\\
&\quad+\left[f(x+2x_0)-\frac{f(2x)+f(4x_0)}{2}\right]+\left[f(y+2x_0)-\frac{f(2y)+f(4x_0)}{2}\right]\\
&\in 6V+6V-2V+2V+2V=18V.
\end{aligned}$$

The proof of our Lemma is completed. ■

Our main result reads as follows.

THEOREM. *Let $(S,+)$ be a commutative semigroup and let X be a sequentially complete, linear topological Hausdorff space. Assume that V is a sequentially closed, bounded, convex and symmetric with respect to zero subset of X. For arbitrary functions $f,g,h : S \to X$ satisfying condition*

$$f(x+y)-g(x)-h(y)\in V, \qquad x,y\in S, \tag{5}$$

there exist functions $f_1, g_1, h_1 : S \to X$ such that

$$f_1(x+y)-g_1(x)-h_1(y)=0, \qquad x,y\in S, \tag{6}$$

$f_1(x+y)-f(x+y)\in 15V$, $g_1(x)-g(x)\in 7V$, and $h_1(x)-h(x)\in 7V$, $x,y\in S$.

Proof. Since $f(2x)-g(x)-h(x)\in V$, $x\in S$, we have

$$\begin{aligned}
f(x+y)-\frac{f(2x)+f(2y)}{2}&\\
&=\frac{1}{2}[f(x+y)-g(x)-h(y)]+\frac{1}{2}[f(x+y)-g(y)-h(x)]\\
&\quad-\frac{1}{2}[f(2x)-g(x)-h(x)]-\frac{1}{2}[f(2y)-g(y)-h(y)]\\
&\in\frac{1}{2}V+\frac{1}{2}V-\frac{1}{2}V-\frac{1}{2}V=2V.
\end{aligned}$$

Let $A : S\to X$ be an additive function obtained in our Lemma and let f_1, g_1 and h_1 be functions defined by the following formulas:

$f_1(x) := A(x)+2f(2x_0)-g(2x_0)-h(2x_0)$, $x\in S$;
$g_1(x) := A(x)+f(2x_0)-h(2x_0)$, $x\in S$;
$h_1(x) := A(x)+f(2x_0)-g(2x_0)$, $x\in S$.

Now condition (6) is an easy consequence of additivity of A. It follows from (3), (4) and (5) that

$$\begin{aligned} g_1(x) - g(x) &= A(x) + f(2x_0) - h(2x_0) - g(x) \\ &= A(x) - a(x) + a(x) + f(2x_0) - h(2x_0) - g(x) \\ &= [A(x) - a(x)] + [f(x + 2x_0) - h(2x_0) - g(x)] \\ &\in 6V + V = 7V. \end{aligned}$$

Similarly, we obtain the relation $h_1(x) - h(x) \in 7V$, $x \in S$. Moreover, according to (6) and (5), we get

$$f_1(x+y) - f(x+y) \in V + [g_1(x) - g(x)] + [h_1(y) - h(y)] \in V + 7V + 7V = 15V,$$

for all $x, y \in S$. This ends the proof of our Theorem. ∎

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