DEMONSTRATIO MATHEMATICA
Vol. XXXVII No 2 2004

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SOME REMARKS ON ITERATION GROUPS ON THE CIRCLE

Abstract. In this paper we give two constructions of some iteration groups on the unit circle $\mathbb{S}^1$. The first one determines rational iteration groups. The other describes all iteration groups $\{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, v \in V\}$ for which $\mathbb{S}^1 \neq \{z \in \mathbb{S}^1 : F^v(z) = z, v \in V\} \neq \emptyset$ (V is a linear space over $\mathbb{Q}$ with $\dim V \geq 1$).

1. Introduction

Let X be a topological space and V be a linear space over $\mathbb{Q}$ with $\dim V \geq 1$. Recall that a family $\{F^v : X \longrightarrow X, \ v \in V\}$ of homeomorphisms such that

$$F^{v_1} \circ F^{v_2} = F^{v_1+v_2}, \qquad v_1, \ v_2 \in V$$

is called an *iteration group* (on X). An iteration group is said to be *rational* (respectively, *real*) if $V = \mathbb{Q}$ (respectively, $V = \mathbb{R}$).

Rational iteration groups on open real intervals have been examined by J. Tabor in [11] and M. Bajger and M. C. Zdun in [3]. The aim of this paper is to give a construction of rational iteration groups on the unit circle $\mathbb{S}^1$ and a construction of all iteration groups $\{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, \ v \in V\}$ for which $\emptyset \neq \{z \in \mathbb{S}^1 : F^v(z) = z, \ v \in V\} \neq \mathbb{S}^1$.

2. Rational iteration groups

First, we will deal with rational iteration groups.

2.1. Preliminaries

We begin by recalling the basic definitions and introducing some notation.

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers. The closure of a set $A \subset \mathbb{S}^1$ will be denoted by cl$A$ while A^{d} stands for the set of

2000 *Mathematics Subject Classification*: 39B12, 37E10, 20F38.

all cluster points of A. By an *open arc* we mean the set $\{e^{2\pi it},\ t \in (t_1,\ t_2)\}$, where $t_1,\ t_2 \in \mathbb{R}$ are such that $0 < t_2 - t_1 < 1$.

It is well-known (see for instance [1], [4] and [13]) that for every continuous mapping $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ there is a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$, which is unique up to translation by an integer, and a unique integer k such that

$$F(e^{2\pi ix}) = e^{2\pi if(x)} \quad \text{and} \quad f(x+1) = f(x) + k, \qquad x \in \mathbb{R}.$$

The integer k is called the *degree* of F, and is denoted by $\deg F$. If $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a homeomorphism, then so is f. Furthermore, $|\deg F| = 1$. We say that a homeomorphism $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *preserves orientation* if $\deg F = 1$, which is equivalent to the fact that f is increasing. For such a homeomorphism F, the number $\alpha(F) \in [0,\ 1)$ defined by

$$\alpha(F) := \lim_{n\to\infty} \frac{f^n(x)}{n} (\mathrm{mod} 1), \qquad x \in \mathbb{R},$$

is called the *rotation number* of F. If $\alpha(F) \notin \mathbb{Q}$, then the set

$$L_F := \{F^n(z),\ n \in \mathbb{Z}\}^{\mathrm{d}}$$

(the *limit set* of F) does not depend on $z \in \mathbb{S}^1$ and either $L_F = \mathbb{S}^1$ or L_F is a non-empty, perfect and nowhere dense subset of $\mathbb{S}^1$ (see for instance [8] and [10]).

LEMMA 1 (see [7]). *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in \mathbb{Q}\}$ is an iteration group for which $\alpha(F^1) \notin \mathbb{Q}$, then there exists a unique pair $(\varphi_{\mathcal{F}},\ c_{\mathcal{F}})$ such that $\varphi_{\mathcal{F}} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a continuous function of degree 1 with $\varphi_{\mathcal{F}}(1) = 1$ and $c_{\mathcal{F}} : \mathbb{Q} \longrightarrow \mathbb{S}^1$ satisfying the following system of functional equations*

$$\varphi_{\mathcal{F}}(F^v(z)) = c_{\mathcal{F}}(v)\varphi_{\mathcal{F}}(z), \qquad z \in \mathbb{S}^1,\ v \in \mathbb{Q}.$$

The mapping $c_{\mathcal{F}}$ is given by

$$c_{\mathcal{F}}(v) = e^{2\pi i\alpha(F^v)}, \qquad v \in \mathbb{Q}, \tag{1}$$

and fulfils the equation

$$c_{\mathcal{F}}(v_1 + v_2) = c_{\mathcal{F}}(v_1)c_{\mathcal{F}}(v_2), \qquad v_1,\ v_2 \in \mathbb{Q}. \tag{2}$$

If moreover $L_{F^1} \neq \mathbb{S}^1$, then

$$\varphi_{\mathcal{F}}[\mathbb{S}^1 \setminus L_{F^1}] \cdot \mathrm{Im} c_{\mathcal{F}} = \varphi_{\mathcal{F}}[\mathbb{S}^1 \setminus L_{F^1}]. \tag{3}$$

2.2. Main results

We start with the following

THEOREM 1. *Let $\psi_0 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be an orientation-preserving homeomorphism and assume that for every $n \in \mathbb{N}$ there exists an orientation-preserving* *homeomorphism* $\psi_n : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *such that* $\psi_n^{n+1} = \psi_{n-1}$. *Then a family* $\{F^w : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ w \in \mathbb{Q}\}$, *where*

$$F^{\frac{1}{(n+1)!}} := \psi_n, \qquad n \in \mathbb{N} \cup \{0\},$$

$$F^{\frac{m}{n}} := (F^{\frac{1}{n!}})^{(n-1)!m}, \qquad n \in \mathbb{N},\ m \in \mathbb{Z},$$

is an iteration group. Moreover, every rational iteration group with $F^1 = \psi_0$ *can be obtained in this way.*

Proof. We shall show that constructed in the above way family $\{F^w : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ w \in \mathbb{Q}\}$ is an iteration group. To do this, let us first observe that the functions $F^w : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ for $w \in \mathbb{Q}$ are well-defined, that is

$$F^{\frac{m}{n}} = F^{\frac{mk}{nk}}, \qquad m \in \mathbb{Z},\ n,\ k \in \mathbb{N},\ k > 1.$$

Indeed, fixing $m \in \mathbb{Z},\ n,\ k \in \mathbb{N},\ k > 1$ we see that

$$\begin{aligned} F^{\frac{mk}{nk}} &= (F^{\frac{1}{nk!}})^{(nk-1)!mk} = ((F^{\frac{1}{nk!}})^{\frac{(nk)!}{n!}})^{(n-1)!m} \\ &= ((F^{\frac{1}{nk!}})^{nk(nk-1)\cdot\ldots\cdot(n+1)})^{(n-1)!m} = (F^{\frac{1}{n!}})^{(n-1)!m} = F^{\frac{m}{n}}. \end{aligned}$$

Next, fix $w_1,\ w_2 \in \mathbb{Q}$ and let $n \in \mathbb{N},\ m_1,\ m_2 \in \mathbb{Z}$ be such that $w_1 = \frac{m_1}{n}$, $w_2 = \frac{m_2}{n}$. Since

$$\begin{aligned} F^{w_1} \circ F^{w_2} &= F^{\frac{m_1}{n}} \circ F^{\frac{m_2}{n}} = (F^{\frac{1}{n!}})^{(n-1)!m_1} \circ (F^{\frac{1}{n!}})^{(n-1)!m_2} \\ &= (F^{\frac{1}{n!}})^{(n-1)!(m_1+m_2)} = F^{\frac{m_1+m_2}{n}} = F^{w_1+w_2} \end{aligned}$$

and the mappings F^w for $w \in \mathbb{Q}$ are homeomorphisms, the family $\{F^w : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ w \in \mathbb{Q}\}$ is an iteration group.

Finally, let $\{F^w : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ w \in \mathbb{Q}\}$ be an iteration group such that $F^1 = \psi_0$. Putting $\psi_n := F^{\frac{1}{(n+1)!}}$ for $n \in \mathbb{N}$ we check at once that $\psi_n^{n+1} = \psi_{n-1}$ for $n \in \mathbb{N}$ and every ψ_n is a homeomorphism of the circle which (see [7]) preserves orientation. ∎

Let us recall that a homeomorphism $F : X \longrightarrow X$ is said to be *embeddable* in a rational (respectively, real) iteration group on X if there exists such an iteration group for which $F^1 = F$.

It is known (see for instance [11]) that if X is a real interval, then every increasing homeomorphism is embeddable in a rational iteration group. For $X = \mathbb{S}^1$ this is no longer true. Theorem 1 gives a necessary and sufficient condition for embeddability of a given homeomorphism of the unit circle in rational iteration group. In the sequel, we present some examples.

Regarding the case when $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a homeomorphism with $L_F = \mathbb{S}^1$ we have the following proposition, which follows immediately from Theorem 2 in [5] (one can also use Theorem 1 and Lemma 1 in [14]).

PROPOSITION 1. *Every homeomorphism $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ such that $L_F = \mathbb{S}^1$ satisfies the assumption of Theorem 1.*

For any homeomorphism $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ with $L_F \neq \mathbb{S}^1$ put

$$K_F := \varphi[\mathbb{S}^1 \setminus L_F],$$

where $\varphi : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a unique continuous mapping such that

$$\varphi(F(z)) = e^{2\pi i\alpha(F)}\varphi(z), \qquad z \in \mathbb{S}^1$$

and $\varphi(1) = 1$ (see [5]). For every sequence $a = (a_n)_{n\in\mathbb{N}}$ satisfying

$$a_n \in \{0, ..., n-1\}, \ \ a_{kn} = a_n (\mathrm{mod}\ n), \qquad k,\ n \in \mathbb{N} \tag{4}$$

define also

$$A_{\alpha(F)}(a) := \{e^{2\pi i(\frac{m}{n}\alpha(F)+\frac{m}{n}a_n)}, \quad n \in \mathbb{N},\ m \in \mathbb{Z}\}.$$

With this notation, we have

PROPOSITION 2. *A homeomorphism $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ with $L_F \neq \mathbb{S}^1$ fulfils the assumption of Theorem 1 if and only if there exists a sequence $a = (a_n)_{n\in\mathbb{N}}$ satisfying* (4) *and such that*

$$K_F \cdot A_{\alpha(F)}(a) = K_F. \tag{5}$$

Proof. Let $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be a homeomorphism for which $L_F \neq \mathbb{S}^1$.

Suppose that F satisfies the assumption of Theorem 1 and let $\mathcal{F} = \{F^w : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ w \in \mathbb{Q}\}$ be an iteration group such that $F^1 = F$. From (2) it follows by induction that

$$c_{\mathcal{F}}(mw) = c_{\mathcal{F}}(w)^m, \qquad m \in \mathbb{Z},\ w \in \mathbb{Q}. \tag{6}$$

Fix k, $n \in \mathbb{N}$. By (1) and (6) we have

$$e^{2\pi i\alpha(F)} = c_{\mathcal{F}}(1) = c_{\mathcal{F}}(\frac{1}{n})^n,$$

and therefore there exists an $a_n \in \{0, ..., n-1\}$ for which $c_{\mathcal{F}}(\frac{1}{n}) = e^{2\pi i(\frac{\alpha(F)}{n}+\frac{a_n}{n})}$. Since

$$c_{\mathcal{F}}\left(\frac{1}{n}\right) = c_{\mathcal{F}}\left(\frac{1}{kn}\right)^k = (e^{2\pi i(\frac{\alpha(F)}{kn}+\frac{a_{kn}}{kn})})^k = e^{2\pi i(\frac{\alpha(F)}{n}+\frac{a_{kn}}{n})},$$

we see that $a_{kn} = a_n(\mathrm{mod}\ n)$. Thus, the sequence $a := (a_n)_{n\in\mathbb{N}}$ satisfies (4). Moreover, from (6) and the equality $c_{\mathcal{F}}(\frac{1}{n}) = e^{2\pi i(\frac{\alpha(F)}{n}+\frac{a_n}{n})}$ we obtain $c_{\mathcal{F}}(\mathbb{Q}) = A_{\alpha(F)}(a)$, and (3) now shows that (5) holds true.

To finish the proof it suffices to apply Theorem 3 in [5]. ∎

As an immediate consequence of Theorems 2, 3 in [5], Theorem 1 and Proposition 2 we obtain

COROLLARY 1. *A homeomorphism* $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *with* $\alpha(F) \notin \mathbb{Q}$ *is embeddable in a rational iteration group if and only if it is embeddable in a real iteration group.*

PROPOSITION 3. *Let* L *be a non-empty, perfect and nowhere dense subset of* $\mathbb{S}^1$ *with* $1 \in \mathbb{S}^1 \setminus \bigcup_{q\in\mathbb{Q}} \mathrm{cl} I_q$, *where* I_q *for* $q \in \mathbb{Q}$ *are open pairwise disjoint arcs such that* $\mathbb{S}^1 \setminus L = \bigcup_{q\in\mathbb{Q}} I_q$. *Assume also that* $\alpha \in [0, 1) \setminus \mathbb{Q}$ *and put*

$$K_1 := \{e^{2\pi i(m\alpha+\frac{1}{4})},\ m \in \mathbb{Z}\}, \qquad K_2 := \{e^{2\pi i(t\alpha+\frac{1}{4})},\ t \in \mathbb{Q}\}.$$

Then there are homeomorphisms $F_1,\ F_2 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *for which*

$$\alpha(F_1) = \alpha = \alpha(F_2),\ \ L_{F_1} = L = L_{F_2},\ \ K_{F_1} = K_1,\ \ K_{F_2} = K_2.$$

Moreover, F_2 *satisfies the assumption of Theorem 1, but* F_1 *does not.*

Proof. The fact that homeomorphisms F_1, $F_2 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ having the stated properties exist can be found in [14].

Let us note that $K_{F_1} \cdot A_{\alpha(F_1)}(a) \neq K_{F_1}$ for any sequence $a = (a_n)_{n\in\mathbb{N}}$ satisfying condition (4). Indeed, if there existed such a sequence for which $K_{F_1} \cdot A_{\alpha(F_1)}(a) = K_{F_1}$, we would have

$$e^{2\pi i(\frac{1}{2}\alpha+\frac{1}{2}a_2+\frac{1}{4})} \in K_{F_1} \cdot A_{\alpha(F_1)}(a) = K_{F_1},$$

and therefore there would be $p,\ m \in \mathbb{Z}$ such that

$$\frac{1}{2}\alpha + \frac{1}{2}a_2 + \frac{1}{4} + p = m\alpha + \frac{1}{4}.$$

Consequently, $\alpha = \frac{\frac{1}{2}a_2+p}{m-\frac{1}{2}} \in \mathbb{Q}$, which is impossible.

Since we also have $K_{F_2} \cdot A_{\alpha(F_2)}(a) = K_{F_2}$ with $a \equiv 0$, our assertion follows from Proposition 2. ∎

The following fact can be concluded from [9].

PROPOSITION 4. *If the set of all* m*-periodic points of a homeomorphism* $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *has exactly* m *elements for an integer* $m \geq 2$, *then* F *does not satisfy the assumption of Theorem 1.*

3. Iteration groups having a nontrivial set of common fixed points of all elements

Now, we give a general construction of all iteration groups $\{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ such that $\emptyset \neq \{z \in \mathbb{S}^1 : F^v(z) = z,\ v \in V\} \neq \mathbb{S}^1$.

3.1. Preliminaries

We begin by recalling the relevant material from [2] and [5].

For any $v, w, z \in \mathbb{S}^1$ there exist unique $t_1, t_2 \in [0, 1)$ such that $we^{2\pi i t_1} = z$ and $we^{2\pi i t_2} = v$, so we can put

$$v \prec w \prec z \quad \text{if and only if} \quad 0 < t_1 < t_2.$$

A set $A \subset \mathbb{S}^1$ is an open arc if there are distinct $v,\ z \in \mathbb{S}^1$ for which

$$A = \{w \in \mathbb{S}^1 : v \prec w \prec z\} =: \overrightarrow{(v,\ z)}\ .$$

Given a subset A of $\mathbb{S}^1$ with card$A \geq 3$ and a function F mapping A into $\mathbb{S}^1$ we say that F is *strictly increasing* if for any $v,\ w,\ z \in A$ such that $v \prec w \prec z$ we have $F(v) \prec F(w) \prec F(z)$. A homeomorphism of the circle is strictly increasing if and only if it preserves orientation.

3.2. Main results

It is easily seen (see also [5]) that for every homeomorphism $F : A \longrightarrow A$, where $A = \{e^{2\pi i t},\ t \in (a,\ b)\}$ with $a,\ b \in \mathbb{R},\ a < b \leq a + 1$, there exists a unique homeomorphism $f : (a,\ b) \longrightarrow (a,\ b)$ such that

$$F(e^{2\pi i x}) = e^{2\pi i f(x)}, \qquad x \in (a,\ b).$$

We say that f *represents* F, and if f is strictly increasing, then we say that the homeomorphism F *preserves orientation.*

The following remarks are easy to check.

Remark 1 (see also [5]). Let $A = \{e^{2\pi i t},\ t \in (a,\ b)\}$, where $a,\ b \in \mathbb{R},\ a < b \leq a + 1$, and assume that $F,\ G : A \longrightarrow A$ are homeomorphisms. If f represents F and g represents G, then:

(i) $g \circ f$ represents $G \circ F$,
(ii) f^{-1} represents F^{-1}.

Remark 2. If $\{F^v : A \longrightarrow A,\ \ v \in V\}$ is an iteration group on the set $A = \{e^{2\pi i t},\ t \in (a,\ b)\}$, where $a,\ b \in \mathbb{R},\ a < b$, then every homeomorphism F^v preserves orientation.

Remark 3. If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ \ v \in V\}$ is an iteration group, then the set

$$E_{\mathcal{F}} := \{z \in \mathbb{S}^1 : F^v(z) = z,\ \ v \in V\}$$

is closed in $\mathbb{S}^1$.

Let $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ \ v \in V\}$ be an iteration group for which $\emptyset \neq E_{\mathcal{F}} \neq \mathbb{S}^1$. Then we have the following decomposition

$$\mathbb{S}^1 \setminus E_{\mathcal{F}} = \bigcup_{q \in M} I_q,$$

where M is a set with card$M \geq 1$ and I_q for $q \in M$ are open pairwise disjoint arcs if card$E_{\mathcal{F}} > 1$, whereas $I_q = \mathbb{S}^1 \setminus \{z_0\}$ for a $z_0 \in \mathbb{S}^1$ if card$E_{\mathcal{F}} = 1$.

LEMMA 2. *If* $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ *is an iteration group such that* $\emptyset \neq E_{\mathcal{F}} \neq \mathbb{S}^1$, *then*

$$F^v[I_q] = I_q, \quad q \in M,\ v \in V.$$

Proof. It is obvious that if $\operatorname{card} E_{\mathcal{F}} = 1$, then our assertion follows. Now, assume that $\operatorname{card} E_{\mathcal{F}} > 1$. Fix $q \in M$, $v \in V$ and let $I_q = \overrightarrow{(a_q, b_q)}$. Clearly, $F^v(a_q) = a_q$, $F^v(b_q) = b_q$, and the fact that the homeomorphism F^v is strictly increasing now yields

$$F^v[I_q] = F^v[\overrightarrow{(a_q, b_q)}] = \overrightarrow{(F^v(a_q), F^v(b_q))} = \overrightarrow{(a_q, b_q)} = I_q. \blacksquare$$

THEOREM 2. *The following construction determines all iteration groups* $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ *for which* $\emptyset \neq E_{\mathcal{F}} \neq \mathbb{S}^1$.

1°. *Take a non-empty closed subset* $E \neq \mathbb{S}^1$ *of* $\mathbb{S}^1$ *and let* M *be a set with* $\operatorname{card} M \geq 1$, *and* I_q *for* $q \in M$ *be open pairwise disjoint arcs if* $\operatorname{card} E > 1$, *whereas* $I_q = \mathbb{S}^1 \setminus \{z_0\}$ *for a* $z_0 \in \mathbb{S}^1$ *if* $\operatorname{card} E = 1$ *such that*

$$\mathbb{S}^1 \setminus E = \bigcup_{q \in M} I_q. \tag{7}$$

2°. *Choose iteration groups* $\{F_q^v : I_q \longrightarrow I_q,\ v \in V\}$ *on* I_q *for* $q \in M$ *such that at least one of them is nontrivial (i.e., different from* $\{\mathrm{id}\}$*).*

3°. *Define*

$$F^v(z) := \begin{cases} F_q^v(z), & z \in I_q,\ q \in M, \\ z, & z \in E, \end{cases} \quad v \in V. \tag{8}$$

Proof. We shall first show that the family $\mathcal{F} := \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is an iteration group with $\emptyset \neq E_{\mathcal{F}} \neq \mathbb{S}^1$. To do this, we prove that the mappings $F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ for $v \in V$ are strictly increasing. Fix $v \in V$ and $x, w, z \in \mathbb{S}^1$ such that $x \prec w \prec z$ and consider the following cases:

(i) $\{x, w, z\} \subset I_q$ for a $q \in M$.

Let $t_x, t_w, t_z \in [0, 1)$ be such that $e^{2\pi i t_x} = x$, $e^{2\pi i t_w} = w$ and $e^{2\pi i t_z} = z$. Then we have either $t_x < t_w < t_z$, or $t_w < t_z < t_x$, or $t_z < t_x < t_w$. Assume, for instance, that $t_x < t_w < t_z$ and let f_q represents the homeomorphism $F^v \mid I_q$. Since, by Remark 2, $F^v \mid I_q$ preserves orientation, we see that $f_q(t_x) < f_q(t_w) < f_q(t_z)$. Moreover, $f_q(t_x), f_q(t_w), f_q(t_z) \in (t_1, t_2)$ for some $t_1, t_2 \in \mathbb{R}$ with $0 < t_2 - t_1 \leq 1$. Therefore $e^{2\pi i f_q(t_w)} \in \overrightarrow{(e^{2\pi i f_q(t_x)}, e^{2\pi i f_q(t_z)})}$, and consequently $F^v(w) \in \overrightarrow{(F^v(x), F^v(z))}$.

(ii) $\operatorname{card}(\{x, w, z\} \cap I_q) = 2$ for a $q \in M$.

According to Lemmas 1 and 2 in [5] we may assume that $x, w \in I_q$. Fixing a $u \in I_q$ for which $w \in \overrightarrow{(x, u)}$ we conclude from (i) and (8) that

$$F^v(w) \in \overrightarrow{(F^v(x), F^v(u))} \subset I_q \quad \text{and} \quad F^v(z) \notin I_q,$$

and, in consequence, $F^v(w) \in \overrightarrow{(F^v(x), F^v(z))}$.

If card$E = 1$, then the proof is complete. From now on we assume that card$E > 1$.

(iii) card$(\{x, w, z\} \cap I_q) = $ card$(\{x, w, z\} \cap I_p) = 1$ for some distinct $p, q \in M$.

We can assume, in view of Lemmas 1 and 2 in [5], that $x \in I_q$, $w \in I_p$. Since $x \prec w \prec z$, we have $I_q \prec I_p \prec I(z)$ (that is, for any $a \in I_q$, $b \in I_p$, $c \in I(z)$, $a \prec b \prec c$), where

$$I(z) := \begin{cases} I_r, & z \in I_r,\ r \in M, \\ \{z\}, & z \in E. \end{cases}$$

Therefore, by (8), we get $F^v[I_q] \prec F^v[I_p] \prec F^v[I(z)]$, and consequently $F^v(x) \prec F^v(w) \prec F^v(z)$.

(iv) card$(\{x, w, z\} \cap E) = 2$.

On account of Lemmas 1 and 2 in [5] we may assume that $x, z \in E$ and $w \in I_q$ for a $q \in M$. As I_q is an open arc, we see that $I_q \subset \overrightarrow{(x, z)}$. Hence, by (8), we obtain

$$F^v[I_q] = I_q \subset \overrightarrow{(x, z)} = \overrightarrow{(F^v(x), F^v(z))},$$

and therefore $F^v(w) \in \overrightarrow{(F^v(x), F^v(z))}$.

(v) $\{x, w, z\} \subset E$.

From (8) we have $F^v(x) = x$, $F^v(w) = w$ and $F^v(z) = z$, and our assertion follows.

We have thus proved that the functions $F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ for $v \in V$ are strictly increasing. Since, by (2), we also have $F^v[\mathbb{S}^1] = \mathbb{S}^1$, Remark 3 in [6] and Lemma 4 in [5] show that these mappings are homeomorphisms.

Fix $z \in \mathbb{S}^1$ and $v_1, v_2 \in V$. If $z \in E$, then, by (8), $(F^{v_1} \circ F^{v_2})(z) = z = F^{v_1+v_2}(z)$. Now, assume that $z \in \mathbb{S}^1 \setminus E$ and let $q \in M$ be such that $z \in I_q$. Then $F^{v_2}(z) \in I_q$, and (8) together with the fact that $\{F_q^v : I_q \longrightarrow I_q, v \in V\}$ is an iteration group gives

$$(F^{v_1} \circ F^{v_2})(z) = (F_q^{v_1} \circ F_q^{v_2})(z) = F_q^{v_1+v_2}(z) = F^{v_1+v_2}(z).$$

We have thus shown that $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is an iteration group. Finally, it is clear that $\emptyset \neq E \subset E_{\mathcal{F}} \neq \mathbb{S}^1$.

Let us next assume that $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is an iteration group for which $\emptyset \neq E_{\mathcal{F}} \neq \mathbb{S}^1$ and put $E := E_{\mathcal{F}}$. From Remark 3 it follows that the set E is closed in $\mathbb{S}^1$. Let M be a set with card$M \geq 1$ and I_q for $q \in M$ be open pairwise disjoint arcs if card$E > 1$, whereas $I_q = \mathbb{S}^1 \setminus \{z_0\}$ for a $z_0 \in \mathbb{S}^1$ if card$E = 1$ such that (7) holds true. Putting $F_q^v := F^v|\, I_q$ for $v \in V$, $q \in M$ we deduce from Lemma 2 that $\{F_q^v : I_q \longrightarrow I_q,\ v \in V\}$ for $q \in M$ are iteration groups. Moreover, it is obvious that at least one of these groups is nontrivial and (8) holds true. ∎

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Received December 13, 2002.