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Nguyen Thanh Long

LINEAR APPROXIMATION AND ASYMPTOTIC EXPANSION ASSOCIATED WITH THE SYSTEM OF FUNCTIONAL EQUATIONS

Abstract. We consider the following perturbed system of functional equations

$$(*) \qquad f_i(x) = \varepsilon \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \Phi(f_j(R_{ijk}(x))) + \sum_{k=1}^{m} \sum_{j=1}^{n} b_{ijk} f_j(S_{ijk}(x)) + g_i(x),$$
$$\forall x \in \Omega \subset R^p;\ i = 1, 2, \ldots, n,$$

where ε is a small parameter, $|\varepsilon| \leq 1; \Omega$ is a compact or non-compact domain of R^p, a_{ijk}, b_{ijk} are the given real constants; $R_{ijk}, S_{ijk} : \Omega \to \Omega, \Phi : R \to R, g_i : \Omega \to R$ are the given continuous functions and $f_i : \Omega \to R$ are unknown functions. By using the Banach fixed point theorem, we prove the system $(*)$ has a unique solution. In the case of $\Phi \in C^2(R;R)$, we also obtain the quadratic convergence of the system $(*)$. Moreover, if $\Phi \in C^N(R;R)$ and $\sum_{k=1}^{m} \sum_{j=1}^{n} \max_{1 \leq j \leq n} \left| b_{ijk} \right| < 1$, then an asymptotic expansion of the solution of the system $(*)$ up to order $N+1$ in ε is obtained, for ε sufficiently small.

1. Introduction

In this paper, we study an asymptotic expansion of solution in the small parameter ε of the following system of functional equations

$$(1.1) \qquad f_i(x) = \varepsilon \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \Phi(f_j(R_{ijk}(x))) + \sum_{k=1}^{m} \sum_{j=1}^{n} b_{ijk} f_j(S_{ijk}(x)) + g_i(x),$$
$$\forall x \in \Omega \subset R^p;\ i = 1, 2, \ldots, n,$$

where ε is a small parameter; Ω is a compact or non-compact domain of R^p, a_{ijk}, b_{ijk} are the given real constants; $R_{ijk}, S_{ijk} : \Omega \to \Omega$, $\Phi : R \to R, g_i :$

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$\Omega \to R$ are the given continuous functions and $f_i : \Omega \to R$ are unknown functions.

In [1], system (1.1) is studied with $p = 1$, $\Omega = [-b, b]$, $m = n = 2$, $a_{ijk} = 0$ and S_{ijk} binomials of first degree. The solution is approximated by a uniformly convergent recurrent sequence and it is stable with respect to the functions g_i.

In [2], we have studied a special case of (1.1) with $p = 1$ and $\Omega = [-b, b]$ or Ω an unbounded interval of R. By using the Banach fixed point theorem, we have obtained the existence, uniqueness and also stability of the solution of the system (1.1) with respect to the functions g_i. In the case of $a_{ijk} = 0$ and S_{ijk} being binomials of first degree, $g \in C^r(\Omega; R^n)$ and $\Omega = [-b, b]$ we have obtained a Maclaurin expansion of the solution of system (1.1) until the order r. Furthermore, if g_i are polynomials of degree r, then the solution of system (1.1) is also a polynomial of degree r. Next, if g_i are continuous functions, the solution f of (1.1) is approximated by a uniformly convergent polynomial sequence. Afterwards, these results have been extended in [3] to the multi-dimensional domain $\Omega \subset R^p, S_{ijk}$ being affine functions. Furthermore, we also give a sufficient condition for the quadratic convergence of the system of functional equations [3].

In this paper, we consider three main parts. In Part 1, by using the Banach fixed point theorem, we prove the existence and uniqueness of the solution of system (1.1). In Part 2, we give a sufficient condition for the quadratic convergence of the system of functional equations. In Part 3, we prove that if $\Phi \in C^N(R; R)$ and $\sum_{k=1}^{m} \sum_{i=1}^{n} \max_{1 \le j \le n} |b_{ijk}| < 1$, then an asymptotic expansion of the solution of system (1.1) up to order $N + 1$ in ε is obtained, for ε sufficiently small. The results obtained here relatively generalize the ones in [1–3].

2. Notations, functions spaces

A point in R^p is denoted by $x = (x_1, \ldots, x_p)$. We call $\alpha = (\alpha_1, \ldots, \alpha_p) \in Z_+^p$ a p -multi- index and denote by x^α the monomial $x_1^{\alpha_1} \ldots x_p^{\alpha_p}$, which has degree $|\alpha| = \sum_{i=1}^{p} \alpha_i$. Similarly, if $D_j = \partial/\partial x_j$ for $1 \le j \le p$, then $D^\alpha = D_1^{\alpha_1} \ldots D_p^{\alpha_p} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_p^{\alpha_p}}$ denotes a differential operator of order $|\alpha|$. We also denote $\alpha! = \alpha_1! \ldots \alpha_p!$.

With Ω a compact subset of R^p, we denote by $X = C(\Omega; R^n)$ the Banach space of functions $f = (f_1, \ldots, f_n) : \Omega \to R^n$ continuous on Ω with respect to the norm

$$\|f\|_X = \sup_{x \in \Omega} \sum_{i=1}^{n} |f_i(x)|. \tag{2.1}$$

When $\Omega \subset R^p$ is a non-compact domain, we denote by $X = C_b(\Omega; R^n)$ the Banach space of functions $f : \Omega \to R^n$ continuous, bounded on Ω with respect to the norm (2.1). We note that, if $\Omega \subset R^p$ is open, the functions in $C(\Omega; R^n)$ need not to be bounded on Ω. If $f \in C(\Omega; R^n)$ is bounded and uniformly continuous on Ω, then it possesses a unique, bounded, continuous extension to the closure $\overline{\Omega}$ of Ω. Hence, we define the vector space $C(\overline{\Omega}; R^n)$ consisting of all those functions $f \in C(\Omega; R^n)$ for which f is bounded and uniformly continuous on Ω. This is a Banach space with norm the given by (2.1).

Similarly, for any non-negative integer m, we put

$$C^m(\Omega; R^n) = \{f = (f_1, \dots, f_n) \in C(\Omega; R^n) : D^\alpha f_i \in C(\Omega; R^n), \quad |\alpha| \le m, i = 1, \dots, n\}$$

for $\Omega \subset R^p$ a domain in R^p, and

$$C^m(\overline{\Omega}; R^n) = \{f = (f_1, \dots, f_n) \in C(\overline{\Omega}; R^n) : D^\alpha f_i \in C(\overline{\Omega}; R^n), \quad |\alpha| \le m, i = 1, \dots, n\}$$

for $\Omega \subset R^p$ an open set in R^p. $C^m(\overline{\Omega}; R^n)$ is also a Banach space with respect to the norm

$$\|f\|_{C^m(\overline{\Omega}; R^n)} = \max_{|\alpha| \le m} \sup_{x \in \Omega} \sum_{i=1}^{n} |D^\alpha f_i(x)| . \tag{2.2}$$

We write system (1.1) in the form of an operational equation in $X = C(\Omega; R^n)$

$$f = \varepsilon A f + B f + g, \tag{2.3}$$

where

$f = (f_1, \dots, f_n), Af = ((Af)_1, \dots, (Af)_n), Bf = ((Bf)_1, \dots, (Bf)_n)$, with

$$(Af)_i(x) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \Phi(f_j(R_{ijk}(x))),$$

$$(Bf)_i(x) = \sum_{k=1}^{m} \sum_{j=1}^{n} b_{ijk} f_j(S_{ijk}(x)), \quad (i = 1, 2, \dots, n) \text{ for all } \forall x \in \Omega.$$

3. Theorem on existence and uniqueness of the solution

Put $\|[b_{ijk}]\| = \sum_{k=1}^{m} \sum_{i=1}^{n} \max_{1 \le j \le n} |b_{ijk}|$. First, we need the following Lemma.

LEMMA 1. *Let* $\|[b_{ijk}]\| < 1$ *and* $S_{ijk} : \Omega \to \Omega$ *be continuous. Then the linear operator* $I - B : X \to X$ *is invertible and*

$$\left\|(I-B)^{-1}\right\| \leq \frac{1}{1-\|[b_{ijk}]\|}.$$

Proof. We easily verify that

$$\|Bf\|_X \leq \|[b_{ijk}]\| \; \|f\|_X \quad \forall f \in X. \tag{3.1}$$

Hence, $\|B\| \leq \|[b_{ijk}]\| < 1$ and then the linear operator $I - B$ is invertible and

$$\left\|(I-B)^{-1}\right\| \leq \frac{1}{1-\|B\|} \leq \frac{1}{1-\|[b_{ijk}]\|}. \quad \blacksquare$$

By Lemma 1, we rewrite the functional equations system (2.1) as follows

$$f = (I-B)^{-1}(\varepsilon Af + g) \equiv Tf. \tag{3.2}$$

We make the following hypotheses:

(H_1) $R_{ijk}, S_{ijk} : \Omega \to \Omega$ are continuous;
(H_2) $g = (g_1, \ldots, g_n) \in X$;
(H_3) $\|[b_{ijk}]\| < 1$;
(H_4) $\Phi : R \to R$ satisfying the following condition: $\forall M > 0$,

$$\exists C_1(M) > 0 : |\Phi(y) - \Phi(z)| \leq C_1(M)\,|y-z| \quad \forall y, z \in [-M, M].$$

(H_5) $M > \frac{2\|g\|_X}{1-\|[b_{ijk}]\|}$ and

$$0 < \varepsilon_0 < \frac{M\,(1-\|[b_{ijk}]\|)}{(MC_1(M) + n\,|\Phi(0)|)\,\|[a_{ijk}]\|}.$$

Given $M > 0$, we put

$$K_M = \{f \in X : \|f\|_X \leq M\}.$$

Then, we have the following Lemma.

LEMMA 2. *Let* (H_1)*–*(H_4) *hold. Then, we have*

(i) $\|Af\|_X \leq \|[a_{ijk}]\|\,(C_1(M)\,\|f\|_X + n\,|\Phi(0)|) \quad \forall f \in K_M$.
(ii) $\left\|Af - A\tilde{f}\right\|_X \leq C_1(M)\,\|[a_{ijk}]\|\,\left\|f - \tilde{f}\right\|_X \quad \forall f, \tilde{f} \in K_M$.

The proof of this Lemma 2 is straightforward and we omit the details. $\blacksquare$

Then, we have the following Theorem.

THEOREM 1. *Let* (H_1)*–*(H_5) *hold. Then, for every* ε*, with* $|\varepsilon| \leq \varepsilon_0$*, the system* (3.2) *has a unique solution* $f \in K_M$.

Proof. It is evident that $Tf \in X$, for every $f \in X$. Considering $f, \widetilde{f} \in K_M$, we easily verify, by Lemma 1 and Lemma 2, that

$$\begin{aligned}(3.3)\quad \|Tf\|_X &= \left\|(I-B)^{-1}(\varepsilon Af + g)\right\|_X \\ &\le \left\|(I-B)^{-1}\right\| (\varepsilon \|Af\|_X + \|g\|_X) \\ &\le \frac{1}{1-\|[b_{ijk}]\|} [\varepsilon_0 \|[a_{ijk}]\| (MC_1(M) + n|\Phi(0)|) + \|g\|_X],\end{aligned}$$

$$\begin{aligned}(3.4)\qquad \left\|Tf - T\widetilde{f}\right\|_X &= \left\|(I-B)^{-1}\varepsilon(Af - A\widetilde{f})\right\|_X \\ &\le \varepsilon_0 \left\|(I-B)^{-1}\right\| \left\|Af - A\widetilde{f}\right\|_X \\ &\le \frac{\varepsilon_0 C_1(M)\|[a_{ijk}]\|}{1-\|[b_{ijk}]\|} \left\|f - \widetilde{f}\right\|_X.\end{aligned}$$

Notice that, from (H_5) we have

$$(3.5)\qquad \frac{1}{1-\|[b_{ijk}]\|} [\varepsilon_0 \|[a_{ijk}]\| (MC_1(M) + n|\Phi(0)|) + \|g\|_X] < M,$$

and

$$\sigma = \frac{\varepsilon_0 C_1(M)\|[a_{ijk}]\|}{1-\|[b_{ijk}]\|} < 1.$$

It follows from (3.3), (3.4), (3.5), that $T : K_M \to K_M$ is a contraction mapping. Then, using Banach fixed point theorem, we have the existence of a unique $f \in K_M$ such that $f = Tf$. ∎

REMARK 1. Theorem 1 gives a consecutive approximate algorithm

$$(3.6)\qquad g^{(\mu)} = Tg^{(\mu-1)}, \mu = 1, 2, \ldots, \text{ where } g^{(0)} \in K_M \text{ is given.}$$

Then the sequence $\{g^{(\mu)}\}$ converges in X to the solution f of (3.2) and we have the error estimation

$$(3.7)\qquad \left\|g^{(\mu)} - f\right\|_X \le \left\|Tg^{(0)} - g^{(0)}\right\|_X \frac{\sigma^\mu}{1-\sigma}, \quad \mu = 1, 2, \ldots \ ∎$$

4. The second order algorithm

In this part, we consider the algorithm for system (1.1)

$$\begin{aligned}(4.1)\qquad f_i^{(\nu)}(x) &= \varepsilon \sum_{k=1}^{m}\sum_{j=1}^{n} a_{ijk}\Phi(f_j^{(\nu-1)}(R_{ijk}(x))) \\ &+ \varepsilon \sum_{k=1}^{m}\sum_{j=1}^{n} a_{ijk}\Phi'(f_j^{(\nu-1)}(R_{ijk}(x)))\end{aligned}$$

$$\times\,[f_j^{(\nu)}(R_{ijk}(x)) - f_j^{(\nu-1)}(R_{ijk}(x))]$$
$$+\sum_{k=1}^{m}\sum_{j=1}^{n} b_{ijk} f_j^{(\nu)}(S_{ijk}(x)) + g_i(x),$$

for $x\in\Omega$, $i=1,2,\dots,n$, and $\nu=1,2,\dots$, where $f^{(0)}=(f_1^{(0)},\dots,f_n^{(0)})\in K_M$ is given. Rewrite (4.1) as linear system of functional equations

$$f_i^{(\nu)}(x) = \sum_{k=1}^{m}\sum_{j=1}^{n} \alpha_{ijk}^{(\nu)}(x) f_j^{(\nu)}(R_{ijk}(x)) + \sum_{k=1}^{m}\sum_{j=1}^{n} b_{ijk} f_j^{(\nu)}(S_{ijk}(x)) + g_i^{(\nu)}(x), \tag{4.2}$$

where

$$\alpha_{ijk}^{(\nu)}(x) = \varepsilon\, a_{ijk}\Phi'(f_j^{(\nu-1)}(R_{ijk}(x))), \tag{4.3}$$

and

$$g_i^{(\nu)}(x) = g_i(x) + \varepsilon\sum_{k=1}^{m}\sum_{j=1}^{n} a_{ijk}[\Phi(f_j^{(\nu-1)}(R_{ijk}(x))) - \Phi'(f_j^{(\nu-1)}(R_{ijk}(x)))f_j^{(\nu-1)}(R_{ijk}(x))]. \tag{4.4}$$

Then, we have the following

THEOREM 2. *Let (H_1)–(H_3) and $\Phi\in C^1(R;R)$ hold. If $f^{(\nu-1)}\in X$ satisfies*

$$\alpha_\nu = \sum_{k=1}^{m}\sum_{i=1}^{n} \max_{1\le j\le n}\sup_{x\in\Omega}\left|\alpha_{ijk}^{(\nu)}(x)\right| + \|[b_{ijk}]\| < 1, \tag{4.5}$$

there exists a unique function $f^{(\nu)}\in X$ being solution of system (4.2)–(4.4).

Proof. We write system (4.2)-(4.4) in the form of an operational equation in $X = C(\Omega;R^n)$

$$f^{(\nu)} = T_\nu f^{(\nu)}, \tag{4.6}$$

where

$$(T_\nu f)_i(x) = \sum_{k=1}^{m}\sum_{j=1}^{n} \alpha_{ijk}^{(\nu)}(x) f_j(R_{ijk}(x)) + \sum_{k=1}^{m}\sum_{j=1}^{n} b_{ijk} f_j(S_{ijk}(x)) + g_i^{(\nu)}(x), \tag{4.7}$$

for $x\in\Omega$, $i=1,2,\dots,n$, and $f=(f_1,\dots,f_n)\in X$.

We can easily check that $T_\nu : X \to X$ and

$$\|T_\nu f - T_\nu \widetilde{f}\|_X \le \alpha_\nu \|f - \widetilde{f}\|_X$$

for all $f, \widetilde{f} \in X$. Then using the Banach fixed point theorem, there exists of a unique function $f^{(\nu)} \in X$ being solution of system (4.2)–(4.4). ■

We make the following hypotheses:

(H_6) $\Phi \in C^2(R; R)$,
(H_7) $\|g\|_X + |\varepsilon| \, \|[a_{ijk}]\| \, (3MM_1 + n\,|\Phi(0)|) \le (1 - \|[b_{ijk}]\|)\, M$,
where $M_1 = \sup_{|y| \le M} |\Phi'(y)|$.

Then, we have the following

THEOREM 3. *Let (H_1)–(H_3), (H_6), (H_7) hold, let f be the solution of system (1.1) and the sequence $\{f^{(\nu)}\}$ be defined by algorithm (4.2)–(4.4).*

(i) *If $\|f^{(0)}\|_X \le M$, then*

$$\|f^{(\nu)} - f\|_X \le \beta_M \|f^{(\nu-1)} - f\|_X^2, \quad \nu = 1, 2, \ldots \tag{4.8}$$

where

$$\beta_M = \frac{\frac{\varepsilon}{2} M_2 \, \|[a_{ijk}]\|}{1 - \|[b_{ijk}]\| - \varepsilon M_1 \, \|[a_{ijk}]\|} > 0, \tag{4.9}$$

and

$$M_2 = \sup_{|y| \le M} |\Phi''(y)|.$$

(ii) *If we choose the first term $f^{(0)}$ sufficiently near f such that*

$$\beta_M \|f^{(0)} - f\|_X < 1,$$

then the sequence $\{f^{(\nu)}\}$ converges quadratically to f and satisfies the error estimation

$$\|f^{(\nu)} - f\|_X \le \frac{1}{\beta_M} \left(\beta_M \|f^{(0)} - f\|_X \right)^{2^\nu}, \quad \nu = 1, 2, \ldots \tag{4.10}$$

Proof. First, we will verify that if $\|f^{(0)}\|_X \le M$ then

$$\|f^{(\nu)}\|_X \le M, \quad \nu = 1, 2, \ldots \tag{4.11}$$

Indeed, supposing

$$\|f^{(\nu-1)}\|_X \le M, \tag{4.12}$$

we deduce from (4.2), (4.12) that

$$\|f^{(\nu)}\|_X \le \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \max_{1 \le j \le n} \sup_{x \in \Omega} |\alpha_{ijk}^{(\nu)}(x)| + \|[b_{ijk}]\| \right) \|f^{(\nu)}\|_X + \|g^{(\nu)}\|_X. \tag{4.13}$$

On the other hand, we have

$$(4.14)\qquad \begin{aligned}|\alpha_{ijk}^{(\nu)}(x)| &\le |\varepsilon|\ |a_{ijk}|\,|\Phi'(f_j^{(\nu-1)}(R_{ijk}(x)))| \\ &\le |\varepsilon|\ |a_{ijk}| \sup_{|y|\le M} |\Phi'(y)| = |\varepsilon|\ M_1\,|a_{ijk}|\,.\end{aligned}$$

Hence, we deduce from (4.13), (4.14) that

$$(4.15)\qquad \|f^{(\nu)}\|_X \le (|\varepsilon|\ M_1\,\|[a_{ijk}]\| + \|[b_{ijk}]\|)\,\|f^{(\nu)}\|_X + \|g^{(\nu)}\|_X.$$

Note that (H_7) implies $|\varepsilon|\ M_1\,\|[a_{ijk}]\| + \|[b_{ijk}]\| < 1$, hence we obtain

$$(4.16)\qquad \|f^{(\nu)}\|_X \le \frac{1}{1-\|[b_{ijk}]\| - |\varepsilon|\ M_1\,\|[a_{ijk}]\|}\left\|g^{(\nu)}\right\|_X.$$

On the other hand, from (4.4) we get

$$(4.17)\qquad \|g^{(\nu)}\|_X \le \|g\|_X + |\varepsilon|\ \|[a_{ijk}]\|\,(n\,|\Phi(0)| + 2MM_1)\,.$$

Hence, from (4.16), (4.17), and (H_7), we obtain

$$(4.18)\qquad \|f^{(\nu)}\|_X \le \frac{\|g\|_X + |\varepsilon|\ \|[a_{ijk}]\|\,(n\,|\Phi(0)| + 2MM_1)}{1-\|[b_{ijk}]\| - |\varepsilon|\ M_1\,\|[a_{ijk}]\|} \le M.$$

Now, we shall estimate $\left\|f - f^{(\nu)}\right\|_X$.

Put $e^{(\nu)} = f - f^{(\nu)}$, we obtain from (1.1) and (4.1) the system

$$(4.19)\qquad \begin{aligned} e_i^{(\nu)}(x) &= f_i(x) - f_i^{(\nu)}(x) \\ &= \varepsilon \sum_{k=1}^{m}\sum_{j=1}^{n} a_{ijk}\left[\Phi(f_j(R_{ijk}(x))) - \Phi'(f_j^{(\nu-1)}(R_{ijk}(x)))f_j^{(\nu-1)}(R_{ijk}(x))\right] \\ &\quad + (Be^{(\nu)})_i(x) + g_i(x) - g_i^{(\nu)}(x) \\ &= (Be^{(\nu)})_i(x) + \varepsilon \sum_{k=1}^{m}\sum_{j=1}^{n} a_{ijk}\left[\Phi(f_j(R_{ijk}(x))) - \Phi(f_j^{(\nu-1)}(R_{ijk}(x)))\right] \\ &\quad + \varepsilon \sum_{k=1}^{m}\sum_{j=1}^{n} a_{ijk}\Phi'(f_j^{(\nu-1)}(R_{ijk}(x)))\left[e_j^{(\nu)}(R_{ijk}(x)) - e_j^{(\nu-1)}(R_{ijk}(x))\right].\end{aligned}$$

Using Taylor's expansion of the function $\Phi(f_j)$ about the point $f_j^{(\nu-1)}$ up to order two, we obtain

$$(4.20)\qquad \begin{aligned}\Phi(f_j(y)) - \Phi(f_j^{(\nu-1)}(y)) &= \Phi'(f_j^{(\nu-1)}(y))\,e_j^{(\nu-1)}(y) \\ &\quad + \frac{1}{2}\Phi''(\lambda_j^{(\nu)}(y))\left|e_j^{(\nu-1)}(y)\right|^2,\end{aligned}$$

where

$$y = R_{ijk}(x),\ \lambda_j^{(\nu)}(y) = f_j^{(\nu-1)}(y) + \theta_j e_j^{(\nu-1)}(y),\ 0 < \theta_j < 1.$$

Substituting (4.20) into (4.19) where the arguments of $f_j, f_j^{(\nu-1)}, e_j^{(\nu-1)}, \lambda_j^{(\nu)}$ appearing in (4.20) are replaced by $y = R_{ijk}(x)$ we obtain

$$(4.21)\quad \begin{aligned} e_i^{(\nu)}(x) &= (Be^{(\nu)})_i(x) \\ &+ \varepsilon \sum_{k=1}^{m}\sum_{j=1}^{n} a_{ijk}\Phi'(f_j^{(\nu-1)}(R_{ijk}(x)))e_j^{(\nu)}(R_{ijk}(x)) \\ &+ \frac{\varepsilon}{2}\sum_{k=1}^{m}\sum_{j=1}^{n} a_{ijk}\left[\Phi''(\ \lambda_j^{(\nu)}(R_{ijk}(x)))\left|e_j^{(\nu-1)}(R_{ijk}(x))\right|^2\right]. \end{aligned}$$

It follows from (4.12), (4.14), (4.21), that

$$(4.22)\quad \begin{aligned} &\|e^{(\nu)}\|_X \\ &\leq \|[b_{ijk}]\|\,\|e^{(\nu)}\|_X + |\varepsilon|\ M_1\|e^{(\nu)}\|_X \\ &\quad + \frac{|\varepsilon|}{2}M_2\sum_{i=1}^{n}\sum_{k=1}^{m}\max_{1\leq j\leq n}|a_{ijk}|\sup_{x\in\Omega}\sum_{j=1}^{n}|e_j^{(\nu-1)}(x)|^2 \\ &\leq (\|[b_{ijk}]\| + |\varepsilon|\ M_1)\,\|e^{(\nu)}\|_X + \frac{|\varepsilon|}{2}M_2\,\|[a_{ijk}]\|\sup_{x\in\Omega}\left(\sum_{j=1}^{n}\left|e_j^{(\nu-1)}(x)\right|\right)^2 \\ &\leq (\|[b_{ijk}]\| + |\varepsilon|\ M_1)\,\|e^{(\nu)}\|_X + \frac{|\varepsilon|}{2}M_2\,\|[a_{ijk}]\|\,\|e^{(\nu-1)}\|_X^2. \end{aligned}$$

Hence, we obtain (4.8) by (4.9), and (4.22). Finally, we deduce easily (4.10) from (4.8). ■

REMARK 2. Choosing the first term $f^{(0)}$ such that $\beta_M\left\|f^{(0)} - f\right\|_X < 1$, Theorem 1 gives a consecutive approximate algorithm (3.6). Then the sequence $g^{(\mu)} \to f$ in X and we have the error estimation (3.7). Choose μ_0 sufficient large such that

$$\beta_M\|g^{(\mu_0)} - f\|_X \leq \beta_M\|Tg^{(0)} - g^{(0)}\|_X\frac{\sigma^{\mu_0}}{1-\sigma} < 1.$$

Then we take $f^{(0)} = g^{(\mu_0)}$. ■

5. Asymptotic expansion of solutions

In this part, we assume that the functions $R_{ijk}, S_{ijk}, g, \Phi$ and the real numbers a_{ijk}, b_{ijk}, M satisfy the assumptions (H_1)–(H_5), respectively.

We make the following hypothese:

$$(H_8)\qquad \Phi \in C^N(R;R).$$

We consider the perturbed system (3.2), where ε is a small parameter $|\varepsilon| \leq \varepsilon_0$. Put $L = I - B$.

Let us consider the sequence of solutions $\{f^{[r]}\}$, $r = 0, 1, 2, \dots, N$, $f^{[r]} \in K_M$ (with suitable constant $M > 0$) defined by the following systems:

$$f^{[0]} = g \equiv P^{[0]}, \tag{5.1}$$

$$Lf^{[1]} = P^{[1]} \equiv Af^{[0]}, \tag{5.2}$$

$$Lf^{[r]} = P^{[r]}, r = 2, 3, \dots, N, \tag{5.3}$$

where

$$P^{[r]} = (P_1^{[r]}, P_2^{[r]}, \dots, P_n^{[r]}), r = 0, 1, \dots, N,$$

$$P_i^{[1]}(x) = (Af^{[0]})_i(x) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \Phi(f_j^{[0]}(R_{ijk}(x))), \tag{5.4}$$

$$P_i^{[2]}(x) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \Phi'(f_j^{[0]}(R_{ijk}(x))) f_j^{[1]}(R_{ijk}(x)). \tag{5.5}$$

For $s = 3, 4, \dots, N$,

(5.6)

$$P_i^{[s]} = \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \sum_{r=1}^{s-1} \Phi^{(r)}(f_j^{[0]}(R_{ijk}(x))) \sum_{|\gamma|=r,\, \eta(\gamma)=s-1} \frac{1}{\gamma!} f^{\rightarrow\gamma}_{j}(R_{ijk}(x)),$$

where, we have used the following notations:

For a multi-index $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_N) \in Z_+^N$,

$$|\gamma| = \sum_{i=1}^{N} \gamma_i, \quad \gamma! = \gamma_1! \gamma_2! \dots \gamma_N!, \ \eta(\gamma) = \sum_{i=1}^{N} i\gamma_i,$$

$$f^{\rightarrow}{}_j = (f_j^{[1]}, f_j^{[2]}, \dots, f_j^{[N]}), \ f^{\rightarrow\gamma}_{j} = (f_j^{[1]})^{\gamma_1} (f_j^{[2]})^{\gamma_2} \dots (f_j^{[N]})^{\gamma_N}.$$

Put

$$h = f^{[0]} + \sum_{r=1}^{N} \varepsilon^r f^{[r]} \equiv f^{[0]} + U, \tag{5.7}$$

then

$$v = f - \sum_{r=0}^{N} \varepsilon^r f^{[r]} \equiv f - h, \tag{5.8}$$

satisfies the system

$$Lv = \varepsilon[A(v + h) - Ah] + E_\varepsilon, \tag{5.9}$$

where

$$E_\varepsilon = \varepsilon[A(f^{[0]} + U) - A(f^{[0]})] - \sum_{r=2}^{N} \varepsilon^r P^{[r]}. \tag{5.10}$$

Then, we have the following lemma.

LEMMA 3. *Let (H_1)-(H_5) hold. Then there exist a constant $C_N^{(1)}$ such that*

$$\|E_\varepsilon\|_X \le C_N^{(1)}\,|\varepsilon|^{N+1}, \tag{5.11}$$

where $C_N^{(1)}$ is a constant depending only on N, $\|[a_{ijk}]\|$, $\|f^{[r]}\|_X$, $r = 0,1,\dots,N$.

Proof. First, we need the following Lemmas.

LEMMA 4. *We have*

$$\left(\sum_{p=1}^{N} x_p\varepsilon^p\right)^r = \sum_{p=r}^{rN} \sum_{|\gamma|=r,\eta(\gamma)=p} \frac{r!}{\gamma!} x^\gamma \varepsilon^p, \tag{5.12}$$

$\forall x = (x_1,\dots,x_N) \in R^N$, $\forall \varepsilon \in R, \forall r, N \in IN$.

LEMMA 5. *We have*

$$\sum_{r=1}^{N-1}\sum_{p=r}^{rN} C_{rp}\varepsilon^p = \sum_{p=1}^{N-1}\sum_{r=1}^{p} C_{rp}\varepsilon^p + \sum_{r=1}^{N-1}\sum_{p=N}^{rN} C_{rp}\varepsilon^p, \tag{5.13}$$

where $\varepsilon, C_{rp} \in R, 1 \le r \le N-1, 1 \le p \le N(N-1), N = 2,3,\dots$.

The proof of the Lemmas 4, 5 is straightforward and we omit the details.

Proof of Lemma 3. In the case of $N = 1$, the proof of Lemma 3 is easy, hence we omit the details, which we only prove with $N \ge 2$. We have

$$(A(f^{[0]} + U) - A(f^{[0]}))_i(x) = \sum_{k=1}^{m}\sum_{j=1}^{n} a_{ijk}[\Phi(f_j^{[0]} + U_j) - \Phi(f_j^{[0]})], \tag{5.14}$$

where, we denote $f_j^{[0]} = f_j^{[0]}(R_{ijk}(x))$.

By using expansion of the function $\Phi(f_j^{[0]} + U_j) - \Phi(f_j^{[0]})$ round the point $f_j^{[0]}$ up to order N, we obtain(we omit the arguments $R_{ijk}(x)$)

$$\Phi(f_j^{[0]}+U_j)-\Phi(f_j^{[0]}) = \sum_{r=1}^{N-1} \frac{1}{r!}\Phi^{(r)}(f_j^{[0]})U_j^r + \frac{1}{N!}\Phi^{(N)}(f_j^{[0]}+\widetilde{\theta}_j U_j)U_j^N. \tag{5.15}$$

Using the Lemma 4, we obtain

$$U_i^r = \left(\sum_{p=1}^{N} \varepsilon^p f_j^{[p]}\right)^r = \sum_{p=r}^{rN} \sum_{|\gamma|=r,\eta(\gamma)=p} \frac{r!}{\gamma!} f^{\rightarrow\gamma}{}_j \varepsilon^p. \tag{5.16}$$

Hence, it follows from (5.15), (5.16) and Lemma 5, that

$$(5.17)\quad \Phi(f_j^{[0]}+U_j)-\Phi(f_j^{[0]})$$
$$= \sum_{r=1}^{N-1}\Phi^{(r)}(f_j^{[0]})\sum_{p=r}^{rN}\sum_{|\gamma|=r,\eta(\gamma)=p}\frac{1}{\gamma!}f^{\to\gamma}_j\varepsilon^p$$
$$+\Phi^{(N)}(f_j^{[0]}+\widetilde{\theta}_jU_j)\sum_{p=N}^{N^2}\sum_{|\gamma|=N,\eta(\gamma)=p}\frac{1}{\gamma!}f^{\to\gamma}_j\varepsilon^p$$
$$= \sum_{p=1}^{N-1}\sum_{r=1}^{p}\Phi^{(r)}(f_j^{[0]})\sum_{|\gamma|=r,\eta(\gamma)=p}\frac{1}{\gamma!}f^{\to\gamma}_j\varepsilon^p$$
$$+\sum_{r=1}^{N-1}\sum_{p=N}^{rN}\Phi^{(r)}(f_j^{[0]})\sum_{|\gamma|=r,\eta(\gamma)=p}\frac{1}{\gamma!}f^{\to\gamma}_j\varepsilon^p$$
$$+\Phi^{(N)}(f_j^{[0]}+\widetilde{\theta}_jU_j)\sum_{p=N}^{N^2}\sum_{|\gamma|=N,\eta(\gamma)=p}\frac{1}{\gamma!}f^{\to\gamma}_j\varepsilon^p.$$

Substituting $\Phi(f_j^{[0]}+U_j)-\Phi(f_j^{[0]})$ in (5.17) into (5.14), we obtain after some rearrangements in order of ε, that

$$(5.18)\quad (A(f^{[0]}+U)-A(f^{[0]}))_i(x)$$
$$= \sum_{p=1}^{N-1}\left(\sum_{k=1}^{m}\sum_{j=1}^{n}a_{ijk}\sum_{r=1}^{p}\Phi^{(r)}(f_j^{[0]})\sum_{|\gamma|=r,\eta(\gamma)=p}\frac{1}{\gamma!}f^{\to\gamma}_j\right)\varepsilon^p$$
$$+\sum_{k=1}^{m}\sum_{j=1}^{n}a_{ijk}\sum_{r=1}^{N-1}\sum_{p=N}^{rN}\Phi^{(r)}(f_j^{[0]})\sum_{|\gamma|=r,\eta(\gamma)=p}\frac{1}{\gamma!}f^{\to\gamma}_j\varepsilon^p$$
$$+\sum_{k=1}^{m}\sum_{j=1}^{n}a_{ijk}\Phi^{(N)}(f_j^{[0]}+\widetilde{\theta}_jU_j)\sum_{p=N}^{N^2}\sum_{|\gamma|=N,\eta(\gamma)=p}\frac{1}{\gamma!}f^{\to\gamma}_j\varepsilon^p$$
$$= \sum_{p=1}^{N-1}P_i^{[p+1]}(x)\varepsilon^p+\varepsilon^NR_N[\Phi,\varepsilon]_i,$$

where

$$(5.19)\quad \varepsilon^NR_N[\Phi,\varepsilon]_i=\sum_{k=1}^{m}\sum_{j=1}^{n}a_{ijk}\sum_{r=1}^{N-1}\sum_{p=N}^{rN}\Phi^{(r)}(f_j^{[0]})\sum_{|\gamma|=r,\eta(\gamma)=p}\frac{1}{\gamma!}f^{\to\gamma}_j\varepsilon^p$$
$$+\sum_{k=1}^{m}\sum_{j=1}^{n}a_{ijk}\Phi^{(N)}(f_j^{[0]}+\widetilde{\theta}_jU_j)\sum_{p=N}^{N^2}\sum_{|\gamma|=N,\eta(\gamma)=p}\frac{1}{\gamma!}f^{\to\gamma}_j\varepsilon^p.$$

We deduce from (5.4)–(5.6), (5.18), (5.19) that

$$(5.20)\quad E_{\varepsilon i}(x) = \varepsilon(A(f^{[0]} + U) - A(f^{[0]}))_i(x) - \sum_{r=2}^{N} \varepsilon^r P_i^{[r]} = \varepsilon^{N+1} R_N[\Phi, \varepsilon]_i.$$

By the boundedness of the functions $f^{[r]}, r = 0, 1, 2, \ldots, N, f^{[r]} \in K_M$, we obtain from (5.19), (5.20) that

$$(5.21)\qquad \|E_\varepsilon\|_X = |\varepsilon|^{N+1} \|R_N[\Phi, \varepsilon]\|_X \leq C_N^{(1)} |\varepsilon|^{N+1}.$$

Lemma 3 is proved completely. ∎

THEOREM 4. *Let (H_1)–(H_3), (H_5), (H_8) hold. Then there exists constant $\varepsilon_1 > 0$ such that, for every ε, with $|\varepsilon| \leq \varepsilon_1$, the system (3.2) has a unique solution $f_\varepsilon \in K_M$ satisfying the asymptotic estimation up to order $N+1$ as follows*

$$(5.22)\qquad \left\| f_\varepsilon - \sum_{r=0}^{N} \varepsilon^r f^{[r]} \right\|_X \leq 2 \left\|L^{-1}\right\| C_N^{(1)} |\varepsilon|^{N+1},$$

the functions $f^{[r]}, r = 0, 1, \ldots, N$ being the solutions of systems (5.1)–(5.6), respectively.

Proof. Put

$$v = f_\varepsilon - \sum_{r=0}^{N} \varepsilon^r f^{[r]} \equiv f_\varepsilon - h.$$

We have

$$(5.23)\qquad \begin{aligned} Lv &= \varepsilon[A(v+h) - Ah] + E_\varepsilon, \\ v &= L^{-1}\left[\varepsilon\left(A(v+h) - Ah\right) + E_\varepsilon\right]. \end{aligned}$$

Hence, it follows from Lemma 3 that

$$(5.24)\qquad \begin{aligned} \|v\|_X &\leq \left\|L^{-1}\right\| \left(|\varepsilon| \|A(v+h) - Ah\|_X + \|E_\varepsilon\|_X\right) \\ &\leq \left\|L^{-1}\right\| \left(|\varepsilon| \|A(v+h) - Ah\|_X + C_N^{(1)} |\varepsilon|^{N+1}\right). \end{aligned}$$

On the other hand

$$(5.25)\qquad \|v+h\|_X = \|f_\varepsilon\|_X \leq M, \quad \|h\|_X \leq \sum_{r=0}^{N} \left\|f^{[r]}\right\|_X \equiv \widetilde{M}.$$

It follows from (4.25) that

$$(5.26)\qquad \|A(v+h) - Ah\|_X \leq \sup_{|y| \leq M + \widetilde{M}} |\Phi'(y)| \, \|[a_{ijk}]\| \, \|v\|_X.$$

From (5.24), (5.26) we see that

(5.27) $$\|v\|_X \leq \|L^{-1}\| \, (\varepsilon_1 \sup_{|y| \leq M+\widetilde{M}} |\Phi'(y)| \, \|[a_{ijk}]\| \, \|v\|_X + C_N^{(1)} \, |\varepsilon|^{N+1}).$$

Choosing $0 < \varepsilon_1 < \varepsilon_0$, such that

(5.28) $$\varepsilon_1 \sup_{|y| \leq M+\widetilde{M}} |\Phi'(y)| \, \|[a_{ijk}]\| \, \|L^{-1}\| \leq 1/2.$$

Hence, we have from (5.27), (5.28) that

(5.29) $$\|v\|_X \leq 2 \, \|L^{-1}\| \, C_N^{(1)} \, |\varepsilon|^{N+1},$$

or

$$\left\| f_\varepsilon - \sum_{r=0}^{N} \varepsilon^r f^{[r]} \right\|_X \leq 2 \, \|L^{-1}\| \, C_N^{(1)} \, |\varepsilon|^{N+1}.$$

Theorem 4 is proved completely. ■

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DEPARTMENT OF MATHEMATICS-INFORMATICS
VIETNAM NATIONAL UNIVERSITY HOCHIMINH CITY
227 Nguyen Van Cu Str., Dist.5
HOCHIMINH CITY, VIETNAM
e-mail: longnt@hcmc.netnam.vn

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