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ON SOME GENERALIZATION OF COEFFICIENT CONDITIONS FOR COMPLEX HARMONIC MAPPINGS

Abstract. Let $h = u + iv$, where u, v are real harmonic functions in the unit disc Δ. Such functions are called complex mappings harmonic in Δ. The function h may be written in the form $h = f + \bar{g}$, where f, g are functions holomorphic in the unit disc, of course. Studies of complex harmonic functions were initiated in 1984 by J. Clunie and T. Sheil-Small ([2]) and were continued by many others mathematicians. We can find some papers on functions harmonic in Δ, satisfying certain coefficient conditions, e.g. [1], [4], [6], [7], [8]. We investigate some more general problem, i. e. a coefficient inequality with any fixed sequence of real positive numbers.

Let us consider functions h harmonic in $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ of the form

$$(1) \quad h = f + \overline{g}, \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \Delta, \ |b_1| < 1.$$

For a fixed sequence $\{\varphi_n\}_{n=2,3,\dots}$ of real positive numbers we denote by $H(\{\varphi_n\})$ the class of functions h of the form (1) and satisfying the condition

$$(2) \qquad |b_1| + \sum_{n=2}^{\infty} \varphi_n(|a_n| + |b_n|) \leq 1.$$

Let next $H^0(\{\varphi_n\})$ be the subclass of $H(\{\varphi_n\})$ of functions h such that $b_1 = 0$.

REMARK 1. a) In the case $\varphi_n = n$, $n = 2, 3, \dots$, we have the classes HS, HS^0 investigated by Y. Avci and E. Złotkiewicz [1], i. e. $H(\{n\}) = HS$, $H^0(\{n\}) = HS^0$. Each function of the class HS^0 is starlike ([1]).

b) If $\varphi_n = n^2$, $n = 2, 3, \dots$, we obtain the classes HC, HC^0 examined also by Y. Avci and E. Złotkiewicz [1], i. e. $H(\{n^2\}) = HC$, $H^0(\{n^2\}) = HC^0$. The functions of the class HC^0 are convex ([1]).

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c) The case $\varphi_n = n^p$, $n = 2, 3, \ldots$, $p > 0$, was considered by A. Ganczar [4].

d) We can also consider the sequence $\varphi_n = \alpha n + (1-\alpha)n^2$, $n = 2, 3, \ldots$, $\alpha \in \langle 0, 1\rangle$, i. e. $H(\{\alpha n + (1-\alpha)n^2\}) = HS(\alpha)$ (see [5]). This case is a kind of generalization of a) and b).

e) In [6] the authors investigated the case $\varphi_n = \frac{2n-1-\alpha}{1-\alpha}$, $n = 2, 3, \ldots$, for $\alpha \in \langle 0, 1)$.

f) If $\{\varphi_n\}_{n=2,3,\ldots}$ is a sequence such that $\varphi_n \geq n$, $n = 2, 3, \ldots$, then by (2) $H(\{\varphi_n\}) \subset HS$, $H^0(\{\varphi_n\}) \subset HS^0$, of course. It concerns the cases examined in [1](a) and b)), [4] (c) with $p > 1$), [5] (d)) and [6] (e)).

g) If $\{\varphi_n\}_{n=2,3,\ldots}$ is such a kind of sequence that $\varphi_n \geq n^2$, $n = 2, 3, \ldots$, then $H(\{\varphi_n\}) \subset HC \subset HS$ and $H^0(\{\varphi_n\}) \subset HC^0 \subset HS^0$, respectively.

Directly from the definition we get

THEOREM 1. *Let $\{\varphi_n\}_{n=2,3,\ldots}$ be a sequence of real positive numbers. If $h \in H(\{\varphi_n\})$, then functions*

$$z \mapsto r^{-1}h(rz), \qquad z \mapsto e^{-it}h(e^{it}z), \qquad z \in \Delta,\ r \in (0,1),\ t \in \mathbf{R},$$

also belong to $H(\{\varphi_n\})$.

Theorem 1 holds also for the class $H^0(\{\varphi_n\})$.

REMARK 2. Let $\{\varphi_n\}_{n=2,3,\ldots}$ be a sequence of real positive numbers. One can easily check that if $h \in H(\{\varphi_n\}))$, then the function h_0 of the form

$$h_0(z) = \frac{h(z) - \overline{b_1 h(z)}}{1 - |b_1|^2}\,, \qquad z \in \Delta \quad (|b_1| < 1),$$

belongs to the class $H^0(\{\varphi_n\})$. However, note that if $h_0 \in H^0(\{\varphi_n\})$ and $|b_1| < 1$, then the function h of the form

$$h(z) = h_0(z) + \overline{b_1 h_0(z)}, \qquad z \in \Delta,$$

does not have to belong to $H(\{\varphi_n\})$ (see e. g. [1], [5]).

We mentioned some inclusions for the considered classes. In view of the definition condition (2) we obtain

THEOREM 2. *Let $\{\varphi_n\}_{n=2,3,\ldots}$, $\{\psi_n\}_{n=2,3,\ldots}$ be sequences of real positive numbers. If*

$$\varphi_n \geq \psi_n\,, \qquad n = 2, 3, \ldots,$$

then the inclusions $H(\{\varphi_n\}) \subset H(\{\psi_n\})$, $H^0(\{\varphi_n\}) \subset H^0(\{\psi_n\})$ hold.

The above property is the generalization of Remark 1, of course.

EXAMPLE 1. For a fixed sequence $\{\varphi_n\}_{n=2,3,\dots}$ of real positive numbers the functions h_1, h_2 of the forms

$$h_1(z) = z + \frac{z^2}{\varphi_2}, \qquad z \in \Delta, \tag{3}$$

$$h_2(z) = z + \frac{\overline{z^2}}{\varphi_2}, \qquad z \in \Delta, \tag{4}$$

belong to the class $H^0(\{\varphi_n\})$.

What is more, because of Theorem 1, superpositions of the functions h_1, h_2 with rotations also belong to $H^0(\{\varphi_n\})$.

Let $\{\varphi_n\}_{n=2,3,\dots}$ be a sequence of real positive numbers such that

$$\frac{\varphi_n}{n} \geq \left(\frac{\varphi_2}{2}\right)^{n-1}, \qquad n = 2, 3, \dots. \tag{5}$$

REMARK 3. If $\varphi_2 \geq 2$, then the condition (5) implies the inequality $\varphi_n \geq n$, $n = 2, 3, \dots$. In view of Remark 1 f) we have inclusions $H(\{\varphi_n\}) \subset HS$ and $H^0(\{\varphi_n\}) \subset HS^0$. It means, among others, that the radius of starlikeness of the class $H^0(\{\varphi_n\})$ in this case is equal to 1.

Obviously, the inequality $\varphi_n \geq n$, $n = 2, 3, \dots$, does not imply the condition (5).

EXAMPLE 2. Fix $\alpha \in \langle 0, 1\rangle$ and set $\varphi_n = \alpha n + (1-\alpha)n^2$, $n = 2, 3, \dots$ (see [5]). Obviously, we have $\varphi_n \geq n$, $n = 2, 3, \dots$. If $\alpha = 1$, then (5) holds, of course.

If $\alpha \in \langle 0, 1)$, then

$$\frac{\varphi_2}{2} = 2 - \alpha \in (1, 2\rangle, \qquad \frac{\varphi_n}{n} = \alpha + (1-\alpha)n, \quad n = 3, 4, \dots.$$

The real function k_1 of the form $k_1(x) = (2-\alpha)^{x-1}$ is increasing and convex in $\mathbf{R}$. The function k_2 of the form $k_2(x) = \alpha + (1-\alpha)x$, $x \in \mathbf{R}$, is an increasing linear function. Moreover, we have $k_1(1) = k_2(1) = 1$ and $k_1(2) = k_2(2) = 2 - \alpha$. From these facts we conclude that

$$k_1(x) > k_2(x), \qquad x > 2.$$

It means that for $\alpha \in \langle 0, 1)$ the considered sequence does not satisfy the inequality (5).

REMARK 4. Fix a sequence $\{\varphi_n\}_{n=2,3,\dots}$, $\varphi_n > 0$, $n = 2, 3, \dots$, and consider the condition

$$\frac{\varphi_n}{n} \geq \frac{\varphi_2}{2}, \qquad n = 2, 3, \dots. \tag{6}$$

If $\varphi_2 = 2$, then the conditions (5) and (6) are equivalent, of course.

Let $\varphi_2 < 2$. If the sequence satisfies (6), then it satisfies the condition (5).

Indeed, in this case we have

$$\frac{\varphi_2}{2} \geq \left(\frac{\varphi_2}{2}\right)^{n-1}, \qquad n = 2, 3, \ldots.$$

The converse implication does not hold.

Let $\varphi_2 > 2$. If the sequence satisfies (5), then it satisfies (6), but it does not have to satisfy the inequality (5).

EXAMPLE 3. Let $\varphi_n = n - \frac{1}{n}$, $n = 2, 3, \ldots$. We have

$$\varphi_2 = \frac{3}{2}, \quad \frac{\varphi_2}{2} = \frac{3}{4}, \quad \frac{\varphi_n}{n} = 1 - \frac{1}{n^2}, \qquad n = 2, 3, \ldots.$$

Note that $\varphi_n < n$, $n = 2, 3, \ldots$.

The function $x \mapsto 1 - \frac{1}{x^2}$ is increasing on the interval $(0, +\infty)$. Hence

$$1 - \frac{1}{x^2} \geq 1 - \frac{1}{2^2} = \frac{3}{4}, \qquad x \geq 2.$$

It means that the considered sequence satisfies the condition (6). In view of Remark 4, the condition (5) holds in this case.

EXAMPLE 4. Let $\varphi_n = n^p$, $n = 2, 3, \ldots$, $p > 0$ (see [4]).

For $p \in (0, 1)$ we can observe that for $\varphi_2 < 2$ the inequalities (5) and (6) are not equivalent (Remark 4). In this situation we have

$$\frac{\varphi_n}{n} = n^{p-1}, \quad \left(\frac{\varphi_2}{2}\right)^{n-1} = 2^{(n-1)(p-1)}, \quad n = 2, 3, \ldots, \quad \varphi_2 < 2.$$

Moreover, we know that $2^{n-1} \geq n$, $n = 2, 3, \ldots$. Hence the sequence satisfies (5). However, the condition (6) does not hold in this case.

If $p > 1$, then $\varphi_n \geq n$, $n = 2, 3, \ldots$, but the sequence does not satisfy (5).

In view of Remark 3 and the above examples, it seems to be interesting to consider a sequence $\{\varphi_n\}_{n=2,3,\ldots}$ of real positive numbers with $\varphi_2 < 2$, satisfying the condition (5). We require neither $\varphi_n < n$, $n = 2, 3, \ldots$ (Example 3), nor $\varphi_n \geq n$, $n = 3, 4, \ldots$ (e. g. $\varphi_2 = 1$, $\varphi_n = n$, $n = 3, 4, \ldots$).

Denote $\Delta_r = \{z \in \mathbf{C} : |z| < r\}$, $r > 0$, with $\Delta_1 = \Delta$.

THEOREM 3. *Assume that $h \in H(\{\varphi_n\})$, where $\{\varphi_n\}_{n=2,3,\ldots}$ is a sequence of real positive numbers satisfying condition* (5) *and such that $\varphi_2 < 2$. Then for any $r \in (0, \frac{\varphi_2}{2})$ the function h is univalent and sens-preserving in the disc Δ_r.*

Proof. Let the assumptions of Theorem 3 hold. From Theorem 1 with $h \in H(\{\varphi_n\})$ the function h_r of the form $h_r = r^{-1}h(rz)$, $z \in \Delta$, $r \in (0, 1)$, belongs to $H(\{\varphi_n\})$. Moreover, for $r \in (0, \frac{\varphi_2}{2})$ by (5) we have

$$\sum_{n=2}^{\infty} n\left(|a_n r^{n-1}| + |b_n r^{n-1}|\right) = \sum_{n=2}^{\infty} n r^{n-1}(|a_n| + |b_n|) \leq$$
$$\leq \sum_{n=2}^{\infty} n\left(\frac{\varphi_2}{2}\right)^{n-1}(|a_n| + |b_n|) \leq \sum_{n=2}^{\infty} \varphi_n(|a_n| + |b_n|) \leq 1 - |b_1|.$$

From this it follows that $h \in HS$. According to the respective theorem from the paper [1], h_r is univalent and sense-preserving in Δ. As $h(rz) = rh_r(z)$, $z \in \Delta$, so h is univalent and sense-preseving in the disc Δ_r, $r \in (0, \frac{\varphi_2}{2}\rangle$.

THEOREM 4. *If $\{\varphi_n\}_{n=2,3,\dots}$ is a sequence of real positive numbers such that $\varphi_2 < 2$ and the condition (5) holds, then for $r \in (0, \frac{\varphi_2}{2}\rangle$ any function $h \in H^0(\{\varphi_n\})$ maps the disc Δ_r onto a domain starlike with respect to the origin. The constant $\frac{\varphi_2}{2}$ is the best possible, so it is the radius of starlikeness for the class $H^0(\{\varphi_n\})$.*

Proof. Under the above assumptions, from Theorem 3 and its proof we observe that $h_r \in HS^0$, $r \in (0, \frac{\varphi_2}{2}\rangle$. Hence ([1]) $h_r(\Delta)$ is a starlike domain. Since

$$h(\Delta_r) = \{w \in \mathbf{C} : w = r\zeta \ \wedge \ \zeta \in h_r(\Delta)\},$$

we conclude that $h(\Delta_r)$ is starlike for $r \in (0, \frac{\varphi_2}{2}\rangle$.

Note that $\frac{\varphi_2}{2}$ is the radius of starlikeness of the class $H^0(\{\varphi_n\})$. To prove this fact we examine the function h_1 of the form (3) with $\varphi_2 \in (0, 2)$. Of course, $h_1 \in H^0(\{\varphi_n\})$ and it is holomorphic in Δ, $h_1(0) = 0$, $h_1'(0) = 1$. In view of the well known theorem ([3], p. 41) it is sufficient to examine the sign of the expression $\operatorname{Re}\frac{zh_1'(z)}{h_1(z)}$ for $z \in \Delta$.

Let $z = \rho e^{it}$, $\rho \in (0,1)$, $t \in \mathbf{R}$. We have

$$\begin{aligned}
\operatorname{Re}\frac{zh_1'(z)}{h_1(z)} &= \operatorname{Re}\frac{\varphi_2 z + 2z^2}{\varphi_2 z + z^2} = \\
&= \frac{\operatorname{Re}\left[(\varphi_2\rho e^{it} + 2\rho^2 e^{2it})(\varphi_2\rho e^{-it} + \rho^2 e^{-2it})\right]}{|\varphi_2\rho e^{it} + 2\rho^2 e^{2it}|^2} = \\
&= \frac{\rho^2\left(\varphi_2^2 + 3\varphi_2\rho\cos t + 2\rho^2\right)}{|\varphi_2\rho e^{it} + 2\rho^2 e^{2it}|^2} = \\
&= \frac{3\varphi_2\rho^3}{|\varphi_2\rho e^{it} + 2\rho^2 e^{2it}|^2}\left(\cos t + d(\rho)\right),
\end{aligned}$$

where

$$d(\rho) = \frac{2\rho^2 + \varphi_2^2}{3\varphi_2\rho} = \frac{2}{3\varphi_2}\rho + \frac{\varphi_2}{3\rho}, \qquad \rho \in (0,1).$$

Observe that $d(\rho) > 0$ for $\rho \in (0,1)$ and

$$\lim_{\rho\to 0^+} d(\rho) = +\infty, \qquad \lim_{\rho\to 1^-} d(\rho) = \frac{2+\varphi_2^2}{3\varphi_2} > 0.$$

We also have

$$d'(\rho) = \frac{2}{3\varphi_2} - \frac{\varphi_2}{3\rho^2} = \frac{2\rho^2 - \varphi_2^2}{3\varphi_2\rho^2}, \qquad \rho \in (0,1).$$

Examining d' and applying elementary methods we can check that

$$d(\rho) > 1 \iff \begin{cases} \rho \in (0, \frac{\varphi_2}{2}) \cup (\varphi_2, 1) & \text{for} \quad \varphi_2 \in (0,1), \\ \rho \in (0, \frac{\varphi_2}{2}) & \text{for} \quad \varphi_2 \in \langle 1, 2). \end{cases}$$

From this we can see that for $\rho \in (0, \frac{\varphi_2}{2})$ we have $\cos t + d(\rho) > 0$, $t \in \mathbf{R}$. Therefore, if $z \in \Delta_r$, $r \in (0, \frac{\varphi_2}{2}\rangle$, then $\operatorname{Re}\frac{zh_1'(z)}{h_1(z)} > 0$.

If $r \in (\frac{\varphi_2}{2}, 1)$, then in the disc Δ_r exists a point $z_0 = \rho_0 e^{it_0}$ such that $d(\rho_0) < 1$ and $\cos t_0 + d(\rho_0) < 0$, what implies $\operatorname{Re}\frac{z_0 h_1'(z_0)}{h_1(z_0)} < 0$. Consequently, for $r \in (\frac{\varphi_2}{2}, 1)$ the domain $h_1(\Delta_r)$ is not starlike.

Remark 5. a) The function h_1 shows that the radius of starlikeness cannot be improved in the respective class of holomorphic functions.

b) In the proof of Theorem 4 we can also consider the function h_2 of the form (4) and examine the expression $\frac{\partial}{\partial t} arg\,(h(\rho e^{it}))$, $\rho \in (0,1)$, $t \in \mathbf{R}$.

From Theorem 4 and Remark 3 we obtain

Corollary 1. *If $\{\varphi_n\}_{n=2,3,\dots}$ is a sequence of real positive numbers satisfying the condition (5), then the radius r_* of starlikeness of the class $H^0(\{\varphi_n\})$ is equal to $r_* = \min(\frac{\varphi_2}{2}, 1)$.*

According to this fact and Example 4 we have

Corollary 2. *The radius r_* of starlikeness of $H^0(\{n^p\})$, $p \in (0,1\rangle$, equals $r_* = 2^{p-1}$ (see [4]).*

We have noted that if $\{\varphi_n\}_{n=2,3,\dots}$ is a sequence of real positive numbers satisfying the condition (5) and $\varphi_2 \geq 2$, then all functions of the class $H^0(\{\varphi_n\})$ are starlike (Corollary 1). We now remind that a function h of the form (1) is called a harmonic function starlike of order $\beta \in \langle 0,1)$ in Δ if $\frac{\partial}{\partial t} arg\,(h(re^{it})) \geq \beta$ for $r \in (0,1)$, $t \in \langle 0, 2\pi)$ (see [7]). Of course, a function starlike of order $\beta \in \langle 0,1)$ is starlike. H. Silverman ([7]) proved that if a function h of the form (1) with $b_1 = 0$ satisfies the condition

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta}(|a_n| + |b_n|) \leq 1 \tag{7}$$

for $\beta \in \langle 0, 1)$, then the function h is starlike of order β. This result we apply to showing the next theorem.

THEOREM 5. *If $\{\varphi_n\}_{n=2,3,\dots}$ is a sequence of real positive numbers satisfying the condition* (5) *and $\varphi_2 \geq 2$, then the functions of the class $H^0(\{\varphi_n\})$ are starlike of order $\beta = \frac{\varphi_2 - 2}{\varphi_2 - 1}$.*

Proof. Let the assumptions of Theorem 5 hold. Obviously, we can see that $\beta = \frac{\varphi_2-2}{\varphi_2-1}$ belongs to the interval $\langle 0, 1)$. According to the mentioned result of H. Silverman, by (2) and (7), it suffices to show that

$$\frac{n-\beta}{1-\beta} \leq \varphi_n \,, \qquad n = 2, 3, \dots .$$

Because of the form of β we have

$$\frac{n-\beta}{1-\beta} = n\left(\frac{2-\varphi_2}{n} + \varphi_2 - 1\right) , \qquad n = 2, 3, \dots .$$

Consider two real functions l_1, l_2 of the forms

$$l_1(x) = \left(\frac{\varphi_2}{2}\right)^{x-1} , \qquad l_2(x) = \frac{2-\varphi_2}{x} + \varphi_2 - 1 \,, \qquad x \in (0, +\infty).$$

We required $\varphi_2 \geq 2$, so the function l_1 is nondecreasing and convex in the interval $(0, +\infty)$ and l_2 is a nondecreasing concave function in the interval $(0, +\infty)$. It implies that the graphs of the functions l_1, l_2 have at most 2 common points. We can see that $l_1(1) = l_2(1) = 1$ and $l_1(2) = l_2(2) = \frac{\varphi_2}{2}$. From these properties of l_1 and l_2 we conclude that

$$l_1(x) \geq l_2(x) \,, \qquad x \geq 2.$$

Therefore by (5) we get

$$\frac{n-\beta}{1-\beta} = n\left(\frac{2-\varphi_2}{n} + \varphi_2 - 1\right) \leq n\left(\frac{\varphi_2}{2}\right)^{n-1} \leq \varphi_n \,, \quad n = 2, 3, \dots .$$

It ends the proof.

In the next part of the paper we make some notes on convexity of functions of the considered classes.

REMARK 6. Let $\{\varphi_n\}_{n=2,3,\dots}$ be a sequence of real positive numbers satisfying the condition (5) and such that $\varphi_2 \geq 4$. Then $\varphi_n \geq n^2$, $n = 2, 3, \dots$.

Indeed, because of the inequality

$$2^{n-1} \geq n \,, \qquad n \in \mathrm{N},$$

for $\varphi_2 \geq 4$ we have

$$\left(\frac{\varphi_2}{2}\right)^{n-1} \geq 2^{n-1} \geq n \,, \qquad n = 2, 3, \dots .$$

From this and by (5) we get the required inequality.

In view of Remark 1g) there hold respective inclusions and the radius of convexity of the class $H^0(\{\varphi_n\})$ in this case equals 1.

THEOREM 6. *If $\{\varphi_n\}_{n=2,3,\dots}$ is a sequence of real positive numbers satisfying the condition* (5) *and $\varphi_2 < 4$, then for $r \in (0, \frac{\varphi_2}{4}\rangle$ each function of the class $H^0(\{\varphi_n\})$ maps the disc Δ_r onto a convex domain. The constant $\frac{\varphi_2}{4}$ is the best possible, so it is the radius of convexity of the class $H^0(\{\varphi_n\})$.*

Proof. Let $\{\varphi_n\}_{n=2,3,\dots}$ be a fixed sequence satisfying the assumtions of Theorem 6. For a function $h \in H^0(\{\varphi_n\})$ the function h_r of the form $h_r = r^{-1}h(rz)$, $z \in \Delta$, $r \in (0,1)$, belongs to $H^0(\{\varphi_n\})$ (Theorem 1). Moreover, by (5) for $r \in (0, \frac{\varphi_2}{4}\rangle$ we have

$$\sum_{n=2}^{\infty} n^2 \left(|a_n r^{n-1}| + |b_n r^{n-1}|\right) = \sum_{n=2}^{\infty} n^2 r^{n-1}(|a_n| + |b_n|) \le$$
$$\le \sum_{n=2}^{\infty} n^2 \left(\frac{\varphi_2}{4}\right)^{n-1} (|a_n| + |b_n|) = \sum_{n=2}^{\infty} \frac{n^2}{2^{n-1}} \left(\frac{\varphi_2}{2}\right)^{n-1} (|a_n| + |b_n|) \le$$
$$\le \sum_{n=2}^{\infty} \frac{n^2}{2^{n-1}} \frac{\varphi_n}{n}(|a_n| + |b_n|) \le \sum_{n=2}^{\infty} \varphi_n(|a_n| + |b_n|) \le 1.$$

According to the respective theorem from the paper [1], $h_r(\Delta)$ is a convex domain. Consequently, $h(\Delta_r)$ is a convex domain.

In order to check that $\frac{\varphi_2}{4}$ is the radius of convexity of the class $H^0(\{\varphi_n\})$ for the fixed sequence we consider again the function h_1 of the form (3) with $\varphi_2 \in (0,4)$. Because of the known results for holomorphic functions it is sufficient to examine the sign of the expression $1 + \operatorname{Re}\frac{zh_1''(z)}{h_1'(z)}$ for $z \in \Delta$ ([3], p. 42).

For $z = \rho e^{it}$, $\rho \in (0,1)$, $t \in \mathbf{R}$, after some computations, we obtain

$$1 + \operatorname{Re}\frac{zh_1''(z)}{h_1'(z)} = 1 + \operatorname{Re}\frac{2\rho e^{it}}{\varphi_2 + 2\rho e^{it}} = 1 + \frac{4\rho^2 + 2\rho\varphi_2 \cos t}{|\varphi_2 + 2\rho e^{it}|^2} =$$
$$= \frac{6\rho\varphi_2}{|\varphi_2 + 2\rho e^{it}|^2}\left(\cos t + p(\rho)\right),$$

where $p(\rho) = \frac{8\rho^2 + \varphi_2^2}{6\rho\varphi_2}$, $\rho \in (0,1)$. Note that $p(\rho) > 0$, $\rho \in (0,1)$, and

$$\lim_{\rho \to 0^+} p(\rho) = +\infty, \qquad \lim_{\rho \to 1^-} p(\rho) = \frac{8 + \varphi_2^2}{6\varphi_2} > 0.$$

Moreover, we have

$$p'(\rho) = \frac{8\rho^2 - \varphi_2^2}{6\rho^2\varphi_2}, \qquad \rho \in (0,1).$$

Examining the properties of the function p we get

$$p(\rho) > 1 \iff \begin{cases} \rho \in (0, \frac{\varphi_2}{4}) \cup (\frac{\varphi_2}{2}, 1) & \text{for} \quad \varphi_2 \in (0, 2), \\ \rho \in (0, \frac{\varphi_2}{4}) & \text{for} \quad \varphi_2 \in \langle 2, 4). \end{cases}$$

Analogously as in the proof of Theorem 4, we conclude that $h_1(\Delta_r)$ is convex for $r \in (0, \frac{\varphi_2}{4}\rangle$ and not convex for $r \in (\frac{\varphi_2}{4}, 1)$.

From Remark 6 and Theorem 6 we obtain

COROLLARY 3. *If $\{\varphi_n\}_{n=2,3,\dots}$ is a sequence of real positive numbers satisfying the condition* (5), *then the radius r_c of convexity of the class $H^0(\{\varphi_n\})$ is equal to* $r_c = \min(\frac{\varphi_2}{4}, 1)$.

REMARK 7. a) As in Remark 4, we note that the radius of convexity cannot be improved in the respective class of holomorphic functions.

b) In the proof of Theorem 6 we can also consider the function h_2 of the form (4) and examine the expression $\frac{\partial}{\partial t}\left[arg\left(\frac{\partial}{\partial t}h(\rho e^{it})\right)\right]$ for $\rho \in (0, 1)$, $t \in \mathbf{R}$.

Theorems 5 and 6 complement and generalize the corresponding results from papers [1], [4], [5]. It seems to be interesting that the mentioned results coincide for the sequence $\{\alpha n + (1-\alpha)n^2\}_{n=2,3,\dots}$, $\alpha \in \langle 0, 1\rangle$ (see [5]), although for $\alpha \in \langle 0, 1)$ it does not satisfy the condition (5) (Example 2). An analogous comment concerns also the case $\varphi_n = n^p$, $n = 2, 3, \dots$, $p > 1$, from Example 4 (see [4]).

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