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ON THE LIMITS OF FUNCTIONS OF TWO VARIABLES

Abstract. We show some conditions concerning sections of real functions of two variables which imply the existence of the double limit of considered function. Moreover we observe that the set of points, where the real function has the finite limit is an G_δ-set and that a function having finite limit except a countable set is continuous except a countable set.

Let $\mathcal{R}$ be the set of all reals and let $\mathcal{R}^2 = \mathcal{R} \times \mathcal{R}$.

It is well known that a function $f : \mathcal{R}^2 \to \mathcal{R}$ having continuous sections $f_x(t) = f(x,t)$ and $f^y(t) = f(t,y)$, $x, y, t \in \mathcal{R}$, can be discontinuous on the set of full plane Lebesgue measure μ_2 ([1]). Evidently, if a function $f : \mathcal{R}^2 \to \mathcal{R}$ with continuous sections f_x and f^y has the finite limit at a point (u,v), then it is contiunous at this point (u,v). So there are functions $f : \mathcal{R}^2 \to \mathcal{R}$ with continuous sections f_x and f^y which has not the finite limits on a set of full plane measure.

On the other hand if sections f_x, $x \in \mathcal{R}$, are equicontinuous at a point u (i.e. for each $\varepsilon > 0$ there is $\delta > 0$ such that for each point y with $|y-u| < \delta$ and for each point x the inequality $|f_x(y) - f_x(u)| < \varepsilon$ holds) and if the section f^u is continuous at a point v then f is continuous at the point (v,u) as the function of two variables ([2]).

In this article we investigate some conditions concerning sections f_x and f^y which imply the existence of the limit of f.

THEOREM 1. *Let a function $f : \mathcal{R}^2 \to \mathcal{R}$ be such that for a point $(u,v) \in \mathcal{R}^2$ there is a positive real r such that for sections f_x, $u \neq x \in (u-r, u+r)$, there are limits*

$$\lim_{y \to v} f(x,y) = b(x,v) \in \mathcal{R}$$

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and for each positive real ε *there is a positive real* δ *such that*

(1) $$|f(x,y) - b(x,v)| < \varepsilon$$
$$\text{for } v \neq y \in (v-\delta, v+\delta) \text{ and } u \neq x \in (u-r, u+r).$$

If there is the limit

$$\lim_{x \to u} b(x,v) = a(u,v) \in \mathcal{R},$$

then there is the limit

$$\lim_{(x,y)\to(u,v),\ x\neq u,\ y\neq v} f(x,y) = a(u,v).$$

Proof. Fix a positive real ε. By (1) there is a positive real δ such that

$$\forall_{x\in(u-r,u+r)\setminus\{u\}} \ \forall_{y\in(v-\delta,v+\delta)\setminus\{v\}} \ |f(x,y) - b(x,v)| < \frac{\varepsilon}{2}.$$

Since there is the limit

$$\lim_{x \to u} b(x,v) = a(u,v),$$

we can find a real $s > 0$ such that $s \leq r$ and

$$|b(x,v) - a(u,v)| < \frac{\varepsilon}{2} \text{ for } u \neq x \in (u-s, u+s).$$

Consequently, for

$$(x,y) \in \Big((u-s,u+s)\setminus\{u\}\Big) \times \Big((v-\delta,v+\delta)\setminus\{v\}\Big)$$

we obtain

$$|f(x,y) - a(u,v)| \leq |f(x,y) - b(x,v)| + |b(x,v) - a(u,v)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This finishes the proof.

As an immediate corollary from the above theorem we obtain:

COROLLARY 1. *Let a function* $f : \mathcal{R}^2 \to \mathcal{R}$ *be such that for a point* $(u,v) \in \mathcal{R}^2$ *there is the limit* $a(u,v) \in \mathcal{R}$ *of the section* f^v *at the point* u *and there is a positive real* r *such that sections* f_x*,* $u \neq x \in (u-r, u+r)$*, are equicontinuous at the point* v*. Then there is the limit*

$$\lim_{(x,y)\to(u,v),\ x\neq u} f(x,y) = a(u,v).$$

Observe that hypotheses of last theorem and the assumption that there is the limit $\lim_{y\to v} f(u,y) \in \mathcal{R}$ do not imply that there is the limit

$$\lim_{(x,y)\to(u,v)} f(x,y) = a(u,v).$$

For example sections f_x of the function

$$f(x,y) = 0 \text{ for } x \neq 0 \text{ and } f(0,y) = 1 \text{ for } y \in \mathcal{R}$$

are equicontinuous and

$$\lim_{x \to 0} f(x, y) = 0 \text{ for all } y \in \mathcal{R},$$

but there is no limit

$$\lim_{(x,y) \to (0,0)} f(x, y).$$

THEOREM 2. *Let a function $f : \mathcal{R}^2 \to \mathcal{R}$ be such that for a point $(u, v) \in \mathcal{R}^2$ there is a real $r > 0$ such that for sections f_x, $x \in (u - r, u + r)$, there are limits*

$$\lim_{y \to v} f(x, y) = b(x, v) \in \mathcal{R}$$

and for each positive real ε there is a positive real δ such that

$|f(x, y) - b(x, v)| < \varepsilon$ for $v \neq y \in (v - \delta, v + \delta)$ and $u \neq x \in (u - r, u + r)$.

Let there exists limits

$$\lim_{x \to u} b(x, v) = \lim_{x \to u} f(x, v) = a(u, v) \in \mathcal{R}.$$

If there is a sequence of points $y_n \neq v$ such that $\lim_{n \to \infty} y_n = v$ and for each positive integer n there is a sequence of points $u_{n,k} \neq u$ with

$$\lim_{k \to \infty} u_{n,k} = u, \quad \lim_{k \to \infty} f(u_{n,k}, y_n) = f(u, y_n),$$

then there is the limit

$$\lim_{(x,y) \to (u,v)} f(x, y) = a(u, v).$$

Proof. If a sequence of points (w_n, z_n), $w_n \neq u$, $z_n \neq v$ for $n \geq 1$, converges to (u, v) then, by Theorem 1, there exists the limit

$$\lim_{n \to \infty} f(w_n, z_n) = a(u, v). \tag{2}$$

By the hypothesis there is the limit $b(u, v) \in \mathcal{R}$ of the section f_u at a point v. We will prove that $b(u, v) = a(u, v)$. For this purpose fix a real $\varepsilon > 0$. By (2) there is a real $\delta > 0$ such that

$$|f(x, y) - a(u, v)| < \frac{\varepsilon}{3} \text{ for } (x, y) \in K((u, v), \delta) \text{ with } x \neq u \text{ and } y \neq v,$$

where

$$K((u, v), \delta) = \{(x, y) : |(x, y) - (u, v)| < \delta\}.$$

Since $\lim_{n \to \infty} f(u, y_n) = b(u, v)$, there is a positive integer j such that

$$|y_j - v| < \frac{\delta}{2} \text{ and } |f(u, y_j) - b(u, v)| < \frac{\varepsilon}{3}.$$

By the hypothesis there is a positive integer i with

$$|u_{j,i} - u| < \frac{\delta}{2} \text{ and } |f(u_{j,i}, y_j) - f(u, y_j)| < \frac{\varepsilon}{3}.$$

Consequently,

$$|b(u,v)-a(u,v)|\le|b(u,v)-f(u,y_j)|+|f(u,y_j)-f(u_{j,i},y_j)|+$$
$$+|f(u_{j,i},y_j)-a(u,v)|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

Since $\varepsilon>0$ was arbitrary chosen we end up with $a(u,v)=b(u,v)$.

If a sequence of points $z_n\neq v$ converges to v, we have

$$\lim_{n\to\infty}f(u,z_n)=b(u,v)=a(u,v),$$

and the proof is completed.

As an immediate consequence we obtain:

COROLLARY 2. *Conserve all hypotheses from Theorem 2 on the function $f:\mathcal{R}^2\to\mathcal{R}$ and on the point $(u,v)\in\mathcal{R}^2$. If there is a sequence of points $y_n\neq v$ such that $\lim_{n\to\infty}y_n=v$ and sections f^{y_n} are continuous at u for $n\ge1$, then there is the limit*

$$\lim_{(x,y)\to(u,v)}f(x,y)=a(u,v).$$

Continuum Hypothesis (or Martin's Axiom) implies that there is a set $A\subset\mathcal{R}^2$ which cuts each closed set of positive plane Lebesgue measure and such that for every straight line l the intersection $l\cap A$ contains at most two points ([4]). Then sections f_x and f^y, $x,y\in\mathcal{R}$, of the function

$$f(x,y)=1\ \text{ for }\ (x,y)\in A\ \text{ and }\ f(x,y)=0\ \text{ for }\ (x,y)\in\mathcal{R}^2\setminus A$$

have the finite limits at all points, but f has not the limit at any point.

REMARK 1. Let $f:\mathcal{R}^2\to\mathcal{R}$ be a function and let $(u,v)\in\mathcal{R}^2$ be a point. Assume that there is the limit

$$\lim_{(x,y)\to(u,v)}f(x,y)=a(u,v)\in\mathcal{R}$$

and that there is a sequence of points $(x_n,y_n)\neq(u,v)$ such that

$$\lim_{n\to\infty}(x_n,y_n)=(u,v)\ \text{ and }\ \lim_{n\to\infty}f(x_n,y_n)=f(u,v).$$

Then the function f is continuous at (u,v).

Proof. The proof is evident. It suffices to observe that

$$\lim_{n\to\infty}f(x_n,y_n)=a(u,v)=f(u,v).$$

The continuity of sections f_x and f^y, $x,y\in\mathcal{R}$, of a function $f:\mathcal{R}^2\to\mathcal{R}$ does not imply the existence of the double limit of f at each point (x,y).

However if sections f_x of a function $f:\mathcal{R}^2\to\mathcal{R}$ belong to some special subfamily of continuous functions and sections f^y have the limit at each point then f has also the double limit at every point (x,y).

For this we will say that a family $\mathcal{A}$ of continuous real functions of one real variable has the property $(*)$ if there are: a positive integer n, functions $f_1, f_2, \ldots, f_n \in \mathcal{A}$ and points $x_1, x_2, \ldots, x_n \in \mathcal{R}$ such that each function $f \in \mathcal{A}$ is a combination

$$f = \sum_{i \le n} a_i f_i \quad \text{and} \quad \det[f_i(x_j)]_{i,j \le n} \neq 0,$$

where det denotes the determinant of a matrix.

The following families have the property $(*)$:

1. the family of polynomials of degree $< k$, where $f_i(x) = x^i$, $i = 0, 1, \ldots, k-1$;

2. the family of trygonometric polynomials of degree $\le k$, i.e. the family of functions

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{k} (a_n \cos(nx) + b_n \sin(nx)),$$

where

$$f_{2i}(x) = \cos(ix), \quad i = 0, 1, \ldots, k \quad \text{and} \quad f_{2i-1}(x) = \sin(ix), \quad i = 1, 2, \ldots, k.$$

3. generally the families of orthonormal polynomials of degree $\le k$.

THEOREM 3. *Let $\mathcal{A}$ be a family of continuous functions having the property $(*)$ and let $f : \mathcal{R}^2 \to \mathcal{R}$ be a function. If sections $f^y \in \mathcal{A}$ and if sections f_x have the finite limit at each point of $\mathcal{R}$ (are continuous), then f has the finite double limit at each point (is continuous).*

Proof. Since the family $\mathcal{A}$ has the property $(*)$, there are a positive integer n, a family of functions $f_1, \ldots, f_n \in \mathcal{A}$ and points $x_1, \ldots, x_n \in \mathcal{R}$ such that each function $g \in \mathcal{A}$ is the combination

$$g = \sum_{i \le n} a_i f_i \quad \text{and} \quad \det[f_i(x_j)]_{i,j \le n} \neq 0.$$

For $(x, y) \in \mathcal{R}^2$ we have

$$f(x, y) = f^y(x) = \sum_{i \le n} a_i(y) f_i(x).$$

Consequently, for $j \le n$ and $y \in \mathcal{R}$ we obtain

$$f(x_j, y) = \sum_{i \le n} a_i(y) f_i(x_j). \tag{3}$$

Since $\det[f_i(x_j)] \neq 0$, the system (3) has unique solution

$$a_i(y) = \frac{\det A_i}{\det[f_i(x_j)]},$$

where the matrix A_i is formed from the matrix $[f_i(x_j)]$ by the replacement of i^{th}-column by the column $[f(x_j, y)]$. Sections $y \to f_{x_i}(y)$, $i = 1, 2, \ldots, n$, have finite limits (are continuous) at each point, as well as functions $y \to a_i(y)$, $i \le n$. Consequently, if

$$\lim_{k\to\infty}(u_k, v_k) = (x, y) \neq (u_k, v_k) \text{ for } k \ge 1,$$

then

$$\lim_{k\to\infty} f(u_k, v_k) = \sum_{i\le n}(f_i(x) \lim_{k\to\infty} a_i(v_k)).$$

If sections f_x are continuous then we obtain the continuity of f. This completes the proof.

Now let $D \subset \mathcal{R}^2$ (or $D \subset \mathcal{R}$) be a nonempty open set and let $f : D \to \mathcal{R}$ be a function.

It is well known that the set $C(f)$ of all continuity points of f is an G_δ-set ([3]). We will prove that the set

$$L(f) = \{(x, y) \in D : \text{there is a finite limit } \lim_{(u,v)\to(x,y)} f(u, v)\}$$

is also an G_δ-set. For this we introduce the following operation:

If $A \subset D$ then let

$$l(A) = \{(x, y) \in D : \underset{r>0}{\exists} K((x, y), r) \setminus \{(x, y)\} \subset int(A)\}.$$

Evidently for each set A the set $l(A)$ is open.

REMARK 2. For arbitrary function $f : D \to \mathcal{R}$ the set $L(f)$ is of G_δ type.

Proof. For $n = 1, 2, \ldots$ and $k = 0, -1, 1, -2, 2, \ldots$ let

$$A_{k,n} = \left\{(x, y) \in D : \frac{k-1}{2^n} < f(x, y) < \frac{k+1}{2^n}\right\}.$$

Observe that

$$L(f) = \bigcap_{n=1}^{\infty} \bigcup_{k=-\infty}^{\infty} l(A_{k,n}),$$

and the proof is completed.

REMARK 3. If the set $D \setminus L(f)$ is countable then the set $D \setminus C(f)$ is also countable.

Proof. If the function f is not continuous at a point (x, y) then there are four rationals $r(x, y), s(x, y), t(x, y), z(x, y)$ such that $r(x, y) < s(x, y) < f(x, y) < t(x, y) < z(x, y)$ and for each open neighbourhood U of (x, y) there is a point $(u, v) \in U$ with $f(u, v) < r(x, y)$ or $f(u, v) > z(x, y)$. Enumerate all systems of four rationals in a sequence

$$(4) \qquad (r_1, s_1, t_1, z_1), \ldots, (r_n, s_n, t_n, z_n), \ldots$$

such that

$$(r_n, s_n, t_n, z_n) \neq (r_m, s_m, t_m, z_m) \text{ for } n \neq m,$$

and for $n \geq 1$ put

$$A_n = \{(x,y) \in D \setminus C(f) : (r(x,y), s(x,y), t(x,y), z(x,y)) = (r_n, s_n, t_n, z_n)\}.$$

If the set $D \setminus C(f)$ is uncountable then there is a positive integer k such that the set A_k is also uncountable. Consequently, the sets

$$A_k \cap L(f)$$

and

$$B_k = \{(x,y) \in A_k \cap L(f) : (x,y) \text{ is a condensation point of } A_k\}$$

are also uncountable. Fix a point $(u,v) \in B_k$. Then $(u,v) \in L(f)$. On the other hand $(u,v) \in A_k$, so in every neighbourhood U of (u,v) there are a point $(a,b) \in A_k \cap U$ with $s_k < f(a,b) < t_k$ and $(a,b) \neq (u,v)$ and a point $(c,d) \in U$ with $f(c,d) \in \mathcal{R} \setminus [r_k, z_k]$. So (u,v) is not in $L(f)$ and this contradiction completes the proof.

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