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OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES SATISFY CERTAIN CONVEXITY ASSUMPTIONS

Abstract. Ostrowski type inequalities for absolutely continuous functions whose derivatives satisfy certain convexity assumptions are pointed out.

1. Introduction

The following Ostrowski type integral inequalities for absolutely continuous functions whose derivatives satisfy certain convexity assumptions have been obtained in [1].

THEOREM 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ such that $|f'|$ is convex on $[a,b]$. Then for any $x \in [a,b]$ we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{1.1}$$

$$\leq \begin{cases} \frac{1}{2}\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[|f'(x)| + \|f'\|_\infty\right] & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{2}\cdot\frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1} + \left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\||f'(x)| + |f'|\right\|_\infty & \\ \qquad \text{if } f' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; & \\ \frac{1}{2}\left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right]\left[(b-a)|f'(x)| + \|f'\|_1\right]. & \end{cases}$$

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The constant $\frac{1}{2}$ in the first and second inequalities is sharp as is the first $\frac{1}{2}$ in the final.

Proceeding in the same spirit as the above theorem, the following result deals with an Ostrowski type inequality where upper and lower bounds for the deviation of a function from its integral mean, namely,

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt$$

are provided (see [2]).

THEOREM 2. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function so that f' is convex on (a,b). Then for any $x \in [a,b]$ one has the inequality*

$$\begin{aligned}(1.2)\quad & \frac{1}{2}\cdot\frac{b-x}{b-a}\left[\frac{1}{b-x}\int_x^b f(t)\,dt - f(b) - \frac{1}{2}(b-x)f'(x)\right] \\ &\quad + 2\cdot\frac{x-a}{b-a}\left[\frac{2}{x-a}\int_{\frac{a+x}{2}}^{x} f(u)\,du - f\left(\frac{a+x}{2}\right)\right] \\ &\leq f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\leq \frac{1}{2}\cdot\frac{x-a}{b-a}\left[\frac{1}{x-a}\int_a^x f(u)\,du - f(a) - \frac{1}{2}(x-a)f'(x)\right] \\ &\quad + 2\cdot\frac{b-x}{b-a}\left[\frac{2}{b-x}\int_x^{\frac{x+b}{2}} f(u)\,du - f\left(\frac{x+b}{2}\right)\right].\end{aligned}$$

In the present paper a different approach is considered where $|f'|$ is assumed to be in turn convex, quasi-convex, or log-convex.

For a comprehensive list of results related to Ostrowski's inequality, see the recent monograph [3] where further references are provided.

2. Some inequalities for $|f'|$ convex

LEMMA 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then we have the representation*

$$\begin{aligned}(2.1)\quad f(x) = {} & \frac{1}{b-a}\int_a^b f(t)\,dt + (x-a)^2\cdot\frac{1}{b-a}\int_0^1 \lambda f'[(1-\lambda)a + \lambda x]\,d\lambda \\ & - (b-x)^2\cdot\frac{1}{b-a}\int_0^1 \lambda f'[\lambda x + (1-\lambda)b]\,d\lambda\end{aligned}$$

for any $x \in [a,b]$.

Proof. We start with the following known identity

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{b-a}\int_a^x (t-a)\,f'(t)\,dt+\frac{1}{b-a}\int_x^b (t-b)\,f'(t)\,dt \tag{2.2}$$

for any $x\in[a,b]$.

If we make the change of variable $t=(1-\lambda)a+\lambda x$, $\lambda\in[0,1]$, then we get

$$\int_a^x (t-a)\,f'(t)\,dt=(x-a)^2\int_0^1 \lambda f'[(1-\lambda)a+\lambda x]\,d\lambda.$$

Also, the change of variable $t=\mu x+(1-\mu)b$, $\mu\in[0,1]$, will provide the equality

$$\int_x^b (t-b)\,f'(t)\,dt=-(b-x)^2\int_0^1 \mu f'[\mu x+(1-\mu)b]\,d\mu.$$

Using (2.2), we then deduce the desired identity (2.1). ■

The following Ostrowski-type inequality holds for $|f'|$ convex.

THEOREM 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x\in[a,b]$. If $|f'|$ is convex on $[a,x]$ and $[x,b]$, then one has the inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{6}\left[|f'(a)|\left(\frac{x-a}{b-a}\right)^2+|f'(b)|\left(\frac{b-x}{b-a}\right)^2+\left[1+2\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]|f'(x)|\right](b-a). \tag{2.3}$$

The constant $\frac{1}{6}$ is best possible in the sense that it cannot be replaced by a smaller value.

Proof. Taking the modulus in (2.1) we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le (x-a)^2\cdot\frac{1}{b-a}\int_0^1 \lambda\,|f'[(1-\lambda)a+\lambda x]|\,d\lambda+(b-x)^2\cdot\frac{1}{b-a}\int_0^1 \lambda\,|f'[\lambda x+(1-\lambda)b]|\,d\lambda:=M(x). \tag{2.4}$$

Since $|f'|$ is convex on $[a,x]$ and $[x,b]$, then obviously

$$\int_0^1 \lambda |f'[(1-\lambda)a+\lambda x]|\,d\lambda \le |f'(a)|\int_0^1 \lambda(1-\lambda)\,d\lambda + |f'(x)|\int_0^1 \lambda^2 d\lambda = \frac{1}{6}|f'(a)| + \frac{1}{3}|f'(x)|$$

and

$$\int_0^1 \lambda |f'[\lambda x+(1-\lambda)b]|\,d\lambda \le |f'(x)|\int_0^1 \lambda^2 d\lambda + |f'(b)|\int_0^1 \lambda(1-\lambda)\,d\lambda = \frac{1}{3}|f'(x)| + \frac{1}{6}|f'(b)|.$$

Thus from (2.4)

$$\begin{aligned} M(x) &\le \frac{1}{3}\left[\frac{1}{2}|f'(a)| + |f'(x)|\right](x-a)^2 + \frac{1}{3}\left[|f'(x)| + \frac{1}{2}|f'(b)|\right](b-x)^2 \\ &= \frac{1}{3}\left[\frac{|f'(a)|(x-a)^2 + |f'(b)|(b-x)^2}{2} + \left[(x-a)^2+(b-x)^2\right]|f'(x)|\right] \\ &= \frac{1}{6}\Bigg[|f'(a)|(x-a)^2 + |f'(b)|(b-x)^2 \\ &\quad + \left[(b-a)^2 + 2\left(x-\frac{a+b}{2}\right)^2\right]|f'(x)|\Bigg] \end{aligned}$$

and the inequality (2.3) is proved.

To prove the sharpness of the constant $\frac{1}{6}$, assume that (2.3) holds with a constant $C>0$. Namely,

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le C\left[|f'(a)|\left(\frac{x-a}{b-a}\right)^2 + |f'(b)|\left(\frac{b-x}{b-a}\right)^2 + \left[1+2\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]|f'(x)|\right](b-a), \tag{2.5}$$

provided that $|f'|$ is convex on $[a,x]$ and $[x,b]$.

If we choose $x=\frac{a+b}{2}$, then by (2.5) we deduce

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le C\left[\frac{|f'(a)|+|f'(b)|}{4} + \left|f'\left(\frac{a+b}{2}\right)\right|\right](b-a), \tag{2.6}$$

where $|f'|$ is convex on $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$.

Consider the function $f_0 : [a,b] \to \mathbb{R}$, given by

$$f_0(x) = \begin{cases} \frac{a+b}{2} - x, & \text{if } x \in [a, \frac{a+b}{2}], \\ x - \frac{a+b}{2}, & \text{if } x \in (\frac{a+b}{2}, b]. \end{cases}$$

The function is absolutely continuous on $[a,b]$ and, obviously, $|f'| = 1$ on $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ showing that it is convex on $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$.

On the other hand, we have

$$f_0\left(\frac{a+b}{2}\right) = 0, \quad \frac{1}{b-a}\int_a^b f_0(x)\,dx = \frac{b-a}{4},$$

$$|f_0'(a)| = |f_0'(b)| = \left|f_0'\left(\frac{a+b}{2}\right)\right| = 1$$

and so from (2.6) we get

$$\frac{b-a}{4} \le C\left(\frac{1}{2} + 1\right)(b-a)$$

giving $C \ge \frac{1}{6}$. ∎

The following corollary is a natural consequence.

COROLLARY 1. *Assume that $f : [a,b] \to \mathbb{R}$ is absolutely continuous such that $|f'|$ is a convex function on $\left[a, \frac{a+b}{2}\right]$ and $\left(\frac{a+b}{2}, b\right]$. Then we have the inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{6}\left[\frac{|f'(a)| + |f'(b)|}{4} + \left|f'\left(\frac{a+b}{2}\right)\right|\right](b-a). \tag{2.7}$$

The $\frac{1}{6}$ is best possible in (2.7) in the sense that it cannot be replaced by a smaller constant.

3. Inequalities for $|f'|$ quasi-convex

Firstly, let us recall the definition of quasi-convex functions.

DEFINITION 1. The function $h : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is said to be *quasi-convex* (QC) on the interval I if

$$h(\lambda x + (1-\lambda)y) \le \max\{h(x), h(y)\} \tag{3.1}$$

for any $x, y \in I$ and $\lambda \in [0,1]$.

Following [4], we say that for an interval $I \subseteq \mathbb{R}$, the mapping $h : I \to \mathbb{R}$ is *quasi-monotone* on I if it is either monotone on $I = [c,d]$ or monotone nonincreasing on a proper subinterval $[c,c'] \subset I$ and monotone nondecreasing on $[c',d]$.

The class $QM(I)$ of quasi-monotone functions on I provides an immediate characterisation of quasi-convex functions [4].

PROPOSITION 1. *Suppose $I \subseteq \mathbb{R}$. Then the following statements are equivalent for a function $h : I \to \mathbb{R}$:*

(a) $h \in QM(I)$;
(b) *On any subinterval of I, h achieves its supremum at an end point;*
(c) $h \in QC(I)$.

As examples of quasi-convex functions we may consider the class of monotonic functions on an interval I for the class of convex functions on that interval.

The following Ostrowski type inequality for absolutely continuous functions for which $|f'|$ is quasi-convex holds.

THEOREM 4. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x \in [a,b]$. If $|f'|$ is quasi-convex on $[a,x]$ and $[x,b]$, then one has the inequality*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{4}\left\{\left(\frac{x-a}{b-a}\right)^2 \left[|f'(a)| + |f'(x)| + ||f'(x)| - |f'(a)||\right] + \left(\frac{b-x}{b-a}\right)^2 \left[|f'(x)| + |f'(b)| + ||f'(x)| - |f'(b)||\right]\right\}. \tag{3.2}$$

The constant $\frac{1}{4}$ is sharp in (3.2) in the sense that it cannot be replaced by a smaller value.

Proof. Since $|f'|$ is quasi-convex on $[a,x]$ and $[x,b]$, then from (3.1)

$$\begin{aligned}\int_0^1 \lambda\,|f'((1-\lambda)a + \lambda x)|\,d\lambda &\le \max\{|f'(a)|, |f'(x)|\}\int_0^1 \lambda d\lambda \\ &= \frac{1}{2}\left[\frac{|f'(a)| + |f'(x)|}{2} + \frac{1}{2}\,||f'(x)| - |f'(a)||\right] \\ &:= M_1(x)\end{aligned}$$

and, similarly

$$\int_0^1 \lambda \left|f'(\lambda x + (1-\lambda) b)\right| d\lambda \le \frac{1}{2}\left[\frac{|f'(x)| + |f'(b)|}{2} + \frac{1}{2}\left||f'(x)| - |f'(b)|\right|\right] := M_2(x).$$

Using (2.4) and the notation $M(x)$ for the right hand side of that inequality, we deduce that

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le M(x) \le \left(\frac{x-a}{b-a}\right)^2 M_1(x) + \left(\frac{b-x}{b-a}\right)^2 M_2(x)$$

and the result (3.2) is thus proved.

The fact that $\frac{1}{4}$ is the best possible constant will be shown in the following. ∎

COROLLARY 2. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $|f'|$ is quasi-convex on $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$, then one has the inequality:*

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le \frac{1}{16}\left\{\left[|f'(a)| + 2\left|f'\left(\frac{a+b}{2}\right)\right| + |f'(b)| + \left|\left|f'\left(\frac{a+b}{2}\right)\right| - |f'(a)|\right|\right]\right. \\ &\quad \left. + \left||f'(b)| - \left|f'\left(\frac{a+b}{2}\right)\right|\right|\right\}(b-a). \end{aligned} \tag{3.3}$$

The constant $\frac{1}{16}$ is best possible.

Proof. The inequality follows by (3.2) on choosing $x = \frac{a+b}{2}$. To prove the sharpness of the constant $\frac{1}{16}$, assume that (3.3) holds with a constant $C > 0$. That is,

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le C\left\{\left[|f'(a)| + 2\left|f'\left(\frac{a+b}{2}\right)\right| + |f'(b)| + \left|\left|f'\left(\frac{a+b}{2}\right)\right| - |f'(a)|\right|\right]\right. \\ &\quad \left. + \left||f'(b)| - \left|f'\left(\frac{a+b}{2}\right)\right|\right|\right\}(b-a). \end{aligned} \tag{3.4}$$

Consider the function $f_0 : [a,b] \to \mathbb{R}$, $f_0(t) = \left|t - \frac{a+b}{2}\right|$. Then f_0 is absolutely continuous and $|f_0'(t)| = 1$, $t \in [a,b]$. Thus, from (3.4), we deduce

$$\frac{b-a}{4} \le C \cdot 4(b-a)$$

giving $C \ge \frac{1}{16}$ and the corollary is proved. ∎

4. Inequalities for $|f'|$ log-convex

In what follows, I will denote an interval of real numbers. A function $f : I \to (0,\infty)$ is said to be *log-convex* or *multiplicatively convex* if $\log f$ is convex, or, equivalently, if for any $x,y \in I$ and $t \in [0,1]$ one has the inequality

$$f(tx+(1-t)y) \le [f(x)]^t [f(y)]^{1-t}. \tag{4.1}$$

We note that if f and g are convex and g is increasing, then $g \circ f$ is convex, moreover, since $f = \exp[\log f]$, it follows that a log-convex function is convex, but the converse may not necessarily be true. This follows directly from (4.1) since, by the arithmetic-geometric mean inequality we have

$$[f(x)]^t [f(y)]^{1-t} \le tf(x) + (1-t) f(y) \tag{4.2}$$

for all $x,y \in I$ and $t \in [0,1]$.

THEOREM 5. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x \in [a,b]$. If $|f'|$ is log-convex on $[a,x]$ and $[x,b]$, then one has the inequality*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\qquad \le \left\{ \left(\frac{x-a}{b-a}\right)^2 |f'(a)| \frac{A\ln A + 1 - A}{(\ln A)^2} \right. \\ &\qquad\qquad \left. + \left(\frac{b-x}{b-a}\right)^2 |f'(b)| \frac{B\ln B + 1 - B}{(\ln B)^2} \right\} (b-a), \end{aligned} \tag{4.3}$$

where

$$A := \left|\frac{f'(x)}{f'(a)}\right|, \quad B := \left|\frac{f'(x)}{f'(b)}\right|.$$

Proof. Using the representation (2.1) and the definition of log-convexity (4.1), we have successively:

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\qquad \le (b-a)\left\{ \left(\frac{x-a}{b-a}\right)^2 \int_0^1 \lambda \left|f'((1-\lambda)a + \lambda x)\right| d\lambda \right. \\ &\qquad\qquad \left. + \left(\frac{b-x}{b-a}\right)^2 \int_0^1 \lambda \left|f'(\lambda x + (1-\lambda) b)\right| d\lambda \right\} \end{aligned} \tag{4.4}$$

$$\leq (b-a)\left\{\left(\frac{x-a}{b-a}\right)^2\int_0^1 \lambda\left|f'(a)\right|^{1-\lambda}\left|f'(x)\right|^{\lambda}d\lambda\right.$$

$$\left.+\left(\frac{b-x}{b-a}\right)^2\int_0^1 \lambda\left|f'(x)\right|^{\lambda}\left|f'(b)\right|^{1-\lambda}d\lambda\right\}$$

$$=(b-a)\left\{\left(\frac{x-a}{b-a}\right)^2\left|f'(a)\right|\int_0^1\lambda A^{\lambda}d\lambda+\left(\frac{b-x}{b-a}\right)^2\left|f'(b)\right|\int_0^1\lambda B^{\lambda}d\lambda\right\}.$$

Since, a simple calculation shows that for any $C>0$, one has

$$\int_0^1\lambda C^{\lambda}d\lambda=\frac{C\ln C+1-C}{(\ln C)^2},$$

then from (4.4) we deduce the desired result (4.3). ■

COROLLARY 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $|f'|$ is log-convex on $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$, then one has the inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{4.5}$$

$$\leq\frac{1}{4}\left[\left|f'(a)\right|\frac{\alpha\ln\alpha+1-\alpha}{(\ln\alpha)^2}+\left|f'(b)\right|\frac{\beta\ln\beta+1-\beta}{(\ln\beta)^2}\right](b-a),$$

where

$$\alpha:=\left|\frac{f'\left(\frac{a+b}{2}\right)}{f'(a)}\right|,\quad \beta:=\left|\frac{f'\left(\frac{a+b}{2}\right)}{f'(b)}\right|.$$

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