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NEW OSTROWSKI TYPE INEQUALITIES FOR MAPPINGS WHOSE DERIVATIVES BELONG TO L_p SPACES

Abstract. The aim of the present paper is to establish two new Ostrowski type inequalities for mappings whose derivatives belongs to L_p -spaces.The analysis used in the proofs is elementary and our results in the special cases yields the Ostrowski type inequality recently established by Dragomir and Wang.

1. Introduction

In 1938, Ostrowski proved the following inequality (See [4, p. 468]):

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)M \tag{1.1}$$

for $x \in [a,b]$, where $f : [a,b] \to R$ is continuous on [a,b], differentiable on (a,b) and $|f'(x)| \leq M$ for all $x \in (a,b)$. In [1] Cerone, Dragomir and Roumeliotis proved the following Ostrowski type inequality:

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\left(x-\frac{a+b}{2}\right)f'(x)\right| \\ \leq \left[\frac{1}{24}(b-a)^2+\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right]\|f''\|_\infty \\ \leq \frac{(b-a)^2}{6}\|f''\|_\infty\end{aligned} \tag{1.2}$$

where $f : [a,b] \to R$ be a twice differentiable mapping on (a,b) and $f'' : [a,b] \to R$ is bounded.

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The following interesting application of (1.2) in numerical integration is also given in [1]. Let $I_n : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$, $\xi_i \in [x_i, x_{i+1}]$. Let $f : [a, b] \to R$ be a twice differentiable mapping on (a,b) whose second derivitive $f'' : [a, b] \to R$ is bounded, i.e. $\|f''\|_\infty < \infty$. Then

$$\int_a^b f(x)dx = A(f, f', \xi, I_n) + R(f, f', \xi, I_n)$$

where

$$A(f, f', \xi, I_n) = \sum_{i=0}^{n-1} h_i f(\xi_i) - \sum_{i=0}^{n-1} f'_{i(\xi_i)}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)h_i$$

and the remiander R satisfies the estimation

$$|R(f, f', \xi, I_n)| \le \left[\frac{1}{24}\sum_{i=0}^{n-1} h_i^3 + \frac{1}{2}\sum_{i=0}^{n-1} h_i\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]\|f''\|_\infty$$
$$\le \frac{\|f''\|_\infty}{6}\sum_{i=1}^{n-1} h_i^3,$$

for all ξ_i as above, where $h_i = x_{i+1} - x_i$ $(i = 0, 1, \ldots, n-1)$.

For further applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical qudrature rules, we refer the interested readers to [3].

In 1998, Dragomir and Wang [2] proved the following new inequality of Ostrowski type in L_p -norm:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\, dt\right| \le L(x)\|f'\|_p \tag{1.3}$$

for $x \in [a, b]$, where $f : [a, b] \to R$ is an absolutely continous mapping for which $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$L(x) = \frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{q+1} + \left(\frac{b-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}\left(b-a\right)^{\frac{1}{q}} \tag{1.4}$$

for $x \in [a, b]$ and

$$\|f'\|_p = \left(\int_a^b |f'(t)|^p\, dt\right)^{\frac{1}{p}}$$

the usual L_p-norm.

For some generalizations of Ostrowski's inequality (1.1) and releted results see the book [4, pp. 468–484] by Mitrinović, Pečarić and Fink, where further references are given . The main object of this paper is to establish new inequalities involving two differentiable mappings whose derivatives belongs to the L_p -spaces. The analysis used in the proofs is elementry and based on the use of Peano kernel approch and Montgomery's identity (see [4, p. 585]).

2. Statement of results

Our main results are given in the following theorems.

THEOREM 1. *Let $f, g : [a,b] \to R$ be absolutely continous mappings for which f', $g' \in L_p[a,b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| f(x)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right) + g(x)\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) - 2\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)\right| \leq L(x)\left[\|f'\|_p\left(\frac{1}{b-a}\int_a^b |g(t)|\,dt\right) + \|g'\|_p\left(\frac{1}{b-a}\int_a^b |f(t)|\,dt\right)\right] \tag{2.1}$$

for $x \in [a,b]$, where $L(x)$ is given by (1.4).

THEOREM 2. *Let f, g, f', g' be as in Theorem* 1. *Then*

$$\left| 2f(x)g(x) - g(x)\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) - f(x)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)\right| \leq L(x)[\|f'\|_p|g(x)| + \|g'\|_p|f(x)|] \tag{2.2}$$

for $x \in [a,b]$, where $L(x)$ is given by (1.4).

3. Proofs of Theorems 1 and 2

From the hypotheses we have the following identities (see [4 , p. 585]):

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b k(x,t)f'(t)\,dt, \tag{3.1}$$

$$g(x) - \frac{1}{b-a}\int_a^b g(t)\,dt = \frac{1}{b-a}\int_a^b k(x,t)g'(t)\,dt, \tag{3.2}$$

for $x \in [a,b]$, where the Peano kernal $k(x,t) : [a,b]^2 \to R$ is given by

$$k(x,t) = \begin{cases} t-a & \text{if } t \in [a,x], \\ t-b & \text{if } t \in (x,b]. \end{cases} \tag{3.3}$$

The identities (3.1) and (3.2) can be eaisly proved dividing the integrals on the right side into two integrals over the intervals $[a, x]$ and $(x, b]$ and using the integration by parts formula. Multiplying both sides of (3.1) and (3.2) respectively by $\frac{1}{b-a}\int_a^b g(t)\,dt$ and $\frac{1}{b-a}\int_a^b f(t)\,dt$ and adding, we get

$$\begin{aligned} f(x)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right) &+ g(x)\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \\ &- 2\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right) \\ &= \left(\frac{1}{b-a}\int_a^b k(x,t)f'(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right) \\ &+ \left(\frac{1}{b-a}\int_a^b k(x,t)g'(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \end{aligned} \tag{3.4}$$

for any $x \in [a, b]$. From (3.4) , using the properties of modules and Hölder's integral inequality, we have

$$\begin{aligned} &\Bigg| f(x)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right) + g(t)\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \\ &\qquad - 2\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)\Bigg| \\ &\le \left(\frac{1}{b-a}\int_a^b |k(x,t)|\,dt|\,|f'(t)|dt\right)\left(\frac{1}{b-a}\int_a^b |g(t)|\,dt\right) \\ &\quad + \left(\frac{1}{b-a}\int_a^b |k(x,t)\,dt|\,|g'(t)|\,dt\right)\left(\frac{1}{b-a}\int_a^b |f(t)|\,dt\right) \\ &\le \frac{1}{b-a}\left(\int_a^b |k(x,t)|^q\,dt\right)^{\frac{1}{q}} \\ &\quad \times \left[\|f'\|_p\left(\frac{1}{b-a}\int_a^b |g(t)|\,dt\right) + \|g'\|_p\left(\frac{1}{b-a}\int_a^b |f(t)|\,dt\right)\right]. \end{aligned} \tag{3.5}$$

Using the elementry calculation we have

$$(3.6)\quad \int_a^b |k(x,t)|^q dt = \int_a^x |t-a|^q\, dt + \int_x^b |b-t|^q\, dt$$
$$= \int_a^x (t-a)^q\, dt + \int_x^b (b-t)^q\, dt$$
$$= \frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1}$$
$$= \frac{1}{q+1}\left[\left(\frac{x-a}{b-a}\right)^{q+1} + \left(\frac{b-x}{b-a}\right)^{q+1}\right](b-a)^{q+1}.$$

Using (3.6) in (3.5) we get the required inequality in (2.1).

Next, multiplying both sides of (3.1) and (3.2) respectively by $g(x)$ and $f(x)$ and adding both sides of obtained equalities, we get

$$(3.7)\quad 2f(x)g(x) - g(x)\left(\frac{1}{b-a}\int_a^b f(t)\, dt\right) - f(x)\left(\frac{1}{b-a}\int_a^b g(t)\, dt\right)$$
$$= g(x)\left(\frac{1}{b-a}\int_a^b k(x,t)f'(t)\, dt\right) + f(x)\left(\frac{1}{b-a}\int_a^b k(x,t)g'(t)\, dt\right)$$

for $x \in [a,b]$. From (3.7), using the properties of modulus and Hölder's integral inequality, we have

$$(3.8)\quad \left|2f(x)g(x) - g(x)\left(\frac{1}{b-a}\int_a^b f(t)\, dt\right) - f(x)\left(\frac{1}{b-a}\int_a^b g(t)\, dt\right)\right|$$
$$\leq \frac{1}{b-a}\left(\int_a^b |k(x,t)|^q\, dt\right)^{\frac{1}{q}} [\|f'\|_p |g(x)| + \|g'\|_p |f(x)|].$$

By making use of (3.6) in (3.8), we get the desired inequality in (2.2).

In concluding we note that , if we take $g(x) = 1$ and hence $g'(x) = 0$ in Theorems 1 and 2, then we recapture the inequality (1.3) given by Dragomir and Wang in [2].

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