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CENTRALIZING MAPPINGS AND DERIVATIONS ON SEMIPRIME RINGS

Abstract. In this paper we study some properties of centralizing mappings on semiprime rings. The main purpose is to prove the result: Let R be a semiprime ring and f an endomorphism of R, g an epimorphism of R such that the mapping $x \to [f(x), g(x)]$ is central. Then $[f(x), g(x)] = 0$ holds for all $x \in R$. We also establish some results about (α, β)-derivations.

1. Introduction

Throughout, R denotes a ring with center $Z(R)$. We write $[x, y]$ for $xy - yx$. Then $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$ hold in R. R is *prime* if $aRb = 0$ implies either $a = 0$ or $b = 0$; it is *semiprime* if $aRa = 0$ implies $a = 0$. A prime ring is obviously semiprime. An additive mapping d from R into itself is called a *derivation* if $d(xy) = xd(y)+d(x)y$ for all $x, y \in R$. Let $a \in R$. Then the mapping $d : R \to R$ defined by $d(x) = [a, x]$ is a derivation and is called an inner derivation of R. In this case, we say that d is an *inner derivation* determined by a. A mapping f from R into itself is *commuting* if $[f(x), x] = 0$; and *centralizing* if $[f(x), x] \in Z(R)$ for all $x \in R$. We call a mapping $f : R \to R$ *central* if $f(x) \in Z(R)$ for all $x \in R$. Recall that if f is an additive commuting mapping from R into itself, then a linearization of $[f(x), x] = 0$ yields $[f(x), y] = [x, f(y)]$ for all $x, y \in R$. The study of centralizing and commuting mappings was initiated by Posner [14]. Considerable work has been done on centralizing and commuting mappings during the last couple of decades (see, e.g., [1, 3–7, 12, 13, 16, 17] and references therein). Derivations are generalized as α- or skew-derivations and (α, β)-derivations and have been applied in various

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situations; in particular, in the solution of some functional equations (see, e.g., Brešar [6]). Let α, β be automorphisms of R. An additive mapping d of R into itself is called an (α, β)-*derivation* if $d(xy) = \alpha(x)d(y) + d(x)\beta(y)$ for all $x, y \in R$. If $\beta = 1$, where 1 is the identity mapping of R, then d is called an α-derivation or a skew-derivation . For instance, $d = \alpha - \beta$ is an (α, β)-derivation and $d = \alpha - 1$ is an α-derivation. Of course, a $(1,1)$-derivation or a 1-derivation is a derivation. For more information on α-derivations and (α, β)-derivations, we refer to [2, 6, 8, 10, 11, 14, 15, 18].

In Section 2, we prove some results on centralizing and commuting mappings on prime and semiprime rings. Our results are inspired by Vukman results [17, 18] (see also recent work of Thaheem and Samman [16, Theorem 2.1]). For instance, Vukman [18, Theorem 1] has proved that if R is a 2-torsion free semiprime ring and $D : R \to R$ is a derivation such that the mapping $x \to [D(x), x]$ is commuting on R, then $[D(x), x] = 0$ holds for all $x \in R$. We prove here an analogous result for a pair of mappings on semiprime rings. We show that if R is an epimorphism of R such that the mapping $x \to [f(x), g(x)]$ is central. Then $[f(x), g(x)] = 0$ holds for all $x \in R$.

Section 3 is devoted to the study of (α, β)-derivations on semiprime rings. We show that if d is a central (α, β)-derivation of a semiprime ring R, then $[x, y]d(u) = d(u)[x, y] = 0$ for all $x, y, u \in R$ (Proposition 3.1). We also show that the same conclusion holds if β is centralizing and d is a commuting (α, β)-derivation of a semiprime ring R (Proposition 3.2). Some well known results about derivations and α-derivations follow as application of these results.

2. The results

We begin with the following lemma which we need in the proof of the main result [Theorem 2.2].

LEMMA 2.1. *Let R be a semiprime ring, f be an additive mapping of R into itself and g an endomorphism of R such that the mapping $x \to [f(x), g(x)]$ is central. Then in this case $2[f(x), g(x)] = 0$ holds for all $x \in R$.*

Proof. By assumption $[f(x), g(x)] \in Z(R)$. Linearizing this, we get

(1) $$[f(x), g(y)] + [f(y), g(x)] \in Z(R) \text{ for all } x, y \in R.$$

Replacing y by x^2 in (1), we have $[f(x), g(x^2)] + [f(x^2), g(x)] \in Z(R)$ for all $x \in R$, which gives

(2) $$2g(x)[f(x), g(x)] + [f(x^2), g(x)] \in Z(R) \text{ for all } x \in R.$$

From (2), we obtain

(3) $$2[g(x), f(x)][f(x), g(x)] + [[f(x^2), g(x)], f(x)] = 0 \text{ for all } x \in R.$$

Thus (3) gives $[[f(x^2), g(x)], f(x)] = -2[g(x), f(x)][f(x), g(x)]$. That is,

(4) $$[[f(x^2), g(x)], f(x)] = 2[f(x), g(x)]^2 \text{ for all } x \in R.$$

We readily see from (4) that $[[f(x^2), g(x)], f(x)] \in Z(R)$. If we replace x by x^2 in $[f(x), g(x)] \in Z(R)$, then $[f(x^2), g(x^2)] \in Z(R)$, which implies

$$g(x)[f(x^2), g(x)] + [f(x^2), g(x)]g(x) \in Z(R)$$

for all $x \in R$. Thus

$$[g(x)[f(x^2), g(x)] + [f(x^2), g(x)]g(x), f(x)] = 0$$

for all $x \in R$. Hence

$$[g(x), f(x)][f(x^2), g(x)] + g(x)[[f(x^2), g(x)], f(x)] \\ +[[f(x^2), g(x)], f(x)]g(x) + [f(x^2), g(x)][g(x), f(x)] = 0,$$

which gives

$$2[g(x), f(x)][f(x^2), g(x)] = -2g(x)[[f(x^2), g(x)], f(x)].$$

Using (4), we get $2[g(x), f(x)][f(x^2), g(x)] = -4g(x)[f(x), g(x)]^2$ for all $x \in R$. Multiplying (4) by $2[f(x), g(x)]$ on the left, we get

$$\begin{aligned} 4[f(x), g(x)]^3 &= 2[[f(x^2), g(x)], f(x)][f(x), g(x)] \\ &= 2[[f(x^2), g(x)][f(x), g(x)], f(x)] \\ &\quad - 2[f(x^2), g(x)][[f(x), g(x)], f(x)] \\ &= -2[[g(x), f(x)][f(x^2), g(x)], f(x)] \\ &= [4g(x)[f(x), g(x)]^2, f(x)] = 4[g(x), f(x)][f(x), g(x)]^2 \\ &= -4[f(x), g(x)]^3. \end{aligned}$$

Hence $8[f(x), g(x)]^3 = 0$. Since the center of a semiprime ring contains no nonzero nilpotents, we conclude that $2[f(x), g(x)] = 0$ for all $x \in R$. ∎

We now prove the main result of this section.

THEOREM 2.2. *Let f be an endomorphism and g an epimorphism of a semiprime ring R such that the mapping $x \to [f(x), g(x)]$ is central. Then $[f(x), g(x)] = 0$ holds for all $x \in R$.*

Proof. By assumption $[f(x), g(x)] \in Z(R)$. Linearizing this, we get

(5) $$[f(x), g(y)] + [f(y), g(x)] \in Z(R) \text{ for all } x, y \in R.$$

By Lemma 2.1, we have

(6) $$2[f(x), g(x)] = 0 \text{ for all } x \in R.$$

Linearizing (6), we get

(7) $$2([f(x), g(y)] + [f(y), g(x)]) = 0 \text{ for all } x, y \in R.$$

Using (5)–(7) and the fact that $[f(x), g(x)] \in Z(R)$, the following identity follows easily

$$(8) \qquad [f(x), g(xy + yx)] + [f(y), g(x^2)] = 0 \text{ for all } x, y \in R.$$

Replacing y by yx in (8), we get $[f(x), g(xyx + yx^2)] + [f(yx), g(x^2)] = 0$, which gives

$$(9) \quad g(xy + yx)[f(x), g(x)] + [f(x), g(xy + yx)]g(x) + f(y)[f(x), g(x^2)] \\ + [f(y), g(x^2)]f(x) = 0 \text{ for all } x, y \in R.$$

Since $[f(x), g(x)] \in Z(R)$ and $2[f(x), g(x)] = 0$, therefore $[f(x), g(x^2)] = g(x)[f(x), g(x)] + [f(x), g(x)]g(x) = 2g(x)[f(x), g(x)] = 0$ for all $x \in R$. When we combine this with (8), then from (9), we get

$$[[g(x), g(y)] + 2g(y)g(x)][f(x), g(x)] - [f(y), g(x^2)]g(x) + [f(y), g(x^2)]f(x) = 0,$$

which implies

$$(10) \quad [g(x), g(y)][f(x), g(x)] + [f(y), g(x^2)](f(x) - g(x)) = 0 \text{ for all } x, y \in R.$$

Now,

$$\begin{aligned} [f(y), g(x^2)] &= g(x)[f(y), g(x)] + [f(y), g(x)]g(x) \\ &= g(x)[f(y), g(x)] + [f(y), g(x)]g(x) + g(x)[f(x), g(y)] \\ &\quad + [f(x), g(y)]g(x) - g(x)[f(x), g(y)] - [f(x), g(y)]g(x) \\ &= g(x)[[f(y), g(x)] + [f(x), g(y)]] + [[f(y), g(x)] \\ &\quad + [f(x), g(y)]]g(x) - g(x)[f(x), g(y)] - [f(x), g(y)]g(x). \end{aligned}$$

That is, $[f(y), g(x^2)] = 2g(x)[[f(y), g(x)] + [f(x), g(y)]] - g(x)[f(x), g(y)] - [f(x), g(y)]g(x)$. By (7), the above expression becomes

$$[f(y), g(x^2)] = -g(x)[f(x), g(y)] - [f(x), g(y)]g(x).$$

So, from (10), we obtain

$$[g(x), g(y)][f(x), g(x)] - (g(x)[f(x), g(y)] + [f(x), g(y)]g(x))(f(x) - g(x)) = 0$$

for all $x, y \in R$. Fix x, then by the ontoness of g, we get $[g(x), y][f(x), g(x)] - (g(x)[f(x), y] + [f(x), y]g(x))(f(x) - g(x)) = 0$ for all $x, y \in R$; in particular, when $y = f(x)$, we get $[g(x), f(x)][f(x), g(x)] = 0$ or $[f(x), g(x)]^2 = 0$. Since x is arbitrary, therefore $[f(x), g(x)]^2 = 0$ for all $x \in R$. Since a semiprime ring has no nontrivial central nilpotents, therefore $[f(x), g(x)] = 0$ for all $x \in R$. ∎

3. (α, β)-derivations

In this section we investigate some properties of (α, β)-derivations on semiprime rings.

PROPOSITION 3.1. *Let R be a semiprime ring and d be a central (α,β)-derivation of R. Then $d(u)[x,y]=0$ holds for all $x,y,u\in R$.*

Proof. Since d is central, therefore $0=[d(y\beta^{-1}(x)),x]=[\alpha(y)d(\beta^{-1}(x))+d(y)\beta(\beta^{-1}(x)),x]=[\alpha(y)d(\beta^{-1}(x)),x]+[d(y)x,x]=\alpha(y)[d(\beta^{-1}(x)),x]+[\alpha(y),x]d(\beta^{-1}(x))+d(y)[x,x]+[d(y),x]x=[\alpha(y),x]d(\beta^{-1}(x))$ for all $x,y\in R$. Since α is onto, we get $[y,x]d(\beta^{-1}(x))=0$. Replacing x by $\beta(x)$, we get

$$(11)\qquad [y,\beta(x)]d(x)=0 \text{ for all } x,y\in R.$$

Replacing y by yz in (11), we have $0=[yz,\beta(x)]d(x)=y[z,\beta(x)]d(x)+[y,\beta(x)]zd(x)$. Using (11), we get

$$(12)\qquad [y,\beta(x)]zd(x)=0 \text{ for all } x,y,z\in R.$$

Linearizing (11) (in x), we get

$$\begin{aligned}0&=[y,\beta(x+u)]d(x+u)\\&=[y,\beta(x)]d(x+u)+[y,\beta(u)]d(x+u)\\&=[y,\beta(x)]d(x)+[y,\beta(x)]d(u)+[y,\beta(u)]d(x)+[y,\beta(u)]d(u).\end{aligned}$$

Using (11), we have

$$(13)\qquad [y,\beta(x)]d(u)=-[y,\beta(u)]d(x).$$

Using (13) and replacing z by $d(u)v(-[y,\beta(u)]d(x))$ in (12), we get $0=[y,\beta(x)]d(u)v[y,\beta(x)]d(u)$. Since β is onto, therefore $[y,x]d(u)v[y,x]d(u)=0$ for all $x,y,u,v\in R$. Since d is central, we may write this expression as

$$(14)\qquad d(u)[x,y]v\,du[x,y]=0 \text{ for all } x,y,u\in R.$$

By (14) and the semiprimeness of R, we get $d(u)[x,y]=0$ for all $x,y,u\in R$. ∎

The following corollary shows that semiprime rings do not admit nontrivial central inner derivations.

COROLLARY 3.2. *Let d be a central inner derivation of a semiprime ring R, then $d=0$.*

Proof. Let d be the central inner derivation of R determined by $a\in R$. Thus, $d(u)=[a,u]\in Z(R)$ for all $u\in R$. Now, Proposition 3.1 implies $[a,u][x,y]=0$; in particular, $[a,u][a,u]=0$ for all $u\in R$. Since R has no nonzero central nilpotents, therefore $[a,u]=d(u)=0$ for all $u\in R$.

We now show that if β is centralizing, then every commuting (α,β)-derivation of a semiprime ring is central.

PROPOSITION 3.3. *Let β be centralizing and d a commuting (α,β)-derivation of a semiprime ring R. Then $[x,y]d(u)=0=d(u)[x,y]$ for all $x,y,u\in R$; in particular, d maps R into its center.*

Proof. Since β is a centralizing automorphism, therefore by Theorem 2.2 (taking $f = \beta, g = 1$), β is commuting. Thus $[\beta(x), y] = [x, \beta(y)]$. Also, $[d(x), y] = [x, d(y)]$ for all $x, y \in R$ (because d is commuting).

We consider

$$(15)\qquad [d(yx), x] = [yx, d(x)] = y[x, d(x)] + [y, d(x)]x = [y, d(x)]x$$

and

$$\begin{aligned}(16)\ [d(yx), x] &= [\alpha(y)d(x) + d(y)\beta(x), x]\\ &= [\alpha(y)d(x), x] + [d(y)\beta(x), x]\\ &= [\alpha(y), x]d(x) + \alpha(y)[d(x), x] + d(y)[\beta(x), x] + [d(y), x]\beta(x)\\ &= [\alpha(y), x]d(x) + [d(y), x]\beta(x) = [\alpha(y), x]d(x) + [y, d(x)]\beta(x).\end{aligned}$$

From (15) and (16), we get $[y, d(x)]x = [\alpha(y), x]d(x) + [y, d(x)]\beta(x)$. Thus

$$(17)\qquad [\alpha(y), x]d(x) = [y, d(x)](x - \beta(x)) = -[d(x), y](x - \beta(x))$$

for all $x, y \in R$. We further consider

$$\begin{aligned}(18)\qquad [x, \beta(xy)] &= [x, \beta(x)\beta(y)] = [x, \beta(x)]\beta(y) + \beta(x)[x, \beta(y)]\\ &= \beta(x)[x, \beta(y)],\end{aligned}$$

and

$$(19)\qquad [x, \beta(xy)] = [\beta(x), xy] = [\beta(x), x]y + x[\beta(x), y] = x[x, \beta(y)].$$

From (18) and (19), we get $\beta(x)[x, \beta(y)] = x[x, \beta(y)]$. Since β is onto, therefore

$$(20)\qquad \beta(x)[x, y] = x[x, y] \text{ for all } x, y \in R.$$

Replacing y by $d(y)$ in (20), we have

$$(21)\qquad 0 = [x - \beta(x)][x, d(y)] = [x - \beta(x)][d(x), y].$$

Using (21), from (17), we get $[\alpha(y), x]d(x) = 0$. Since α is onto, therefore

$$(22)\qquad [y, x]d(x) = 0 \text{ for all } x, y \in R.$$

Replacing y by yz in (22), we get $y[z, x]d(x) + [y, x]zd(x) = 0$, which along with (22) yields

$$(23)\qquad [y, x]zd(x) = 0 \text{ for all } x, y, z \in R.$$

Linearizing (22) (in x), we get

$$(24)\qquad [y, x]d(u) = [u, y]d(x) \text{ for all } x, y, u \in R.$$

Replacing z by $d(u)z[u, y]$ in (23), we have $0 = [y, x]d(u)z[u, y]d(x) = [y, x]d(u)z[y, x]d(u)$. The semiprimeness of R implies

$$(25)\qquad [y, x]d(u) = 0 \text{ for all } x, y, u \in R.$$

Replacing x by $d(u)$ in (25), we get $[y, d(u)]d(u) = 0$ for all $y \in R$. Thus $d(u) \in Z(R)$ from Herstein [9, Lemma 1.1.4]. Hence (25) implies $[x, y]d(u) = 0 = d(u)[x, y]$ for all $x, y, u \in R$. ∎

We close the paper with the following remarks.

REMARK 3.4. We observe that some well known results in the literature follow as application of our results.

(a) If we take $\beta = 1$ in Proposition 3.3, then it follows that every commuting α-derivation d of a semiprime ring R is central and satisfies

$$d(u)[x, y] = [x, y]d(u) = 0 \quad \text{for all } \ x, y, u \in R.$$

(b) Let α be a centralizing automorphism of a semiprime ring R, then $d = \alpha - 1$ is a commuting α-derivation. So by (a), $(\alpha - 1)$ maps R into its center and $(\alpha - 1)(u)[x, y] = [x, y](\alpha - 1)(u) = 0$; that is, $\alpha(u)[x, y] = u[x, y]$ and $[x, y]\alpha(u) = [x, y]u$ for all $x, y, u \in R$.

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