DEMONSTRATIO MATHEMATICA
Vol. XXXVII No 2 2004

Mohammad Ashraf, Nadeem-ur-Rehman

ON SEMIDERIVATIONS OF PRIME RINGS

Abstract. A semiderivation of a ring R is an additive mapping $f : R \longrightarrow R$ together with a function $g : R \longrightarrow R$ such that $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$ and $f(g(x)) = g(f(x))$, for all $x, y \in R$. If f is a non-zero semiderivation of a prime ring R, then it is well known that g must necessarily be an endomorphism. Let R be a prime ring with center $Z(R)$, f a non-zero semiderivation with associated endomorphism g which is one-one & onto, and σ, τ be two automorphisms of R such that $f\sigma = \sigma f, f\tau = \tau f, g\sigma = \sigma g, g\tau = \tau g$. Suppose that U is a non-zero (σ, τ)-Lie ideal of R and $C(R)_{\sigma,\tau} = \{c \in R \mid c\sigma(x) = \tau(x)c, \text{ for all } x \in R\}$. In the present paper it is shown that (i) if char $R \neq 2$ and $f(U) \subset C(R)_{\sigma,\tau}$, then R is commutative or $U \subset C(R)_{\sigma,\tau}$ (ii) if char $R \neq 2$ and $f^2(U) = 0$, then $U \subset Z(R)$ (iii) if char $R \neq 2$ and $f(U) \subset Z(R)$, then $U \subset Z(R)$ (iv) if char $R \neq 2, 3$ and $f(U) \subset U, f^2(U) \subset Z(R)$, then $U \subset Z(R)$.

1. Introduction

Let R be a ring with center $Z(R)$, and U an additive subgroup of R. For any $x, y \in R$; $[x, y]$ will denote the commutator $xy - yx$. Recall that a ring R is prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. An additive subgroup U of R is said to be a Lie ideal of R if $[U, R] \subset U$. Let $\sigma, \tau : R \longrightarrow R$ be two mappings. We set $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$. Then U is called a (σ, τ)-right Lie ideal (resp. (σ, τ)-left Lie ideal) if $[U, R]_{\sigma,\tau} \subset U$ (resp. $[R, U]_{\sigma,\tau} \subset U$). U is said to be (σ, τ)-Lie ideal of R if U is both a (σ, τ)-right Lie ideal as well as (σ, τ)-left Lie ideal of R. Note that every Lie ideal is a $(1, 1)$- right(left) Lie ideal of R. But there exist (σ, τ)-Lie ideals of R which are not Lie ideals of R. For example, let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| \, a, b \in \mathbb{Z} \right\}, \quad U = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| \, a \in \mathbb{Z} \right\},$$

1991 *Mathematics Subject Classification*: 16W25, 16U80.
Key words and phrases: prime rings, derivations, semiderivations, Lie ideals, torsion free rings.

$$\sigma\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad \tau\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.$$

Then σ and τ are automorphisms of R and U is a (σ,τ)-Lie ideal of R, but not a Lie ideal of R. Following Bergen [6], an additive mapping $f : R \longrightarrow R$ is called a semiderivation if there exists a function $g : R \longrightarrow R$ such that (i) $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$, and (ii) $f(g(x)) = g(f(x))$ hold for all $x, y \in R$. If $g = 1$ -i.e., an identity mapping of R, then all semiderivations associated with g are merely ordinary derivations. If g is any endomorphism of R, then other examples of semiderivations are of the form $f(x) = x - g(x)$. For an example of a semiderivation which is not a derivation, let $R = R_1 \oplus R_2$ where R_1 and R_2 are any rings. Let $\alpha_1 : R_1 \longrightarrow R_1$ be an additive map and $\alpha_2 : R_2 \longrightarrow R_2$ be a left and right R_2-module map which is not a derivation. Define $f : R \longrightarrow R$ such that $f((r_1, r_2)) = (0, \alpha_2(r_2))$ and $g : R \longrightarrow R$ such that $g((r_1, r_2)) = (\alpha_1(r_1), 0)$, $r_1 \in R_1$, $r_2 \in R_2$. Then it can be easily seen that f is a semiderivation on R (with associated map g) which is not a derivation. In case R is prime and $f \neq 0$, it has been shown by Chang [7, Theorem1] that g must necessarily be a ring endomorphism.

Let d be a non-zero derivation of R. Then for a Lie ideal U of R, Bergen et al. [5] proved the following: (i) If $d(U) \subset Z(R)$, then $U \subset Z(R)$. (ii) If $ad(U) = 0$ (or $d(U)a = 0$) for $a \in R$, then $a = 0$ or $U \subset Z(R)$. (iii) If $d^2(U) = 0$, then $U \subset Z(R)$. Further, the above results were extended to (σ,τ)-Lie ideals of R. (cf. [3], [16]). In the present paper our objective is to generalize these results for semiderivations. In fact our theorems generalize the results obtained in [1, Theorem 5], [3, Theorems 1 & 2] and [16, Theorem].

Throughout the present paper R will represent a prime ring with automorphisms σ, τ and a non-zero semiderivation f (with associated endomorphism g) such that $f\sigma = \sigma f, f\tau = \tau f, g\sigma = \sigma g$, $g\tau = \tau g$, and $C(R)_{\sigma,\tau} = \{x \in R \mid x\sigma(y) = \tau(y)x, \text{ for all } y \in R\}$. We shall use the following relations frequently:

$$[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y$$

and

$$[x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z).$$

2. Main results

We begin our discussion with the following theorem.

THEOREM 2.1. *Let R be 2-torsion free, U a non-zero (σ,τ)-right Lie ideal of R. If associated endomorphism g of f is onto and $f(U) \subset C(R)_{\sigma,\tau}$, then R is commutative or $U \subset C(R)_{\sigma,\tau}$.*

For easy references, we state the following known lemmas which will be used in our subsequent discussion.

LEMMA 2.1 ([2, Lemma 2]). *Let U be a non-zero (σ,τ)-right Lie ideal of R and $a \in R$. If $[U,a]_{\sigma,\tau} \in C(R)_{\sigma,\tau}$, then $a \in Z(R)$ or $U \subset C(R)_{\sigma,\tau}$.*

LEMMA 2.2 ([2, Corollary 2]). *Let U be a non-zero (σ,τ)- Lie ideal of R such that $U \not\subset Z(R)$ and $U \not\subset C(R)_{\sigma,\tau}$, for every $a,b \in R$. If $aUb = \{0\}$, then $a = 0$ or $b = 0$.*

LEMMA 2.3 ([3, Lemma 4]). *Let U be a non-zero (σ,τ)-left Lie ideal of R such that $U \subset C(R)_{\sigma,\tau}$, then $U \subset Z(R)$.*

LEMMA 2.4 ([16, Lemma1]). *Let U be a non-zero (σ,τ)-left Lie ideal of R. If $[R,U]_{\sigma,\tau} \subset Z(R)$, then $U \subset Z(R)$.*

The following lemma has its independent interest. It can also be regarded as a generalization of the main theorem due to Herstein [10] for semiderivation in the case when char $R \neq 2$.

LEMMA 2.5. *Let R be 2-torsion free, and associated endomorphism g of f be onto. If $a \in R$ such that $[a, f(x)] = 0$, for all $x \in R$, then $a \in Z(R)$.*

Proof. By our hypothesis, we have

$$[a, f(x)] = 0, \quad \text{for all } x \in R. \tag{2.1}$$

Replace x by xy in (2.1) and use (2.1), to get

$$f(x)[a,y] + [a, g(x)]f(y) = 0, \quad \text{for all } x,y \in R. \tag{2.2}$$

Now, replacing y by $y + f(y)$ in (2.2), and using (2.1) & (2.2), we get

$$[a, g(x)]f^2(y) = 0, \quad \text{for all } x,y \in R. \tag{2.3}$$

Replacing x by zx in (2.3) and using (2.3), we get $[a, g(z)]g(x)f^2(y) = 0$, for all $x,y,z \in R$. Hence $[a,g(z)]Rf^2(y) = \{0\}$, and the primeness of R implies that either $[a, g(z)] = 0$ or $f^2(y) = 0$. Now suppose that

$$f^2(y) = 0, \quad \text{for all } y \in R. \tag{2.4}$$

Replacing y by xy in (2.4), we get

$$f^2(x)g^2(y) + f(x)d(g(y)) + f(x)g(f(y)) + xf^2(y) = 0.$$

Now, applying (2.4) and the fact that $f(g(y)) = g(f(y))$, we have $2f(x)f(g(y)) = 0$, for all $x,y \in R$. This yields that

$$f(x)f(g(y)) = 0, \quad \text{for all } x,y \in R. \tag{2.5}$$

Replace x by yx in (2.5) and use (2.5), to get $f(y)xf(g(y)) = 0$, for all $x,y \in R$ and hence either $f(y) = 0$ or $f(g(y)) = 0$. But since g is onto in both the cases we find that $f(x) = 0$, for all $x \in R$, a contradiction. Hence $[a, g(z)] = 0$, for all $z \in R$ and since g is onto it implies the required result.

Proof of Theorem 2.1. Since U is a (σ,τ)-right Lie ideal of R; $[u,x]_{\sigma,\tau} \in U$, for all $x \in R, u \in U$. By our hypothesis, we have $f(U) \subset C(R)_{\sigma,\tau}$ and hence $f([x,u]_{\sigma,\tau}) \in C(R)_{\sigma,\tau}$ -i.e., $[f([u,x]_{\sigma,\tau}),y]_{\sigma,\tau} = 0$, for all $x,y \in R, u \in U$. This can be rewritten as

$$[[f(u),g(x)]_{\sigma,\tau},y]_{\sigma,\tau} + [[u,f(x)]_{\sigma,\tau},y]_{\sigma,\tau} = 0.$$

Since g is onto, we find that $[f(u),g(x)]_{\sigma,\tau} = 0$, and hence $[[u,f(x)]_{\sigma,\tau},y]_{\sigma,\tau} = 0$, for all $x,y \in R$, $u \in U$. This implies that $[u,f(x)]_{\sigma,\tau} \in C(R)_{\sigma,\tau}$ that is $[U,f(R)]_{\sigma,\tau} \subset C(R)_{\sigma,\tau}$. Hence application of Lemma 2.1 gives that $f(R) \subset Z(R)$ or $U \subset C(R)_{\sigma,\tau}$. If $f(R) \subset Z(R)$, then by Lemma 2.5 R is commutative.

Combining Lemma 2.3 with the above theorem we get the following:

COROLLARY 2.1. *Let R be 2-torsion free, and U a non-zero (σ,τ)-Lie ideal of R. If associated endomorphism g of f is onto and $f(U) \subset C(R)_{\sigma,\tau}$, then $U \subset Z(R)$.*

LEMMA 2.6. *Let R be 2-torsion free, U a non-zero (σ,τ)-Lie ideal of R, and associated endomorphism g of f be onto. If $a \in R$ such that $f(U)a = 0$ (or $af(U) = 0$), then $a = 0$ or $U \subset Z(R)$.*

Proof. Since U is a (σ,τ)-Lie ideal of R, $[x,u]_{\sigma,\tau} \in U$, for all $x \in R, u \in U$. Now replace x by $\tau(u)x$, to get $\tau(u)[x,u]_{\sigma,\tau} \in U$. Hence by our hypothesis, we find that $f(\tau(u)[x,u]_{\sigma,\tau})a = 0$, for all $x \in R, u \in U$. This yields that

$$f(\tau(u))[x,u]_{\sigma,\tau}a = 0, \quad \text{for all } x \in R, u \in U. \tag{2.6}$$

Replacing x by $xf(v)$, where $v \in U$ in (2.6) and using the hypothesis, we obtain $f(\tau(u))x[f(v),u]_{\sigma,\tau}a = 0$, for all $x \in R, u,v \in U$. Thus primeness of R forces that either $f(\tau(u)) = 0$ or $[f(v),u]_{\sigma,\tau}a = 0$. This implies that for each $u \in U$ either $f(u) = 0$ or $[f(v),u]_{\sigma,\tau}a = 0$. Define $H = \{u \in U \mid f(u) = 0\}, K = \{u \in U \mid [f(v),u]_{\sigma,\tau}a = 0, \text{ for all } v \in U\}$. Clearly H and K are additive subgroups of U and $U = H \cup K$. Hence by using Brauer's trick $K = U$ or $H = U$. Since $f(U) \neq 0, H \neq U$ and hence $K = U$ i.e., $[f(v),u]_{\sigma,\tau}a = 0$, for all $u,v \in U$. Now, in view of our hypothesis we get $f(v)\sigma(u)a = 0$ and hence $\sigma^{-1}(f(v))U\sigma^{-1}(a) = 0$. Hence application of Lemma 2.3 and Lemma 2.2 yields that $\sigma^{-1}(f(v)) = 0$ or $\sigma^{-1}(a) = 0$. This implies that $f(U) = 0$ or $a = 0$. But since $f(U) \neq 0$, we get the required result.

Using similar arguments with necessary variations, we get the required result in case if $af(U) = 0$.

LEMMA 2.7. *Let R be 2-torsion free, and U a non-zero (σ,τ)-Lie ideal of R. If associated endomorphism g of f is one-one & onto and $f^2(U) = 0$, then $f(U) \subset Z(R)$.*

Proof. Using the similar arguments as used in the begining of the proof of Lemma 2.6, we find that $\tau(u)[x,u]_{\sigma,\tau} \in U$, for all $x \in R, u \in U$. By our hypothesis, we have $f^2(\tau(u)[x,u]_{\sigma,\tau}) = 0$, for all $x \in R, u \in U$. This implies that

$$f^2(\tau(u))g^2([x,u]_{\sigma,\tau}) + f(\tau(u))f(g([x,u]_{\sigma,\tau}))+ \\ f(\tau(u))g(f([x,u]_{\sigma,\tau})) + \tau(u)f^2([x,u]_{\sigma,\tau}) = 0.$$

Since $f^2(U) = 0$ and $f(g(u)) = g(f(u))$, the above relation reduces to $2f(\tau(u))f(g([x,u]_{\sigma,\tau})) = 0$. This yields that

$$(2.7) \qquad f(\tau(u))f(g([x,u]_{\sigma,\tau})) = 0, \quad \text{for all } x \in R, u \in U.$$

It is eassy to show that $f(U) + U$ is a (σ,τ)-Lie ideal of R. In fact for any $u, v \in U, x \in R$, we have

$$\begin{aligned}[f(u)+v,x]_{\sigma,\tau} &= [f(u),x]_{\sigma,\tau} + [v,x]_{\sigma,\tau} \\ &= f([u,g(x)]_{\sigma,\tau}) + [v,x]_{\sigma,\tau} - [u,f(x)]_{\sigma,\tau} \in f(U) + U.\end{aligned}$$

This implies that $f(U) + U$ is a (σ,τ)-right Lie ideal of R. Similarly we can show that $f(U)+U$ is a (σ,τ)-left Lie ideal of R, and hence a (σ,τ)-Lie ideal of R. Further more if $f^2(U) = 0$, then $f(f(U) + U) \subset f(U) \subset f(U) + U$ and $f^2(f(U)+U) = 0$. Therefore, without loss of generality we may assume that if U is a (σ,τ)-Lie ideal of R such that $f^2(U) = 0$, then $f(U) \subset U$.

Now replace u by $u + f(v)$ in (2.7), to get $f(\tau(u))f(g([x,f(v)]_{\sigma,\tau})) = 0$, for all $x \in R, u, v \in U$, and hence $f(u)\tau^{-1}(f(g([x,f(v)]_{\sigma,\tau}))) = 0$. Now application of Lemma 2.6 gives that $U \subset Z(R)$ or $\tau^{-1}(f(g([x,f(v)]_{\sigma,\tau}))) = 0$. If $\tau^{-1}(f(g([x,f(v)]_{\sigma,\tau}))) = 0$, then $g(f([x,f(v)]_{\sigma,\tau})) = 0$, for all $x \in R, v \in U$. Since g is one-one, the last equation gives that $f([x,f(v)]_{\sigma,\tau}) = 0$, for all $x \in R, v \in U$. Thus if $U \subset Z(R)$, then $f(U) \subset Z(R)$. On the other hand if $f([x,f(v)]_{\sigma,\tau}) = 0$, then in view of our hypothesis the above relation reduces to

$$(2.8) \qquad [f(x),f(v)]_{\sigma,\tau} = 0, \quad \text{for all } x \in R, v \in U.$$

Replacing x by $xf(u)$ in (2.8), we get $f(x)[f(u),f(v)]_{\sigma,\tau} + [f(x),\tau(f(v))]f(u) = 0$, and in view of equation (2.8), we have $[f(x),\tau(f(v))]f(u) = 0$, for all $x \in R, u, v \in U$. Again application of Lemma 2.6 yields that $U \subset Z(R)$ or $[f(x),\tau(f(v))] = 0$. If $[f(x),\tau(f(v))] = 0$, then by using Lemma 2.5, we get $\tau(f(v)) \in Z(R)$, for all $v \in U$. This implies that $f(v) \in Z(R)$, for all $v \in U$ i.e., $f(U) \subset Z(R)$. On the other hand if $U \subset Z(R)$, then again $f(U) \subset Z(R)$.

THEOREM 2.2. *Let R be 2-torsion free, and U a non-zero (σ,τ)-Lie ideal of R. If associated endomorphism g of f is one-one & onto and $f^2(U) = 0$, then $U \subset Z(R)$.*

Proof. Since U is a (σ,τ)-Lie ideal of R, $[x,u]_{\sigma,\tau}\in U$, for all $x\in R, u\in U$. Now, replace x by $x\sigma(u)$, to get $[x,u]_{\sigma,\tau}\sigma(u)\in U$, for all $x\in R, u\in U$. Hence by our hypothesis we find that $f^2([x,u]_{\sigma,\tau}\sigma(u))=0$. This yields that

$$f^2([x,u]_{\sigma,\tau})g^2(\sigma(u))+f([x,u]_{\sigma,\tau})f(g(\sigma(u)))+\\ f([x,u]_{\sigma,\tau})g(f(\sigma(u)))+[x,u]_{\sigma,\tau}f^2(\sigma(u))=0.$$

Since $f^2(U)=0$ and $f(g(u))=g(f(u))$, the above relation reduces to

$$(2.9)\qquad f([x,u]_{\sigma,\tau})g(f(\sigma(u)))=0,\quad \text{for all } x\in R, u\in U.$$

Now, replacing u by $u+v$ in (2.9) and using (2.9), we get

$$f([x,v]_{\sigma,\tau})g(f(\sigma(u)))+f([x,u]_{\sigma,\tau})g(f(\sigma(v)))=0,\ \text{for all}\ \ x\in R, u,v\in U.$$

Multiplying from right by $g(f(\sigma(u)))$ in the last equation , we get

$$f([x,v]_{\sigma,\tau})g(f(\sigma(u))^2)+f([x,u]_{\sigma,\tau})g(f(\sigma(v))f(\sigma(u)))=0,\\ \text{for all}\ \ x\in R, u,v\in U.$$

Now application of Lemma 2.7 and (2.9) yields that

$$(2.10)\qquad f([x,v]_{\sigma,\tau})g(f(\sigma(u))^2)=0,\quad \text{for all } x\in R, u,v\in U.$$

Replacing x by $\tau(v)x$ in (2.10) and using (2.10),we get

$$(2.11)\qquad f(\tau(v))[x,v]_{\sigma,\tau}g(f(\sigma(u))^2)=0,\quad \text{for all } x\in R, u,v\in U.$$

Linearize (2.11) on v and use (2.11), to get

$$(2.12)\qquad f(\tau(v))[x,w]_{\sigma,\tau}g(f(\sigma(u))^2)+f(\tau(w))[x,v]_{\sigma,\tau}g(f(\sigma(u))^2)=0,\\ \text{for all } x\in R, u,v,w\in U.$$

Multiplying (2.12) from left by $f(\tau(v))$ and applying Lemma 2.7 and (2.11), we get

$$(2.13)\qquad f(\tau(v))^2[x,w]_{\sigma,\tau}g(f(\sigma(u))^2)=0,\quad \text{for all } x\in R, u,v,w\in U.$$

Replace x by $yf([x,w_1]_{\sigma,\tau})$ in (2.13), to get

$$f(\tau(v))^2\{y[f([x,w_1]_{\sigma,\tau}),\sigma(w)]+[y,w]_{\sigma,\tau}f([x,w_1]_{\sigma,\tau})\}g(f(\sigma(u))^2)=0.$$

Now in view of (2.10), we find that $f(\tau(v))^2R[f([x,w_1]_{\sigma,\tau}),\sigma(w)]g(f(\sigma(u))^2)$ $=\{0\}$, for all $x\in R, u,v,w,w_1\in U$ and hence primeness of R implies that either $f(\tau(v))^2=0$ or $[f([x,w_1]_{\sigma,\tau}),\sigma(w)]g(f(\sigma(u))^2)=0$. If $f(\tau(v))^2=0$ for all $v\in U$, then $\tau(f(v)^2)=0$ and hence $f(U)^2=0$. Thus for all $u,v\in U$ $0=f(u+v)^2=f(u)^2+2f(u)f(v)+f(v)^2$. Hence this yields that $f(u)f(v)=0$, for all $u,v\in U$, by Lemma 2.6 we get $f(U)=0$, and hence by Corollary 2.1, we have $U\subset Z(R)$. On the other hand if $[f([x,w_1]_{\sigma,\tau}),\sigma(w)]g(f(\sigma(u))^2)=0$, then application of (2.10) gives that $f([x,w_1])\sigma(w)g(f(\sigma(u))^2)=0$, for all $x\in R, u,w,w_1\in U$, and hence $\sigma^{-1}(f([x,w_1]_{\sigma,\tau}))Ug(f(u)^2)=0$. Thus by Lemma 2.2, we find that $\sigma^{-1}(f([x,w_1]_{\sigma,\tau}))=0$ or $g(f(u)^2)=0$. If

$g(f(u)^2) = 0$, then $f(u)^2 = 0$, for all $u \in U$ i.e., $f(U)^2 = 0$. Now using the similar arguments as above we get the required result. On the other hand if $\sigma^{-1}(f([x, w_1]_{\sigma,\tau})) = 0$, then

$$f([x, w_1]_{\sigma,\tau}) = 0, \quad \text{for all } x \in R, w_1 \in U. \tag{2.14}$$

Replace x by $x\sigma(w_1)$ in (2.14) and use (2.14), to get

$$[x, w_1]_{\sigma,\tau} f(\sigma(w_1)) = 0, \quad \text{for all } x \in R, w_1 \in U. \tag{2.15}$$

Replacing x by xy in (2.15) and using (2.15), we get $[x, \tau(w_1)]yf(\sigma(w_1)) = 0$, for all $x, y \in R, w_1 \in U$. Hence for each $w_1 \in U$ primeness of R forces that either $f(\sigma(w_1)) = 0$ or $[x, \tau(w_1)] = 0$, for all $x \in R$. Thus we find that for each $w_1 \in U$ either $f(w_1) = 0$ or $w_1 \in Z(R)$. Now we define $H = \{w_1 \in U \mid f(w_1) = 0\}, K = \{w_1 \in U \mid w_1 \in Z(R)\}$. Then it can be seen that H and K are additive subgroups of U. Moreover, $U = H \cup K$. But a group can not be a set theoretic union of two of its proper subgroups and hence $H = U$ or $K = U$. By assumption $U \not\subset Z(R)$ and therefore $U = H$. This gives that $f(U) = 0$ and by Corollary 2.1, $U \subset Z(R)$, a contradiction. This completes the proof of the above theorem.

THEOREM 2.3. *Let R be 2-torsion free, and U a non-zero (σ, τ)-Lie ideal of R. If associated endomorphism g of f is one-one & onto and $f(U) \subset Z(R)$, then $U \subset Z(R)$.*

Proof. By our hypothesis, we have $f([x, u]_{\sigma,\tau}) \in Z(R)$, for all $x \in R$, $u \in U$. Hence, replacing x by $xf(v)$ and using the fact that $f(U) \subset Z(R)$, we arrive at $g([x, u]_{\sigma,\tau})f^2(v) \in Z(R)$, for all $x \in R, u, v \in U$. Since $f(U) \subset Z(R)$ implies that $f^2(U) \subset Z(R)$ and R is prime, we find that either $f^2(v) = 0$ or $g([x, u]_{\sigma,\tau}) \in Z(R)$. If $f^2(v) = 0$, for all $v \in U$, then by using Theorem 2.2 we get the required result. On the other hand if $g([x, u]_{\sigma,\tau}) \in Z(R)$, then $[x, u]_{\sigma,\tau} \in Z(R)$, for all $x \in R, u \in U$ and by Lemma 2.4 we get the required result.

It can be eassily seen that in case associated endomorphism g of f is onto, $f(U) \subset Z(R)$ implies that $f^2(U) \subset Z(R)$. Thus it is natural to ask whether the conclusion of the above theorem remains true if the hypothesis $f(U) \subset Z(R)$ is replaced by a weaker hypothesis that $f^2(U) \subset Z(R)$. The following theorem, under some additional condition, provides an affirmative answer to this question and improve the results obtaind in [1, Theorems 1 & 5] and [16, Theorem].

THEOREM 2.4. *Let R be 2-torsion free and 3-torsion free, U a non-zero (σ, τ)-Lie ideal of R. If associated endomorphism g of f is one-one & onto and $f(U) \subset U$, $f^2(U) \subset Z(R)$, then $U \subset Z(R)$.*

Proof. Since U is a (σ, τ)-Lie ideal of R, $[x, u]_{\sigma,\tau} \in U$, for all $x \in R, u \in U$. Thus by our hypothesis, we find that $f^2([x, u]_{\sigma,\tau}) \in Z(R)$, for all $x \in R$,

$u \in U$. This yields that

$$(2.16) \qquad [f^2(x), g^2(u)]_{\sigma,\tau} + 2[f(x), g(f(u))]_{\sigma,\tau} + [x, f^2(u)]_{\sigma,\tau} \in Z(R), \quad \text{for all } x \in R, u \in U.$$

Replacing x by $xf^2(v)$ in (2.16) and using (2.16) together with the fact that $f^2(U) \subset Z(R)$, we get

$$2[g(f(x)), g^2(u)]_{\sigma,\tau} f^3(v) + 2g(f(x))[f^3(v), \sigma(g^2(u))] + [g^2(x), g^2(u)]_{\sigma,\tau} f^4(v) + g^2(x)[f^4(v), \sigma(g(u))] + 2[g(x), g(f(u))]_{\sigma,\tau} f^3(v) + 2g(x)[f^3(v), \sigma(g(f^2(u)))] \in Z(R).$$

Since $f^2(v), f^3(v)$ and $f^4(v)$ are in $Z(R)$, the above relation reduces to

$$2[g(f(x)), g^2(u)]_{\sigma,\tau} f^3(v) + [g^2(x), g^2(u)]_{\sigma,\tau} f^4(v) + 2[g(x), g(f(u))]_{\sigma,\tau} f^3(v) \in Z(R).$$

This implies that $2f([g(x), g(u)]_{\sigma,\tau}) f^3(v) + [g^2(x), g^2(u)]_{\sigma,\tau} f^4(v) \in Z(R)$, and hence

$$(2.17) \qquad 2f^3(v) g(f([x,u]_{\sigma,\tau})) + g^2([x,u]_{\sigma,\tau}) f^4(v) \in Z(R).$$

Replacing x by $xf^2(w)$ in (2.17) and using the fact that $f^2(U) \subset Z(R)$, we get

$$(2.18) \qquad \{2f^3(v) g(f([x,u]_{\sigma,\tau})) + g^2([x,u]_{\sigma,\tau}) f^4(v)\} g^2(f^2(w)) + 2f^3(v) g([x,u]_{\sigma,\tau}) g(f^3(w)) \in Z(R), \quad \text{for all } x \in R, u, v, w \in U.$$

Since $f^2(w)$ is central, we find that

$$(2.19) \qquad g^2(f^2(w)) \in Z(R), \quad \text{for all } w \in U.$$

Combining (2.17) and (2.19) with (2.18), we get $f^3(v) g([x,u]_{\sigma,\tau} f^3(w)) \in Z(R)$, for all $x \in R, u, v, w \in U$. But since R is prime and $f^3(U) \subset Z(R)$; the above relation yields that either $f^3(v) = 0$ or $g([x,u]_{\sigma,\tau} f^3(w)) \in Z(R)$. If $g([x,u]_{\sigma,\tau} f^3(w)) \in Z(R)$, then $[x,u]_{\sigma,\tau} f^3(w) \in Z(R)$, and again either $f^3(w) = 0$ or $[x,u]_{\sigma,\tau} \in Z(R)$. If $[x,u]_{\sigma,\tau} \in Z(R)$, for all $x \in R, u \in U$ then by Lemma 2.4, we find that $U \subset Z(R)$. Now, suppose that $f^3(U) = 0$. In view of the arguments given in the first paragraph of the proof of Lemma 2.6, we have $\tau(u)[x,u]_{\sigma,\tau} \in U$, for all $x \in R, u \in U$, and hence $f^3(\tau(u)[x,u]_{\sigma,\tau}) = 0$. This yields that

$$f^3(\tau(u)) g^3([x,u]_{\sigma,\tau}) + f^2(\tau(u)) f(g^2([x,u]_{\sigma,\tau})) + f^2(\tau(u)) g(f(g([x,u]_{\sigma,\tau}))) + rf(\tau(u)) f^2(g([x,u]_{\sigma,\tau})) + f^2(\tau(u)) g(g(f([x,u]_{\sigma,\tau}))) + f(\tau(u)) f(g(f([x,u]_{\sigma,\tau}))) + f(\tau(u)) g(f^2([x,u]_{\sigma,\tau})) + \tau(u) f^3([x,u]_{\sigma,\tau}) = 0.$$

Since $f^3(U) = 0$ and $f(g(u)) = g(f(u))$, the above gives that

$$3f^2(\tau(u))g^2(f([x,u]_{\sigma,\tau})) + 3f(\tau(u))g(f^2([x,u]_{\sigma,\tau})) = 0, \quad \text{for all } x \in R, u \in U.$$

This implies that $f^2(\tau(u))g^2(f([x,u]_{\sigma,\tau}))+f(\tau(u))g(f^2([x,u]_{\sigma,\tau})) = 0$. Now, replacing u by $f(u)$ in the above equation and using the fact that $f^3(U) = 0$, we have $f^2(\tau(u))g(f^2([x,f(u)]_{\sigma,\tau})) = 0$, for all $x \in R, u \in U$, and hence $f^2(u)\tau^{-1}(g(f^2([x,f(u)]_{\sigma,\tau}))) = 0$. Since $f^2(U) \subset Z(R)$ and R is prime, we have either $f^2(u) = 0$ or $\tau^{-1}(g(f^2([x,f(u)]_{\sigma,\tau}))) = 0$. This implies that for each $u \in U$ either $f^2(u) = 0$ or $f^2([x,f(u)]_{\sigma,\tau}) = 0$. Thus the set $H = \{u \in U \mid f^2(u) = 0\}, K = \{u \in U \mid f^2([x,f(u)]_{\sigma,\tau}) = 0, \text{ for all } x \in R\}$ are additive subgroups of U whose union is U. Hence we find that $U = H$ or $U = K$. If $U = H$, then we find that $f^2(U) = 0$. Hence by Theorem 2.2 we get the required result. On the other hand if $U = K$ then

$$f^2([x,f(u)]_{\sigma,\tau}) = 0, \quad \text{for all } x \in R, u \in U. \tag{2.20}$$

Replacing x by $x\sigma(f(u))$ in (2.20), we get $f^2([x,f(u)]_{\sigma,\tau}\sigma(f(u))) = 0$, for all $x \in R, u \in U$. -i.e.

$$f^2([x,f(u)]_{\sigma,\tau})\sigma(f(u)) + g(f([x,f(u)]_{\sigma,\tau}))f(\sigma(f(u)))+$$
$$f(g([x,f(u)]_{\sigma,\tau}))f^2(\sigma(u)) + g^2([x,f(u)]_{\sigma,\tau})f^3(\sigma(u)) = 0.$$

Now applying (2.20) and using the fact that $f^3(U) = 0$, we have $g(f([x,f(u)]_{\sigma,\tau}))f^2(\sigma(u)) = 0$, and hence $\sigma^{-1}(g(f([x,f(u)]_{\sigma,\tau})))f^2(u) = 0$, for all $x \in R, u \in U$. Since $f^2(U) \subset Z(R)$ and R is prime, we find that for each $u \in U$ either $f^2(u) = 0$ or $\sigma^{-1}(g(f([x,f(u)]_{\sigma,\tau}))) = 0$. Hence again using Brauer's trick we have either $f^2(u) = 0$ for all $u \in U$ or $g(f([x,f(u)]_{\sigma,\tau})) = 0$ for all $u \in U, x \in R$. If $f^2(u) = 0$, for all $u \in U$, then again by Theorem 2.2 we get $U \subset Z(R)$. On the other hand if $g(f([x,f(u)]_{\sigma,\tau})) = 0$, then

$$f([x,f(u)]_{\sigma,\tau}) = 0, \quad \text{for all } x \in R, u \in U. \tag{2.21}$$

Now, replace x by $x\sigma(f(u))$ in (2.21) and use (2.21), to get

$$[x,f(u)]_{\sigma,\tau}\sigma(f^2(u)) = 0, \quad \text{for all } x \in R, u \in U. \tag{2.22}$$

Again replacing x by xy in (2.22) and using (2.22), we find that $[x,\tau(f(u))]y\sigma(f^2(u)) = 0$, for all $x,y \in R, u \in U$. Now primeness of R implies that either $\sigma(f^2(u)) = 0$ or $[x,\tau(f(u))] = 0$. If $\sigma(f^2(u)) = 0$, then $f^2(u) = 0$, for all $u \in U$. Hence again by Theorem 2.2, we get the requried result. On the other hand if $[x,\tau(f(u))] = 0$, then $[\tau^{-1}(x), f(u)] = 0$, for all $x \in R, u \in U$. Thus $[y,f(u)] = 0$, for all $y \in R, u \in U$, implies that $f(U) \subset Z(R)$. Hence, by Theorem 2.3 we get the required result.

Acknowledgement. The authors are thankful to Professor Murtaza A. Quadri for encouragement and fruitful dicussions. They are also indebted to the referee for his valuable suggestions.

References

[1] R. Awtar, *Lie stucture in prime rings with derivations*, Publ. Math. (Debrecen) 31 (1984), 209–215.
[2] N. Aydin and H. Kandamer, *(σ,τ)-Lie ideals in prime rings*, Turkish J. Math. 18 (1994), 143–148.
[3] N. Aydin and M. Soyturk, *(σ,τ)-Lie ideals in prime rings with derivation*, Turkish J. Math. 19 (1995), 239–244.
[4] H. E. Bell and W. S. Martindale, *Semiderivations and commutativity in prime rings*, Canad. Math. Bull. 31(4) (1988), 500–508.
[5] J. Bergen, I. N. Herstein and J. W. Ker, *Lie ideals and derivations of prime rings*, J. Algebra 71 (1981), 259–267.
[6] J. Bergen, *Derivations in prime rings* , Canad. Math. Bull. 26 (1983), 267–270.
[7] J. C. Chang, *On semiderivations of prime rings*, Chinese J. Math. 12 (1984), 255–262.
[8] C. L. Chuang, *On the structure of semiderivations in prime rings*, Proc. Amer. Math. Soc. 108(4) (1990), 867–869.
[9] I. N. Herstein, *A note on derivations* , Canad. Math. Bull. 21(3) (1978), 369-370.
[10] I. N. Herstein, *A note on derivations* II, Canad. Math. Bull. 22(4) (1979), 509–511.
[11] A. Kaya, *On a generalisation of Lie ideals in prime rings* , Turkish J. Math. 21 (1997), 285–294.
[12] K. Kaya, *(σ,τ)-right Lie ideals in prime rings* , Proc. 4th National Math. Symposium Antakya (1991).
[13] K. Kaya, *(σ,τ)-derivation of prime ring* , Douga Mat.12(2) (1988), 46–51.
[14] P. H. Lee and T. K. Lee, *Lie ideals of prime rings with derivations*, Bull. Inst. Math. (Academia Sinica) 11 (1983), 75–79.
[15] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. 8 (1957), 1093–1100.
[16] M. Soyturk, *(σ,τ)-Lie ideals in prime rings with derivation*, Turkish J. Math. 20 (1996), 233–236.

Mohammad Ashraf
DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH - 202 002, INDIA
e-mail : mashraf80@hotmail.com

Nadeem-ur-Rehman
MATHEMATICS GROUP
BIRLA INSTITUTE OF TECHNOLOGY & SCIENCE
PILANI (RAJASTHAN) - 333031, INDIA
e-mail : rehman100@postmark.net

Received September 16, 2002; revised version September 16, 2003.