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NOTE ON LOGICS OF IDEMPOTENTS

Abstract. The main result of this paper is the characterization of certain logics of idempotents by Boolean semirings. Moreover some interesting examples are likewise added.

1. Introduction

Let R be a ring with identity 1. Denote by $U(R)$ the set of all idempotents of the ring R. The following definition will play an important role in the sequel:

DEFINITION 1. Let $(L, \leq, 0, 1, ')$ be a poset with 0 and 1 as the least and the greatest element, respectively, and a unary operation $': L \to L$ (the *orthocomplementation*) such that:

(i) $p \leq q \Rightarrow q' \leq p', \ p, q \in L$
(ii) $(p')' = p, \ p \in L$
(iii) $p \vee p' = 1, \ p \in L$
(iv) $p \leq q' \Rightarrow p \vee q$ exists in L, $p, q \in L$
(v) $p \leq q \Rightarrow q = p \vee (p' \wedge q), \ p, q \in L.$

Then L will be called a *logic* or also an *orthomodular poset*. If L is also a lattice, then L is called an *orthomodular lattice*.

DEFINITION 2. Let L be a logic. A subset S of L is said to be a *sublogic* of L if the following conditions are satisfied:

(i) If $p \in S$ then $p' \in S$.
(ii) If $p, q \in S$ and $p \leq q'$, then $p \vee q \in S$.

Let L be a logic. We say that $p, q \in L$ are *orthogonal* ($p \perp q$) if $p \leq q'$.

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Let R be an associative ring with identity. In this paper we will furthermore suppose that the order in the set $U(R)$ is defined always by setting

$$(p \leq q) \Leftrightarrow (pq = qp = p), \; p, q \in U(R) \tag{1}$$

and the orthocomplement by

$$p' = 1 - p, \; p \in U(R). \tag{2}$$

It is well-known (see [2], [4], [5]) that the set $U(R)$ is a logic with regard to conditions (1) and (2). In the next section we give a generalization of this result.

Now we give some useful definitions and notions. An orthomodular lattice is a Boolean algebra if and only if it is distributive.

DEFINITION 3. Let R be an associative ring with identity. A bijective mapping $\alpha: R \to R$ is said to be an *automorphism* of R if

(i) $\alpha(a + b) = \alpha(a) + \alpha(b), \; a, b \in R$
(ii) $\alpha(ab) = \alpha(a)\alpha(b), \; a, b \in R$
(iii) $\alpha(1) = 1$.

If $p \in U(R)$ then the element $1 - 2p$ is invertible and it can be shown that the mapping $\alpha_p: R \to R$ defined by setting $\alpha_p(x) = (1-2p)\, x\, (1-2p)$, $x \in R$, is an automorphism of R.

Let α be an automorphism of the ring R. We denote by $\alpha \mid U(R)$ the restriction of α to $U(R)$.

PROPOSITION 1. *Let R be an associative ring with identity. If α is a ring automorphism of R, then $\alpha \mid U(R)$ is an automorphism of the logic $(U(R), \leq, 0, 1, ')$ onto itself.*

Proof. The proof is clear. ■

Notice that many results which concern automorphisms of logics can be found in [12].

DEFINITION 4 ([8]). The algebra $(S, +, \cdot, 0, 1)$ is said to be a *semiring* if the following conditions are satisfied:

(i) The algebra $(S, +, 0)$ is a commutative monoid with a neutral element 0.
(ii) The algebra $(S, \cdot, 1)$ is a monoid with a neutral element 1.
(iii) If $x, y, z \in S$ then

$$x(y + z) = xy + xz, \quad (y + z)x = yx + zx.$$

(iv) $x \cdot 0 = 0 \cdot x = 0, \; x \in S$.

DEFINITION 5. A semiring $(S, +, \cdot, 0, 1)$ is said to be a *Boolean semiring* if all its elements are idempotents.

Concerning semirings see the book [8] and the papers [14], [15].

Let R be an associative commutative ring with identity. Let us now provide R with the following operations:

$$x \oplus y = x + y - xy, \quad x \cdot y = xy, \quad x, y \in R. \tag{3}$$

Then $(U(R), \oplus, \cdot, 0, 1)$ is a commutative semiring. The Boolean algebra $B = (\exp X, \cup, \cap, \emptyset, X)$ is a Boolean semiring.

Let us introduce another example of a Boolean semiring.

EXAMPLE 1. Let R be an associative ring with identity and suppose that $p, q \in U(R) \backslash \{0, 1\}$, $p \neq q$ and $pq = qp$. Then the set

$$B_{p,q} = \{0, p, q, 1-p, 1-q, pq, 1-pq, p-pq, q-pq, 1-p+pq,$$
$$1-q+pq, p+q-pq, 1-p-q+pq, p+q-2pq, 1-p-q+2pq, 1\}$$

is a Boolean semiring which is generated by two idempotents $p, q \in U(R)$, i.e. $(B_{p,q}, \oplus, \cdot, 0, 1)$ is a Boolean semiring which is equipped with the operations $\oplus, \cdot$ defined by (3).

2. Characterization of logics of idempotents

In this section we introduce first of all some conditions which guarantee that a subset S of a logic L is a sublogic of L. We introduce at the same time some examples and also certain consequences and conclusions which immediately follow.

THEOREM 1. *Let R be an associative ring with identity and let S be a subset of $U(R)$. The following condition is sufficient for S to be a sublogic of $U(R)$:*

$$\text{If } p, q \in S \text{ and if } pq = qp \text{ then } B_{p,q} \subset S. \tag{4}$$

Proof. The proof is given in [6] and [7]. Remark furthermore that condition (4) guarantees the existence of the elements $p \vee q$, $p \wedge q$ if $p, q \in B_{p,q}$. In this case $p \vee q = p + q - pq$ and $p \wedge q = pq$. ∎

Suppose now that C is the commutative field of all complex numbers. Denote by $\mathcal{M}_{22}(C)$ the set of all $(2, 2)$-matrices over C. The set $\mathcal{M}_{22}(C)$ is a noncommutative ring with identity.

We introduce now some examples of logics.

EXAMPLE 2. The idempotents of the ring $\mathcal{M}_{22}(C)$ are the following matrices:

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where

$$a, b, c, d \in C, \quad Tr(A) = 1, \qquad \det(A) = 0.$$

It can be shown that $U(\mathcal{M}_{22}(C))$ is a logic which is a lattice.

The following example has undeniable connections with the foundation of the set of all states of the spin of an electron and therefore it belongs to the branch of quantum theory.

EXAMPLE 3. Let H_2 be a two-dimensional Hilbert space over the complex numbers. The space H_2 among others corresponds to the set of all states of the spin of one electron. Let S be the set of $(2,2)$-matrices of the following forms:

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} \frac{1}{2} & \frac{-i}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_i' = E - M_i\,, \quad i = 1,2,3.$$

It can be shown that the set S satisfies condition (4) of Theorem 1. Therefore $(S, \leq, O, E, {}')$ is a sublogic of the logic $U(\mathcal{M}_{22}(C))$ of all $(2,2)$-matrices over the field C. This sublogic S is an orthomodular lattice. Furthermore it is possible to show that

$$S = \bigcup_{i=1,2,3} B_{M_i, M_i'}\,. \tag{5}$$

The sublogic S is generated by the idempotent matrices M_1, M_2, M_3. Notice that there exists a connection between the matrices M_i, $i = 1,2,3$, and the Pauli matrices δ_i, $i = 1,2,3$. (See [6], [7].) Remember that the logic S can be generated also by the $(0,1)$-*pasting* of Boolean subalgebras $B_i = \{M_i, M_i', 0, E\}$, $i = 1,2,3$ (see [11] or a special case of Definition 7 below).

It is important to give some conditions for a given sublogic S of $U(R)$ to be a Boolean algebra. The following proposition gives a sufficient condition.

PROPOSITION 2. *Let R be an associative ring with identity and let S be a subset of $U(R)$ satisfying condition* (4) *of Theorem* 1 *and, moreover, assume that*

(6) *all elements of S are pairwise commutative.*

Then the sublogic $(S, \leq, 0, 1, {}')$ is a Boolean algebra.

Proof. The proof is given in [7]. ∎

Condition (6) of Proposition 2 is a solution of an open problem introduced in [6].

COROLLARY 1. *Let R be an associative ring with identity and suppose that $p, q \in U(R) \backslash \{0,1\}$, $p \neq q$, and $pq = qp$. Then the set $B_{p,q}$ (see Example 1) is a Boolean subalgebra of the logic $U(R)$.*

Further examples of (Boolean) subalgebras of logics of idempotents can be found in [3], [4], [5]. Determination of certain Boolean subalgebras of logics of idempotents may play an important role in the investigation of physical systems, because then it is possible to use the methods of classical physics for the solution of some problems which appear in physical systems.

In the following example we show the structure of the set $B_{p,q}$, $p, q \in U(R)$.

EXAMPLE 4. Let $B_{p,q}$ be the set introduced in Example 1. According to the commutativity of the elements p, q it follows that all elements of $B_{p,q}$ must be pairwise commutative. Therefore, by Proposition 2, the set $B_{p,q}$ is a Boolean subalgebra of the logic $U(R)$. The Boolean subalgebra $B_{p,q}$ is drawn in Figure 1.

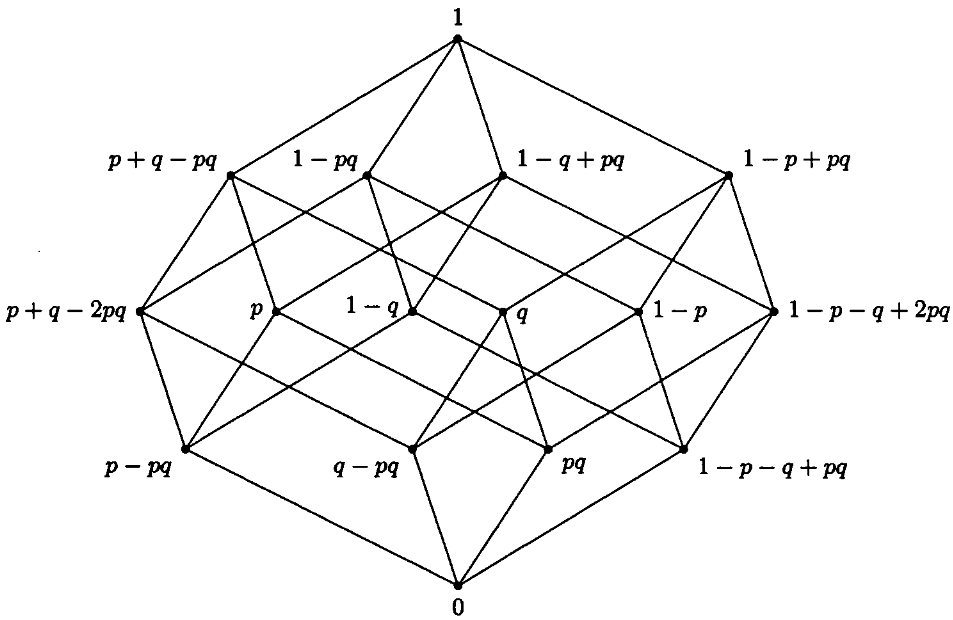

Fig. 1.

Notice that the structure of every sublogic S of the logic $U(R)$ can be completely described by its Boolean subalgebras $B_{p,q}$, $p, q \in S$.

REMARK. It is well-known (see [11]) that to each physical system S it is possible to associate a logic L of propositions so that to the elements of S correspond to the propositions which can be verified by experiments. The ordering of the logic L, resp. the orthocomplementation on L, correspond to the implication, resp. to the negation, of the propositions. In accord with this interpretation to the lattice operations $\wedge$, resp. $\vee$, correspond the logical operations of conjunction, resp. disjunction, of propositions. Then of course all elements of $B_{p,q}$ can be expressed with the help of $\wedge, \vee$, and $'$. Therefore for example we have $p + q - pq = p \vee q$, $1 - pq = p' \vee q'$, $1 - p + pq = p' \vee q$, $1 - q + pq = p \vee q'$, $p, q \in U(R)$. This last result is completely in accord with results of the monograph [11].

As a conclusion of this section we introduce a characterization of a sublogic S of $U(R)$.

THEOREM 2. *Let R be an associative ring with identity and let S be a sublogic of the logic $U(R)$. Suppose furthermore that S satisfies condition* (4) *introduced by Theorem 1. Then*

$$S = \bigcup_{p,q \in S} B_{p,q}. \tag{7}$$

Proof. By condition (4) we have $B_{p,q} \subset S$ if $p, q \in S$. Therefore $\bigcup_{p,q \in S} B_{p,q} \subset S$. But then we have also the reverse inclusion, i.e.,

$$S \subset \bigcup_{p,q \in S} B_{p,q}.$$

The equality (7) is also satisfied, so the proof is finished. ■

3. Concluding notices

In this section we consider some results which concern the pasting.

DEFINITION 6. *Let L be a logic. A maximal Boolean subalgebra B of L is called a block of L.*

Now we introduce the general pasting technique from [10].

DEFINITION 7. Let $\mathcal{L}$ be a family of logics such that all $P, Q \in \mathcal{L}$, $P \neq Q$, satisfy the following conditions:

(i) $P \not\subseteq Q$,

(ii) $P \cap Q$ is a sublogic of both P and Q and the partial orderings and the orthocomplementations of P and Q coincide on $P \cap Q$.

Endow the set $L = \bigcup_{M \in \mathcal{L}} M$ with the relation $\leq_L$ and the unary operation $'^L$ defined as follows:

$$a \leq_L b \quad (a = b'^L, \text{resp.}) \text{ iff } a \leq_M b \quad (a = b'^M, \text{resp.}) \text{ for some } M \in \mathcal{L}$$

The set L with $\leq_L$, $'^L$ is called the *pasting* of the family $\mathcal{L}$.

Historically, R. Greechie was the first who utilized the pasting technique for constructing "stateless logics". In the paper [13], V. Rogalewicz has shown that every orthomodular poset is a pasting of Boolean algebras. In this paper we introduce only one example using the pasting technique for the description of the structure of the system of n electrons which do not mutually interact.

A pasting of n blocks $B_1, B_2, \ldots, B_n$ which intersect only in the elements $0, 1$ will be called the $(0,1)$-pasting of $B_1, B_2, \ldots, B_n$ and denoted by $B_1 \oplus B_2 \oplus \ldots \oplus B_n$. For further information see [10] and [16]. Remember that the logic S from Example 3 has as blocks the sets $B_i = \{M_i, M_i', 0, E\}$, $i = 1, 2, 3$, so that S can be obtained by pasting of its blocks B_i, i.e. $S = B_1 \oplus B_2 \oplus B_3$. If we investigate the system which consists of n electrons (without mutual interactions) then each electron e_i, $i = 1, 2, \ldots, n$, can be characterized by three proper Boolean subalgebras $B_{3i}, B_{3i+1}, B_{3i+2}$. The logic S which corresponds to the system of n electrons can be expressed by pasting of its blocks B_k, $k = 1, 2, 3, \ldots, 3n$. The following figure shows the Hasse diagram of that logic S.

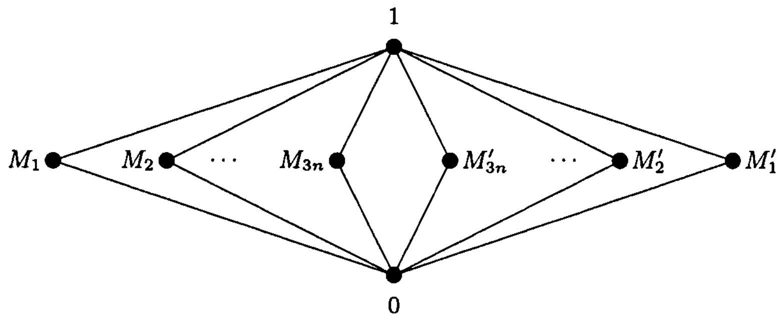

Fig. 2.

As introduced above (see Example 2), the logic $U(\mathcal{M}_{22}(R))$ is an orthomodular lattice if R is the field of all complex numbers. This proposition is also valid if R is either a commutative field or an integral domain with identity. In contrast to this, the logic $U(\mathcal{M}_{nn}(R))$ with $n \geq 3$ is not a lattice. This fact follows immediately from [9], Prop. 1. Moreover notice that all blocks of the logic $U(\mathcal{M}_{nn}(R))$ have only the following form: $\mathcal{B}_A = \{0, E, A, E - A\}$, $A \in U(\mathcal{M}_{nn}(R))$.

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