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BOOLEAN CARRIED HOMOMORPHISMS IN ORTHOMODULAR LATTICES

Abstract. Let L, L_1 be orthomodular lattices. Let us say that a surjective homomorphism $f : L \to L_1$ is Boolean carried if for any maximal Boolean subalgebra B_1 of L_1 there is a maximal Boolean subalgebra B of L such that $f(B) = B_1$. In this note we investigate the class $\mathcal{H}_{\mathcal{OML}}$ of all L's such that all surjective homomorphisms from L to orthomodular lattices are Boolean carried. We prove as a main result that if L possesses at most countably many infinite maximal Boolean subalgebras then $L \in \mathcal{H}_{\mathcal{OML}}$. We also relate the class $\mathcal{H}_{\mathcal{OML}}$ to the classes previously studied and provide some model-theoretic properties of $\mathcal{H}_{\mathcal{OML}}$.

1. Preliminaries

We assume the basic notions of the theory of OMLs, universal algebra and model theory to be known; the reader can find the necessary information in e.g. [11], [6], [7]. For the convenience of the reader, let us briefly review the basic notions of the theory of OMLs as we shall use them in the sequel.

DEFINITION 1.1. An *orthomodular lattice* (abbr., an OML) is an algebra $L = (X, \wedge, \vee, ^\perp, \mathbf{0}, \mathbf{1})$ of the type (2,2,1,0,0) such that L is an orthocomplemented lattice satisfying the orthomodular law: If $x \leq y$, then $y = x \vee (y \wedge x^\perp)$.

Let us denote by $\mathcal{OML}$ the class of all orthomodular lattices, and let us denote by $\mathcal{BA}$ the class of all Boolean algebras.

A subset K of an OML L is called a *subOML* of L if K is a subalgebra of L. If K is Boolean, then it is called a *Boolean subalgebra.*

DEFINITION 1.2. Let L be an OML. For $x, y \in L$, let $com(x, y)$ denote the *commutator* of x, y, i.e. $com(x, y) = (x \vee y) \wedge (x \vee y^\perp) \wedge (x^\perp \vee y) \wedge (x^\perp \vee y^\perp)$. (It should be noted that the notion of commutator is often defined in the

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dual way. For instance, in [1] this notion is defined in terms of lower and upper commutator.)

Elements x, y of L are called *commutative* (abbr., xCy), if $com(x, y) = \mathbf{0}_L$. It is easily seen that x, y are commutative if and only if they are contained in a Boolean subalgebra of L (see e.g. [11]). Let us put $C(L) = \{a \in L; aCb \text{ for any } b \in L\}$ and call $C(L)$ the *centre* of L. As known ([11]), $C(L)$ is a Boolean subalgebra of L.

PROPOSITION 1.3. *Suppose that $L \in \mathcal{OML}$ and $c \in C(L)$. Then $L \cong [\mathbf{0}, c] \times [\mathbf{0}, c^{\perp}]$. More explicitly, the mapping $h : L \to [\mathbf{0}, c] \times [\mathbf{0}, c^{\perp}]$ defined by putting $h(x) = (x \wedge c, x \wedge c^{\perp})$ is an isomorphism of L onto $[\mathbf{0}, c] \times [\mathbf{0}, c^{\perp}]$.*

Proof is easy (see [11, p. 20]).

PROPOSITION 1.4. *Suppose that $L_1, L_2, L \in \mathcal{OML}$ and $f : L_1 \times L_2 \to L$ is a surjective homomorphism. Put $c_1 = f(\mathbf{1}_{L_1}, \mathbf{0}_{L_2})$, $c_2 = f(\mathbf{0}_{L_1}, \mathbf{1}_{L_2})$. Then $c_1, c_2 \in C(L)$ and $c_2 = c_1^{\perp}$. Moreover, if we define mappings $f_i : L_i \to L$ $(i = 1, 2)$ by putting $f_1(x) = f(x, \mathbf{0})$, $f_2(y) = f(\mathbf{0}, y)$, when $x \in L_1$ and $y \in L_2$, then f_i $(i = 1, 2)$ become surjective homomorphisms from L_i onto $[\mathbf{0}, c_i]$.*

Proof. Let us sketch the proof of Prop. 1.4 for L_1. If $x \in L_1$, then obviously $(x, \mathbf{0}) \leq (\mathbf{1}, \mathbf{0})$. Since f preserves the ordering, we have $f(x, \mathbf{0}) \leq f(\mathbf{1}, \mathbf{0})$. Thus, $f_1(x) \leq c_1$ and therefore f_1 maps L_1 into the interval $[\mathbf{0}, c_1]$.

To show that f_1 is a morphism, take $x, y \in L_1$. Then we obtain

$$\begin{aligned} f_1(x \wedge y) &= f(x \wedge y, \mathbf{0}) = f((x, \mathbf{0}) \wedge (y, \mathbf{0})) = \\ &= f(x, \mathbf{0}) \wedge f(y, \mathbf{0}) = f_1(x) \wedge f_2(y); \\ f_1(x^{\perp}) &= f(x^{\perp}, \mathbf{0}) = f((x^{\perp}, \mathbf{1}) \wedge (\mathbf{1}, \mathbf{0})) = f(x^{\perp}, \mathbf{1}) \wedge f(\mathbf{1}, \mathbf{0}) = \\ &= (f(x, \mathbf{0}))^{\perp} \wedge c_1 = (f_1(x))^{\perp} \wedge c_1 = (f_1(x))^{\perp_{[\mathbf{0}, c_1]}}. \end{aligned}$$

We see that f_1 preserves the operations $\wedge$ and $^{\perp}$ and therefore it has to be a morphism in OML's.

Finally, let us verify that f_1 is surjective. Suppose that $d \in [\mathbf{0}, c_1]$ and choose such element (x, y) that $f(x, y) = d$. Then $f_1(x) = f(x, \mathbf{0}) = f((x, y) \wedge (\mathbf{1}, \mathbf{0})) = f(x, y) \wedge f(\mathbf{1}, \mathbf{0}) = d \wedge c_1 = d$.

DEFINITION 1.5. Let $L \in \mathcal{OML}$ and let $I \subseteq L$. Let us call I an *ideal* in L if the following conditions are satisfied:

(i) $a \in I,\ b \leq a \Rightarrow b \in I$,
(ii) $a,\ b \in I \Rightarrow a \vee b \in I$.

If I is an ideal and if there is an element $a \in L$ such that $I = [\mathbf{0}, a]$, where $[\mathbf{0}, a] = \{b \in L; b \leq a\}$, then the ideal I is called *principal*. An important example of an ideal in L is the *commutator ideal*, I_c, where I_c is the ideal generated by all elements of the form $com(a,b)$ $(a,b \in L)$.

DEFINITION 1.6. Let V be a nontrivial variety of OML's and let X be a set. Let us denote by $\mathbf{F}_X(V)$ the free OML over the set X in the variety V. For simplicity, let us agree to write $\mathbf{F}_X$ (resp. $\mathbf{B}_X$) instead of $\mathbf{F}_X(\mathcal{OML})$ (resp. $\mathbf{F}_X(\mathcal{BA})$).

DEFINITION 1.7. Let V be a variety of algebras and let $L \in V$. Let us say that L is a *projective algebra* in V if the following statement holds:

If $K \in V$ and if $f : K \to L$ is a surjective homomorphism, then there is a homomorphism $g : L \to K$ such that $g \circ f = id_L$, where $(g \circ f)(x) := f(g(x))$.

PROPOSITION 1.8. *Let V be a variety of algebras, and let $F \in V$ be a free algebra in V. Then F is a projective algebra in V.*

PROPOSITION 1.9. *Let X be an uncountable set, $f : \mathbf{F}_X \to \mathbf{B}_X$ be the uniquely defined homomorphism with $f(x) = x$ for any $x \in X$. Then there is no homomorphism $g : \mathbf{B}_X \to \mathbf{F}_X$ such that $g \circ f = id_{\mathbf{B}_X}$.*

Proof. See the proof of the main theorem of the paper [5].

THEOREM 1.10 (Bruns, Roddy [4, 5]). *Let $B \in \mathcal{BA}$. Then B is a projective algebra in the variety $\mathcal{OML}$ if and only if B is at most countable.*

Let us recall [7] that a subalgebra G of an algebra F in a language $\mathcal{L}$ is said to be an *elementary subalgebra*, $G \preceq F$, if for any formula $\varphi(x_1, \ldots, x_n)$ of $\mathcal{L}$ and any $a_1, \ldots, a_n \in G$, $a_1, \ldots, a_n$ satisfies φ in G if and only if it satisfies φ in F. The following two facts are easy to prove.

THEOREM 1.11. *Let V be a variety of algebras, and let $F \in V$ be a free algebra in V over an infinite set X. Let $Y \subseteq X$ be an infinite subset and G be the subalgebra of F generated by the set Y. Then $G \preceq F$.*

COROLLARY 1.12. *Let V be a variety of algebras, and let $F_X, F_Y \in V$ be free algebras in V over infinite sets X, Y. Then $F_X \equiv F_Y$ (i.e. the algebras F_X and F_Y are* elementarily equivalent).

2. Blocks in OMLs

DEFINITION 2.1. Let L be an OML. A maximal Boolean subalgebra of L is called a *block* in L. The collection of all blocks in L will be denoted by $Bl(L)$.

The following two propositions can be found in e.g. [11, p. 38, 39].

PROPOSITION 2.2. *Suppose that $L \in \mathcal{OML}$, $X \subseteq L$ and the elements of the set X are pairwise commutative. Then there exists a block B of L such that $X \subseteq B$.*

PROPOSITION 2.3. *If B is a block of the orthomodular lattice L then the atoms of B are atoms of L.*

PROPOSITION 2.4. (i) *Suppose that $L, L_1 \in \mathcal{OML}$ and suppose that L_1 is a subOML of L. Suppose further that $B_1 \subseteq L_1$. Then $B_1 \in Bl(L_1)$ if and only if $B_1 = L_1 \cap B$ for a block $B \in Bl(L)$.*

(ii) *Suppose that $L_i \in \mathcal{OML}$ $(i \in I)$. Put $L = \mathbf{X}_{i\in I} L_i$, where $\mathbf{X}_{i\in I} L_i$ is the Cartesian product of L_i $(i \in I)$ (endowed with the operations coordinatewise). Then $B \in Bl(L)$ if and only if $B = \mathbf{X}_{i\in I} B_i$, where every B_i is a block in the corresponding L_i.*

Proof is easy (see e.g. [3, 12]).

THEOREM 2.5. *Let V be a nontrivial variety of OML's and let X be an infinite set. Then the algebra $\mathbf{F}_X(V)$ is atomless.*

Proof. Write $F = \mathbf{F}_X(V)$. Let a be an arbitrary element of F different from $\mathbf{0}_F$. Then there exists a term t and elements $x_1, \ldots, x_n \in X$ such that $a = t_F(x_1, \ldots, x_n)$. Let us choose $y \in X$ different from all $x_1, \ldots, x_n$. Such a choice is possible since X is infinite. We are going to show that $\mathbf{0}_F < a \wedge y < a$. Clearly, $\mathbf{0}_F \leq a \wedge y \leq a$. Let $f : F \to F$ be the uniquely defined homomorphism such that $f(y) = \mathbf{1}_F$ and $f(x) = x$ for any element $x \in X$, $x \neq y$. Then $f(a \wedge y) = f(a) \wedge f(y) = t_F(f(x_1), \ldots, f(x_n)) \wedge \mathbf{1}_F = t_F(x_1, \ldots, x_n) = a$. Thus, $f(a \wedge y) \neq \mathbf{0}_F = f(\mathbf{0}_F)$. It follows that $\mathbf{0}_F \neq a \wedge y$. On the other hand, let $g : F \to F$ be the uniquely defined homomorphism such that $g(y) = \mathbf{0}_F$ and $f(x) = x$ for any element $x \in X$, $x \neq y$. Then $g(a \wedge y) = g(a) \wedge g(y) = \mathbf{0}_F$, $g(a) = t_F(g(x_1), \ldots, g(x_n)) = t_F(x_1, \ldots, x_n) = a$. Thus, $g(a \wedge y) \neq g(a)$. It follows that $a \wedge y \neq a$ and the proof is complete.

COROLLARY 2.6. *Let V be a nontrivial variety of OML's and let X be an infinite set. Then every block of $\mathbf{F}_X(V)$ is infinite.*

Proof. Suppose that B is a finite block of $\mathbf{F}_X(V)$. Then B posesses an atom, a. According to Prop. 2.3, the element a is an atom in L. This is a contradiction with Theorem 2.5.

THEOREM 2.7. *Let V be a variety of OML's such that $\mathcal{BA}$ is a proper subclass of V. Let X be an infinite set. Then the free algebra $\mathbf{F}_X(V)$ posesses uncountably many blocks.*

Proof. Write $F = \mathbf{F}_X(V)$. Consider a countable infinite subset $Y = \{x_1, x_2, \ldots\} \subseteq X$. Let $M = \{m_1, m_2, \ldots\}$ be an infinite subset of the set $\mathbf{N}$ of natural numbers, $m_1 < m_2 < \ldots$. Let us set $t_k = x_{m_1} \vee x_{m_2} \vee \ldots \vee x_{m_k}$ (where the operation $\vee$ is taken in F), $k = 1, 2, \ldots$. Then we have $t_1 < t_2 < \ldots$ in the algebra F. Thus, all the elements $t_1, t_2, \ldots$ of the algebra F are pairwise commutative and therefore (see Prop. 2.2) there exists a block B_M in F such that $\{t_1, t_2, \ldots\} \subseteq B_M$.

Let now $P, Q \subseteq \mathbf{N}$ be an infinite increasing subsets, $P \neq Q$. We shall prove that $B_P \neq B_Q$. Let us suppose $P = \{p_1, p_2, \ldots\}$, $Q = \{q_1, q_2, \ldots\}$. Because $P \neq Q$, there is $k \geq 1$ such that $p_1 = q_1$, ..., $p_{k-1} = q_{k-1}$, $p_k \neq q_k$. Let us show that the elements $t = x_{p_1} \vee x_{p_2} \vee \ldots \vee x_{p_{k-1}} \vee x_{p_k}$ and $s = x_{p_1} \vee x_{p_2} \vee \ldots \vee x_{p_{k-1}} \vee x_{q_k}$ do not commute.

Since $\mathcal{BA}$ is a proper subclass of V, there is an algebra $L \in V$ such that $L \notin \mathcal{BA}$. As L is not Boolean, there exist elements $a, b \in L$ which do not commute. Let $f : F \to L$ be the uniquely determined homomorphism such that $f(x_{p_k}) = a$, $f(x_{q_k}) = b$ and $f(x) = \mathbf{0}_L$ for any $x \in X \setminus \{x_{p_k}, x_{q_k}\}$. Then we have $f(t) = f(x_{p_k}) = a$, $f(s) = f(x_{q_k}) = b$. The elements $f(t)$ and $f(s)$ are not commutative in L and therefore the elements t and s cannot be commutative in F.

Since the elements t and s do not commute, we see that $t \notin B_Q$, thus $B_P \neq B_Q$. Obviously, the set of all increasing sequences is uncountable and this completes the proof.

3. Preservation of blocks

The result from Thm. 3.2 has already been proved in [4]. Since its proof is short, we provide it here for the convenience of the reader. We first need the following auxiliary result. As before, xCy stands for x commutes with y.

LEMMA 3.1. *Suppose that $a_1, \ldots, a_n \in L$ and choose indices $i_1, i_2, \ldots, i_k$ with $1 \leq i_1 \leq \ldots \leq i_k \leq n$. Then $a_1^{\varepsilon_1} \wedge \ldots \wedge a_n^{\varepsilon_n} \; C \; a_{i_1}^{\xi_1} \wedge \ldots \wedge a_{i_k}^{\xi_k}$, where $\varepsilon_1, \ldots, \varepsilon_n, \xi_1, \ldots, \xi_k \in \{0, 1\}$ and $a^0 = a$, $a^1 = a^\perp$ for any element $a \in L$.*

Proof. Put $a = a_1^{\varepsilon_1} \wedge \ldots \wedge a_n^{\varepsilon_n}$. Choose an index i_j . Then $a \leq a_{i_j}^{\varepsilon_{i_j}}$ and therefore $aCa_{i_j}^{\varepsilon_{i_j}}$. Thus, aCa_{i_j}, and this means that $aCa_{i_j}{}^{\xi_j}$. Since j was arbitrary, we infer that $aCa_{i_1}^{\xi_1}$, ..., $aCa_{i_k}^{\xi_k}$ and this implies that $aCa_{i_1}^{\xi_1} \wedge \ldots \wedge a_{i_k}^{\xi_k}$.

THEOREM 3.2. *Let $L \in \mathcal{OML}$, $B \in \mathcal{BA}$ and $f : L \to B$ be a surjective morphism. Let B be at most countable. Then there is a block $B_1 \in Bl(L)$ such that $f(B_1) = B$.*

Proof. Write $B = \{b_1, b_2, \ldots\}$. Choose elements $a_1, a_2, \ldots \in L$ such that $f(a_i) = b_i$ $(i = 1, 2, \ldots)$. Put

$c_1 = a_1,$

$c_2 = (a_1 \wedge a_2) \vee (a_1^\perp \wedge a_2),$

$$c_{i+1} = \bigvee_{\varepsilon\in\{0,1\}^i} (a_1^{\varepsilon_1} \wedge \ldots \wedge a_i^{\varepsilon_i} \wedge a_{i+1}),\ i = 1, 2, \ldots,\ \text{where } (\varepsilon_1, \ldots, \varepsilon_i) = \varepsilon.$$

Making use of Lemma 3.1, we see that $c_{i+1}Cc_j$ for any i, j with $1 \le j \le i$. It follows that elements $c_1, c_2, \ldots$ are mutually commutative. Moreover, we see that

$$\begin{aligned} f(c_1) &= f(a_1) = b_1, \\ f(c_{i+1}) &= f\Big(\bigvee_{\varepsilon\in\{0,1\}^i} (a_1^{\varepsilon_1} \wedge \ldots \wedge a_i^{\varepsilon_i} \wedge a_{i+1})\Big) = \\ &= \bigvee_{\varepsilon\in\{0,1\}^i} [(f(a_1))^{\varepsilon_1} \wedge \ldots \wedge (f(a_i))^{\varepsilon_i} \wedge f(a_{i+1})] = \\ &= b_{i+1} \wedge \Big(\bigvee_{\varepsilon\in\{0,1\}^i} (b_1^{\varepsilon_1} \wedge \ldots \wedge b_i^{\varepsilon_i})\Big) = b_{i+1} \wedge \mathbf{1}_B = b_{i+1}. \end{aligned}$$

If B_1 is a block in L that contains all of the elements $c_1, c_2, \ldots$, then $f(B_1) = B$. The proof is complete.

PROPOSITION 3.3. *Let $L \in \mathcal{OML}$ and let L possess a finite block. If $f : L \to B$ ($B \in \mathcal{BA}$) is a surjective homomorphism, then B is finite.*

Proof. Let B_1 be a finite block of L and let $b_1, \ldots, b_n$ be all atoms of B_1. Then $b_1, \ldots, b_n$ are atoms in L (see Prop. 2.3). Obviously, the element $f(b_i) = a_i$ is either $\mathbf{0}_B$ or an atom of B. Since $\mathbf{1}_L = b_1 \vee \ldots \vee b_n$, we also have $f(\mathbf{1}_L) = f(b_1) \vee \ldots \vee f(b_n)$. Thus, $\mathbf{1}_B = a_1 \vee \ldots \vee a_n$, where any element a_i is either $\mathbf{0}_B$ or an atom of B. Then for any element $b \in B$ we have $b = (b \wedge a_1) \vee \ldots \vee (b \wedge a_n)$, where any element $b \wedge a_i$ is either $\mathbf{0}_B$ or a_i. Thus, B has at most 2^n elements and is therefore finite.

Let us introduce some more notions which we shall use in the sequel.

Definition and notation 3.4. Put

$\mathcal{H}_{\mathcal{OML}} = \{L \in \mathcal{OML}$; for any OML L_1 and any surjective homomorphism $f : L \to L_1$, if $B_1 \in Bl(L_1)$ is a block in L_1, then there exists a block $B \in Bl(L)$ in L such that $f(B) = B_1\}$,

$\mathcal{H}_{\mathcal{BA}} = \{L \in \mathcal{OML}$; for any BA B_1 and any surjective homomorphism $f : L \to B_1$ there exists a block $B \in Bl(L)$ such that $f(B) = B_1\}$.

We shall now study the size of $\mathcal{H}_{\mathcal{OML}}$ and $\mathcal{H}_{\mathcal{BA}}$.

COROLLARY 3.5. *If $L \in \mathcal{OML}$ and if L possesses a finite block, then $L \in \mathcal{H}_{\mathcal{BA}}$.*

Proof. It follows from Prop. 3.3 and Thm. 3.2.

THEOREM 3.6. *Both of the following inclusions are proper: $\mathcal{H}_{\mathcal{OML}} \subset \mathcal{H}_{\mathcal{BA}} \subset \mathcal{OML}$.*

Proof. The following proof of the inclusion $\mathcal{H}_{\mathcal{BA}} \subset \mathcal{OML}$ being proper was communicated to us by J.Harding [10]. Let X be an uncountable set, $\mathbf{F}_X$ and $\mathbf{B}_X$ be the free algebras from Definition 1.6. Let $f : \mathbf{F}_X \to \mathbf{B}_X$ be a morphism such that $f|X = id_X$. Suppose that there exists $B_1 \in Bl(\mathbf{F}_X)$ such that $f(B_1) = \mathbf{B}_X$. Write $f_1 = f|B_1$. Then $f_1 : B_1 \to \mathbf{B}_X$ is a surjective homomorphism. Because $\mathbf{B}_X$ is a free Boolean algebra, $\mathbf{B}_X$ is a projective algebra in the variety $\mathcal{BA}$. Thus, there exists a homomorphism $g : \mathbf{B}_X \to B_1$ such that $g \circ f_1 = id_{\mathbf{B}_X}$. Then g is a homomorphism $\mathbf{B}_X \to \mathbf{F}_X$ and $g \circ f = id_{\mathbf{B}_X}$. This is a contradiction with Prop. 1.9. We see that there is no such block B_1. As a consequence, $\mathbf{F}_X \notin \mathcal{H}_{\mathcal{BA}}$.

Let B_2 be a four-element BA. Let L be the horizontal sum (see [11, p. 306]) of the algebras B_2 and $\mathbf{F}_X$. Let L_1 be the horizontal sum of the algebras B_2 and $\mathbf{B}_X$. According to Corollary 3.5, $L \in \mathcal{H}_{\mathcal{BA}}$. Let $\varphi : L \to L_1$ be such a homomorphism that $\varphi|B_2 = id_{B_2}$, $\varphi|\mathbf{F}_X = f$ (f defined above). By the previous part of this proof, there is no block in L which can be mapped onto $\mathbf{B}_X$ by φ. Thus, $L \notin \mathcal{H}_{\mathcal{OML}}$. The proof is complete.

THEOREM 3.7. *The classes $\mathcal{H}_{\mathcal{OML}}$ and $\mathcal{H}_{\mathcal{BA}}$ are closed under the formation of homomorphic images and finite products.*

Proof. The first statement is obvious. To prove the second, let us assume that $L_1, L_2 \in \mathcal{H}_{\mathcal{OML}}$. Suppose that $f : L_1 \times L_2 \to L$ is a surjective homomorphism and suppose further that $B \in Bl(L)$. Adopt the notation of Prop. 1.4 and assume that $h : L \to [\mathbf{0}, c_1] \times [\mathbf{0}, c_2]$ is the isomorphism defined in Prop. 1.3 (for $c_1 = c$, $c_2 = c^\perp$). Then $h(B) \in Bl([\mathbf{0}, c_1] \times [\mathbf{0}, c_2])$ and therefore there are blocks $B_i \in Bl([\mathbf{0}, c_i])$ $(i = 1, 2)$ such that $h(B) = B_1 \times B_2$ (Prop. 2.4). Since f_i is a homomorphism from L_i onto $[\mathbf{0}, c_i]$ and $L_i \in \mathcal{H}_{\mathcal{OML}}$, there is a block $B_i' \in Bl(L_i)$ such that $f_i(B_i') = B_i$ $(i = 1, 2)$. Since $B_1' \times B_2' \in Bl(L_1 \times L_2)$, it suffices to prove $f(B_1' \times B_2') = B$. Since h is an isomorphism, this means that we have to verify the equality $h(f(B_1' \times B_2')) = h(B)$. Now

$$\begin{aligned} h(B) &= B_1 \times B_2, \\ h(f(B_1' \times B_2')) &= \{h(f(x, y)); x \in B_1', y \in B_2'\} \\ &= \{h(f_1(x) \vee f_2(y); x \in B_1', y \in B_2'\} \\ &= \{(f_1(x), f_2(y)); x \in B_1', y \in B_2'\} \\ &= f_1(B_1') \times f_2(B_2') = B_1 \times B_2 = \\ &= h(B). \end{aligned}$$

This completes the proof for $\mathcal{H}_{\mathcal{OML}}$. The closedness of $\mathcal{H}_{\mathcal{BA}}$ under the formation of finite products can be proved analogously.

The following proposition will be applied in Thm. 3.9.

PROPOSITION 3.8. *If $L \in \mathcal{OML}$ and if the commutator ideal I_c is principal, then $L \in \mathcal{H}_{\mathcal{BA}}$.*

Proof. Suppose that $I_c = [\mathbf{0}, a]$. Then $a \in C(L)$ (see e.g. [11]) and therefore $L \cong [\mathbf{0}, a] \times [\mathbf{0}, a^\perp]$. Moreover, $[\mathbf{0}, a^\perp] \cong L/[\mathbf{0}, a] = L/I_c \in \mathcal{BA}$. We shall now show that $[\mathbf{0}, a]$ does not admit a Boolean quotient. To this end, assume that $f_1 : [\mathbf{0}, a] \to B_1$ is surjective. Put $B_2 = [\mathbf{0}, a^\perp]$ and $f_2 = id_{[\mathbf{0},a^\perp]}$. Define a morphism $f : L \to B_1 \times B_2$ by setting

$$f(x) = (f_1(x \wedge a), f_2(x \wedge a^\perp)).$$

Then we obtain $f(a) = (f_1(a), \mathbf{0}_{B_2})$. As $B_1 \times B_2$ is a Boolean algebra, $a \in I_c$, it follows that $f(a) = \mathbf{0}_{B_1 \times B_2}$. Thus, $f_1(a) = \mathbf{0}_{B_1}$ and B_1 has to be trivial, which completes the proof.

Let us now recall two classes of OMLs which appeared naturally in the course of developing the theory of OMLs. Let L be an OML. Let us say that L is *commutator-finite* if the set $Com(L) = \{com(x, y); x, y \in L\}$ is finite, and let us say that L satisfies the *relative centre property* if $C([\mathbf{0}, a]) = \{a \wedge c; c \in C(L)\}$ for any $a \in L$.

THEOREM 3.9. (i) *If L is commutator finite, then $L \in \mathcal{H}_{\mathcal{BA}}$.*

(ii) *If L is complete and satisfies the relative centre property, then $L \in \mathcal{H}_{\mathcal{BA}}$. A consequence: The lattice of all projections in a von Neumann algebra belongs to $\mathcal{H}_{\mathcal{BA}}$.*

Proof. If L satisfies either (i) or (ii), then it satisfies the assumption of Prop. 3.8 (see [2] and [8]).

THEOREM 3.10. *The class $\mathcal{H}_{\mathcal{BA}}$ is not closed under the formation of subalgebras.*

Proof. Let $\mathbf{F}_X$ and L denote the same OML's as in the proof of Thm 3.6. Then $\mathbf{F}_X$ is a subalgebra of L, $L \in \mathcal{H}_{\mathcal{BA}}$, but $\mathbf{F}_X \notin \mathcal{H}_{\mathcal{BA}}$.

PROPOSITION 3.11. *Let $L \in \mathcal{OML}$ and let $K \in \mathcal{H}_{\mathcal{BA}}$ for any subalgebra K of L. Then $L \in \mathcal{H}_{\mathcal{OML}}$.*

Proof. Let $f : L \to L_1$ be a surjective homomorphism, $L_1 \in \mathcal{OML}$ and let B_1 be a block in L_1. Let us write $K = f^{-1}(B_1)$. Then K is a subalgebra of L and therefore $K \in \mathcal{H}_{\mathcal{BA}}$. It follows that there is a block, $\bar{B}$, in K such that $f(\bar{B}) = B_1$. If we extend $\bar{B}$ to a block, B, of L, we see that $f(B) = B_1$.

THEOREM 3.12. *Let $L \in \mathcal{OML}$ and let* $\mathrm{card}(L) \leq \aleph_0$. *Then $L \in \mathcal{H}_{\mathcal{OML}}$.*

Proof. Let us first show that $L \in \mathcal{H}_{\mathcal{BA}}$. Suppose that $f : L \to B$ is a surjective morphism. Then $\mathrm{card}(B) \leq \aleph_0$, and we can use Thm. 3.2.

To complete the proof, let K be a subalgebra of L. Then we also have $\mathrm{card}(K) \leq \aleph_0$. From the previous part of this proof it follows that $K \in \mathcal{H}_{\mathcal{BA}}$. According to Prop. 3.11 we see that $L \in \mathcal{H}_{\mathcal{OML}}$.

Prior to the formulation of our main result, let us agree to denote by $Bl_\infty(L)$ the set of all infinite blocks of L.

THEOREM 3.13. *Let $L \in \mathcal{OML}$ and let $\mathrm{card}(Bl_\infty(L)) \leq \aleph_0$. Then $L \in \mathcal{H}_{\mathcal{OML}}$.*

Proof. Let us first show that $L \in \mathcal{H}_{\mathcal{BA}}$. Suppose that $f : L \to B$ is a surjective morphism. If $\mathrm{card}(B) \leq \aleph_0$, then we can use Thm. 3.2. Suppose therefore that $\mathrm{card}(B) > \aleph_0$. Let $B_1, B_2, \ldots$ be all infinite blocks in L and let us assume that $f(B_i) \neq B$ for any $i \in \mathbf{N}$. Then there are elements $b_1, b_2, \ldots \in B$ such that $b_i \in B \setminus f(B_i)$. Without any loss of generality, we may assume that the set $\{b_i; i \in \mathbf{N}\}$ is infinite (otherwise we can extend it to a countable subset of B). Consider now the Boolean algebra, $\bar{B}$, generated by $\{b_i; i \in \mathbf{N}\}$ in B. Put $K = f^{-1}(\bar{B})$ and set $g = f|K$. According to Thm. 3.2, there is a block $B^* \in Bl(K)$ such that $g(B^*) = \bar{B}$. Since $\bar{B}$ is infinite, B^* has to be infinite, too. It follows that there is an i ($i \in \mathbf{N}$) such that $B^* \subseteq B_i$. Then $f(B_i) \supseteq f(B^*) = \bar{B}$. But $b_i \notin f(B_i)$ which is a contradiction. Thus, $L \in \mathcal{H}_{\mathcal{BA}}$.

To complete the proof, let K be a subalgebra of L. Then we also have (see Prop. 2.4) $\mathrm{card}(Bl_\infty(K)) \leq \aleph_0$. From the previous part of this proof it follows that $K \in \mathcal{H}_{\mathcal{BA}}$. According to Prop. 3.11 we see that $L \in \mathcal{H}_{\mathcal{OML}}$ and the proof is complete.

In order to express our next result in a lucid form, let us further refine the classes investigated so far.

DEFINITION 3.14. Let us write

$\mathcal{OML}_{\mathsf{fb}} = \{L \in \mathcal{OML} \ ; \ \mathrm{card}(Bl(L)) < \aleph_0\}$ (an $L \in \mathcal{OML}_{\mathsf{fb}}$ is sometimes called *block-finite*, see [11]);

$\mathcal{OML}_{\mathsf{cb}} = \{L \in \mathcal{OML} \ ; \ \mathrm{card}(Bl(L)) \leq \aleph_0\}$;

$\mathcal{OML}_{\mathsf{cnb}} = \{L \in \mathcal{OML} \ ; \ \mathrm{card}(Bl_\infty(L)) \leq \aleph_0\}$.

THEOREM 3.15. *All of the following inclusions are proper:*

$$\mathcal{BA} \subset \mathcal{OML}_{\mathsf{fb}} \subset \mathcal{OML}_{\mathsf{cb}} \subset \mathcal{OML}_{\mathsf{cnb}} \subset \mathcal{H}_{\mathcal{OML}}.$$

Proof. For the inclusions $\mathcal{BA} \subset \mathcal{OML}_{\mathsf{fb}} \subset \mathcal{OML}_{\mathsf{cb}} \subset \mathcal{OML}_{\mathsf{cnb}}$ it is obvious. According to Thm. 3.13, there is $\mathcal{OML}_{\mathsf{cnb}} \subseteq \mathcal{H}_{\mathcal{OML}}$. Now, let X be a countable infinite set. Then the free OML $\mathbf{F}_X$ over the set X is also countable and therefore (Thm. 3.12) $\mathbf{F}_X \in \mathcal{H}_{\mathcal{OML}}$. Finally, from Thm. 2.7 and Corollary 2.6 it follows that $\mathbf{F}_X \notin \mathcal{OML}_{\mathsf{cnb}}$. This completes the proof.

PROPOSITION 3.16. *The classes $\mathcal{OML}_{\mathsf{fb}}$, $\mathcal{OML}_{\mathsf{cb}}$ and $\mathcal{OML}_{\mathsf{cnb}}$ are closed under the formation of homomorphic images.*

Proof. Suppose that L is block-finite and $f : L \to L_1$ is surjective. Then $L \in \mathcal{H}_{\mathcal{OML}}$ and therefore any block $B_1 \in Bl(L_1)$ is a homomorphic image of certain block of L. It follows that L_1 has at most finitely many blocks. For $L \in \mathcal{OML}_{\mathsf{cb}}$ or $L \in \mathcal{OML}_{\mathsf{cnb}}$ the proof is analogous.

In the paper [2] the authors denoted by *WBEP* the class of the OMLs determined by the following properties:

$L \in WBEP$ if and only if every subOML of L generated by the union of finitely many blocks is block-finite.

We can now contribute to the investigation carried in [2] by proving the following result:

PROPOSITION 3.17. *The class $WBEP \cap \mathcal{H}_{\mathcal{OML}}$ is closed under the formation of homomorphic images.*

Proof. Suppose that $L \in WBEP \cap \mathcal{H}_{\mathcal{OML}}$ and consider a surjective homomorphism $f : L \to L_1$. According to Thm. 3.7, $L_1 \in \mathcal{H}_{\mathcal{OML}}$.

Let us assume that $B_1, \ldots, B_n$ are blocks of L_1. Then there are blocks $B'_1, \ldots, B'_n \in Bl(L)$ such that $f(B'_i) = B_i$ $(i = 1, \ldots, n)$. Suppose that S' is the (block-finite) subOML of L generated by the set $\bigcup_{i=1}^{n} B'_i$. Put $S = f(S')$. Then S is a block-finite subOML in L (see Prop. 3.16) generated by the set $f\left(\bigcup_{i=1}^{n} B'_i\right) = \bigcup_{i=1}^{n} B_i$.

Recall for our final result that a class K of algebras for a language $\mathcal{L}$ is called *axiomatizable* if there exists a theory T in $\mathcal{L}$ such that K is exactly the class of all models of T.

THEOREM 3.18. *The classes $\mathcal{H}_{\mathcal{OML}}$ and $\mathcal{H}_{\mathcal{BA}}$ are not axiomatizable.*

Proof. Every axiomatizable class is closed under elementary equivalence. Let us choose two infinite sets, say X, Y, where X is countable and Y is uncountable. From Corollary 1.12 we infer that $\mathbf{F}_X \equiv \mathbf{F}_Y$. The algebra $\mathbf{F}_X$ is countable and therefore $\mathbf{F}_X \in \mathcal{H}_{\mathcal{OML}}$ according to Thm. 3.12. On the other hand, $\mathbf{F}_Y \notin \mathcal{H}_{\mathcal{BA}}$ according to the proof of the Theorem 3.6.

Let us finally formulate two open questions related to our investigation.

(1) Let us introduce a new class, $\mathcal{X}$, of BAs by setting

$\mathcal{X} = \{B \in \mathcal{BA}$; for any orthomodular lattice L and for any surjective $f : L \to B$ there exists a block $B_1 \in Bl(L)$ such that $f(B_1) = B\}$.

Observe that each at most countable BA does belong to $\mathcal{X}$ (Thm. 3.2). Is there an uncountable BA which belongs to $\mathcal{X}$?

(2) Is there a variety of OMLs, some V, such that $\mathcal{BA} \subset V \subseteq \mathcal{H}_{\mathcal{BA}}$?

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