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FREE $(n+1)$-VALUED ŁUKASIEWICZ BCK-ALGEBRAS

Abstract. The cardinal of the finitely generated free $(n+1)$-valued Łukasiewicz BCK-algebras has been determined by different authors only for some values of n. In this article we find the formula that allows its calculus for every value of n. By the application of this formula for $n = 1$, $n = 2$, we corroborate the results obtained by L. Iturrioz and A. Monteiro (Rev. Un. Mat. Argentina, 22 (1966), 146) and L. Iturrioz and O.Rueda (Discrete Math., 18 (1977), 35–44). In addition we generalize the results found by A. V. Figallo (Rev. Un. Mat. Argentina, 41, 4 (2000), 33–43).

1. Preliminaries

In 1966, Y. Imai and K. Iseki [7] defined an important class of algebras which they called BCK-algebras. Later, S. Tanaka [18] studied the class of commutative BCK-algebras and H. Yutani [21] proved that they constitute a variety.

An important subvariety of the commutative BCK-algebras are those which M. Pałasiński called Łukasiewicz BCK-algebras in [15]. On the other hand, it is well–known that these algebras coincide with the dual algebras defined by Y. Komori in [11] under the name of C-algebras with the objective of obtaining the algebraic counterpart to the infinite–valued Łukasiewicz implicative calculus, which was investigated by A. J. Rodriguez Salas in [16].

Then, following M. Pałasiński's notation [15], we shall say that the Łukasiewicz BCK-algebras (or $BCKL$-algebras) are algebras $\langle A, \to, 1\rangle$ of type $(2,0)$, which satisfy these identities:

(C1) $1 \to x = x$,
(C2) $x \to (y \to x) = 1$,
(C3) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(C4) $(x \to y) \to y = (y \to x) \to x$,

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(C5) $((x \to y) \to (y \to x)) \to (y \to x) = 1$.

Notice that the previous introduction of the $BCKL$-algebras is unusual, since most of the literature referring to them uses the dual notion, i.e. $\alpha \star \beta$ and 0 instead of $\beta \to \alpha$ and 1, respectively.

In these algebras the relation $\leq$ defined by $x \leq y$ if, and only if, $x \to y = 1$ is a partial order on A and $x \leq 1$, for every $x \in A$. In addition $(A, \leq)$ is a join semilattice where $x \vee y = (x \to y) \to y$ is the supremum of the elements x and y.

A bounded $BCKL$-algebra (or $BCKL^0$-algebra) is an algebra $\langle A, \to, 0, 1\rangle$ of type $(2,0,0)$ such that $\langle A, \to, 1\rangle$ is a $BCKL$-algebra and 0 is the first element for $\leq$.

We shall denote the varieties of $BCKL$-algebras and $BCKL^0$-algebras by $\boldsymbol{BCKL}$ and $\boldsymbol{BCKL^o}$, respectively.

In [16] it was proved that the variety $\boldsymbol{BCKL^o}$ coincide with that of Wasjberg algebras. Also, it is well–known that the variety of Wasjberg algebras is polynomially equivalent to the variety of Chang's MV-algebras [2].

If $\mathbf{K}$ is one of the $\boldsymbol{BCKL}$ or $\boldsymbol{BCKL^o}$ varieties, we shall denote the set of $\mathbf{K}$–congruences and $\mathbf{K}$–homomorphisms by $Con_{\mathbf{K}}(A)$ and $Hom_{\mathbf{K}}(A, B)$, respectively. Besides, if $S \subseteq A$ is a $\mathbf{K}$–subalgebra of A, we shall write $S \lhd_{\mathbf{K}} A$ and indicate by $[G]_{\mathbf{K}}$ the $\mathbf{K}$–subalgebra of A generated by G. The subindex $\mathbf{K}$ will be omitted where no confusion might arise.

Let $A \in \mathbf{K}$. $D \subseteq A$ is a deductive system of A if $1 \in D$ and if $x, x \to y \in D$, imply $y \in D$. If $\mathcal{D}(A)$ is the set of all deductive systems of A, then $Con_{\mathbf{K}}(A) = \{R(D) : D \in \mathcal{D}(A)\}$, where $R(D) = \{(x, y) \in A^2 : x \to y, y \to x \in D\}$ ([3, 11, 16]). If $R = R(D)$, we shall denote the quotient algebra by A/D.

Let $h \in Hom_{\mathbf{K}}(A, B)$. The set $Ker(h) = \{x \in A : h(x) = 1\}$ is called the kernel of h. It is simple to verify that, if $h \in Hom_{\mathbf{K}}(A, B)$, then $Ker(h) \in \mathcal{D}(A)$.

Let n be an integer, $n \geq 1$. A $BCKL_{n+1}$-algebra (or $BCKL^0_{n+1}$-algebra) is a $BCKL$-algebra (or $BCKL^0$-algebra) which satisfies the identity:

(C6) $(x^n \to y) \vee x = 1$,

where $x^1 \to y = x \to y$ and $x^{n+1} \to y = x \to (x^n \to y)$, for $n = 1, 2, \ldots$.

We shall denote the varieties of the $BCKL_{n+1}$-algebras and $BCKL^0_{n+1}$-algebras by $\boldsymbol{BCKL_{n+1}}$ and $\boldsymbol{BCKL^0_{n+1}}$, respectively.

Now we shall indicate some properties of the $BCKL_{n+1}$-algebras, for the demonstration of which the reader is referred to the bibliography included below:

(P1) Let $C_{n+1} = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$ be the set of rational fractions. Then, $\langle C_{n+1}, \rightarrow, 1\rangle \in \boldsymbol{BCKL_{n+1}}$, where $x \rightarrow y = min\ \{1, 1 - x + y\}$, for every $x, y \in C_{n+1}$.

(P2) If $S \lhd_{BCKL_{n+1}} C_{n+1}$ and $|S| > 1$, then $S \simeq_{BCKL^0_{t+1}} C_{t+1}$, where $t \leq n$ ([3]).

In what follows, we shall denote the set $\{x \in A : a \leq x \leq b\}$ by $[a, b]$.

(P3) $[\frac{n-t}{n}, 1] \lhd C_{n+1}$ for every t, $0 \leq t \leq n$. In addition, $[\frac{n-t}{n}, 1] \simeq C_{t+1}$ since the operation $\rightarrow$ is determined by the order over a finite chain ([3, 16, 19]).

(P4) If $A \in \boldsymbol{BCKL_{n+1}}$ is non–trivial, then A is isomorphic to a subalgebra of $P = \prod_{M \in \mathcal{M}(A)} A/M$, where $\mathcal{M}(A)$ is the set of the maximal deductive systems of A. Also, if $A \in \boldsymbol{BCKL^o_{n+1}}$ is finite, then $A \simeq P$ ([3, 16]).

(P5) $\boldsymbol{BCKL_{n+1}}$ has the congruence extension property ([3]).

(P6) If $A \in \boldsymbol{BCKL_{n+1}}$, then the following conditions are equivalent ([3]):

(i) $M \in \mathcal{M}(A)$,

(ii) there exists $h \in Hom(A, C_{n+1})$ such that $Ker(h) = M$,

(iii) $A/M \simeq C_{j+1}$ for some j, $1 \leq j \leq n$.

2. Free $BCKL_{n+1}$-algebras

In what follows, we shall denote by $\mathcal{L}(n, c)$ the $BCKL_{n+1}$-algebra with a set G of free generators, such that $|G| = c$, where c is a cardinal number.

The cardinal of $\mathcal{L}(n, c)$ was determined for some particular cases by different authors. In 1966, L. Iturrioz and A. Monteiro in [9], determined it for $n = 1$ and showed that the formula is

$$|\mathcal{L}(1, m)| = \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} \cdot 2^{2^{m-k}}.$$

Later, in 1977, L. Iturrioz and O. Rueda, in [10], obtained the cardinal of $\mathcal{L}(n, c)$ for $n = 2$, giving the formula:

$$|\mathcal{L}(2, m)| = \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} \cdot 2^{(2^k-1)\cdot 3^{m-k}} \cdot 3^{3^{m-k} - 2^{m-k}}.$$

In 1989, A.V. Figallo [4] determined a method to calculate $|\mathcal{L}(n, m)|$ for some particular values of n, generalizing the results obtained in [9] and [10].

The aim of this paper is to determine the cardinal of $|\mathcal{L}(n, m)|$ for every pair n, m of positive integers. For this purpose we shall use some results obtained in [4], which will be revised now.

Let $X \subseteq A$ be an ordered set. We shall denote the set of minimal elements of X by $\mu(X)$.

(P7) Let $A \in \boldsymbol{BCKL}$, $X \subseteq A$ and $[X]_{\boldsymbol{BCKL}} = A$. Then

(i) $\mu(X) = \mu(A)$,

(ii) the following conditions are equivalent:

(a) $X = \mu(A)$,

(b) X is an antichain.

(P8) If G is a set of free generators of $\mathcal{L}(n,c)$ such that $|G| = c$, then $G = \mu(\mathcal{L}(n,c))$.

(P9) $\mathcal{L}(n,c) = \bigcup_{g \in G} [g, 1]$.

(P10) $\mathcal{L}(n,m)$ is finite.

Let m be an integer, $m \geq 1$ and $G = \{g_1, g_2, \ldots, g_m\}$ a set of free generators of $\mathcal{L}(n,m)$.

By (P9) we have

$$|\mathcal{L}(n,m)| = \sum_{i=1}^{m} (-1)^{k+1} a_k, \tag{1}$$

where

$$a_k = \sum_{1 \leq i_1 < \ldots < i_k \leq m} |\bigcap_{t=1}^{k} [g_{i_t}, 1]|.$$

By the symmetry of the problem, it is sufficient to compute $|\bigcap_{i=1}^{k} [g_i, 1]|$.

Let $G_k = \{g_1, g_2, \ldots, g_k\}$, $G_{m-k} = G \setminus G_k$ and $g_k^* = \bigvee_{i=1}^{k} g_i$. Then, the verification of $B_k = \bigcap_{i=1}^{k} [g_i, 1] = [g_k^*, 1]$ is simple. Therefore,

$$|\mathcal{L}(n,m)| = \sum_{k=1}^{m} (-1)^{k+1} \binom{l}{k} |B_k|. \tag{2}$$

On the other hand, it is not difficult to prove that B_k is a finite subalgebra of $\mathcal{L}(n,m)$ with first element g_k^*. Then by (P4)

$$B_k \simeq \prod_{D \in \mathcal{M}(B_k)} B_k/D. \tag{3}$$

Let $\mathcal{M}_i(B_k) = \{D \in \mathcal{M}(B_k) : B_k/D \simeq C_{i+1}\}$, $1 \leq i \leq n$, $1 \leq k \leq m$, then

$$\beta_{i,k}^n = |\mathcal{M}_i(B_k)|. \tag{4}$$

From (3), (4) and (P6), we obtain

$$|B_k| = \prod_{i=1}^{n} (i+1)^{\beta_{i,k}^n}. \tag{5}$$

From (2) and (5)

$$|\mathcal{L}(n,m)| = \sum_{k=1}^{m}(-1)^{k+1}\binom{m}{k}\prod_{i=1}^{n}(i+1)^{\beta^n_{i,k}}. \tag{6}$$

By (P5), for every $D \in \mathcal{M}_i(B_k)$, there exists a unique $M \in \mathcal{M}(\mathcal{L}(n,m))$ such that $D = M \cap B_k$. Then, in order to compute $\beta^n_{i,k}$, we must determine the number of maximal deductive systems M of $\mathcal{L}(n,m)$ that satisfy

(M1) $B_k \not\subseteq M$,
(M2) if $D = M \cap B_k$, then $B_k/D \simeq C_{i+1}$.

Let

$$\mathcal{M}^n_{i,k} = \{M \in \mathcal{M}(\mathcal{L}(n,m)) : M \text{ verifies M1 and M2}\}.$$

In [4] it was shown that for every $M \in \mathcal{M}^n_{i,k}$ there exists a unique $h \in Hom\ (\mathcal{L}(n,m), C_{n+1})$ that satisfies

(H0) $M = Ker(h)$,
(H1) $B_k \not\subseteq Ker(h)$,
(H2) $h(B_k) = [\frac{n-i}{n}, 1]$,
(H3) $h(\mathcal{L}(n,m)) = [\frac{n-j}{n}, 1] \supseteq [\frac{n-i}{n}, 1]$.

In addition, if

$$\mathcal{H}^n_{i,k} = \{h \in Hom(\mathcal{L}(n,m), C_{n+1}) : h \text{ verifies H1 and H2}\},$$

then

$$\beta^n_{i,k} = |\mathcal{H}^n_{i,k}| \tag{7}$$

and, for each $h \in \mathcal{H}^n_{i,k}$, the restriction $f = h|G$ verifies

(F1) $f(G_k) \subseteq [0, \frac{n-i}{n}]$,
(F2) there exists $g \in G_k$, such that $f(g) = \frac{n-i}{n}$,
(F3) $[f(G)]_{\mathbf{BCKL}} \supseteq [\frac{n-i}{n}, 1]$.

Finally, if C^G_{n+1} is the set of functions from G to C_{n+1} and

$$\mathcal{F}^n_{i,k} = \{f \in C^G_{n+1} : f \text{ satisfies F1, F2 and F3}\}, \tag{8}$$

then,

$$\beta^n_{i,k} = |\mathcal{F}^n_{i,k}|. \tag{9}$$

3. Calculus of $\beta^n_{i,k}$

Let s be an integer, $s \geq 1$ and for every i, $1 \leq i \leq s$ let us consider the set

$$\mathcal{F}_{i,s} = \{f \in C^{o\,G}_{s+1} : f \text{ satisfies } F1^*, F2^* \text{ and } 0 \in f(G)\},$$

where

(F1*) $f(G_k) \subseteq [0, \frac{s-i}{s}]$,
(F2*) there exists $g \in G_k$, such that $f(g) = \frac{s-i}{s}$.

It is well–known that, for every divisor t of s, there exists a unique subalgebra S_t^o of C_{s+1}^o isomorphic to C_{t+1}^o. Besides, these are all the subalgebras of C_{s+1}^o. Let

$$\mathcal{S}_{t,s} = \{f \in {C_{s+1}^o}^G : [f(G)]_{BCKL^o} = S_t^o\}. \tag{10}$$

Then, it is clear that

$$\mathcal{F}_{i,s} = \bigcup_{t/s} \mathcal{S}_{t,s},$$

therefore,

$$|\mathcal{F}_{i,s}| = \sum_{t/s} |\mathcal{S}_{t,s}|. \tag{11}$$

From the properties of Wajsberg algebras, we conclude that $f \in \mathcal{F}_{i,s}$ is such a function that $[\frac{s-i}{s}, 1] \subseteq [f(G)]_{BCKL}$ if, and only if, $[f(G)]_{BCKL^o} = S_s^o$. Then, if we take $s = n$, all the functions of $\mathcal{S}_{s,s}$ verify F1, F2, F3 and, besides, $0 \in f(G)$.

From (11), we obtain the following result

$$|S_{s,s}| = |\mathcal{F}_{i,s}| - \sum_{\substack{t/s \\ t \neq s}} |\mathcal{S}_{t,s}|. \tag{12}$$

On the other hand, for every $f \in \mathcal{F}_{i,s}$, let $G_k^i = \{g \in G_k : f(g) = \frac{s-i}{s}\}$, $G_k^o = \{g \in G_k : f(g) = 0\}$ and $G_{m-k}^o = \{g \in G_{m-k} : f(g) = 0\}$, such that their cardinals are r_1, r_2 and r_3, respectively.

Then, to calculate the cardinal of $\mathcal{F}_{i,s}$ we shall distinguish the following cases:

(a) If $i = s$, then $r_1 = k = r_2$, $0 \leq r_3 \leq m - k$ and $\mathcal{F}_{i,s} = \{f : G \longrightarrow C_{s+1}^o : f(G_k) = 0\}$. Therefore,

$$|\mathcal{F}_{i,s}| = |{C_{s+1}^o}^{G_{m-k}}| = (s+1)^{m-k}. \tag{13}$$

(b) If $i \neq s$, then, $1 \leq r_1 \leq k$, $0 \leq r_2 \leq k - r_1$, $0 \leq r_3 \leq m - k$. Therefore,

$$\begin{aligned} |\mathcal{F}_{i,s}| &= |\{f \in {C_{s+1}^o}^G : f \text{ verifies F1* and F2*}\}| - \\ &\quad |\{f \in {C_{s+1}^o}^G : f \text{ verifies F1*, F2* and } G_k^o = G_{m-k}^o = \emptyset\}| \\ &= \sum_{r_1=1}^{k} \sum_{r_2=0}^{k-r_1} \binom{k}{r_1} \binom{k-r_1}{r_2} (s-i-1)^{k-(r_1+r_2)} \end{aligned}$$

$$\times \sum_{r_3=0}^{m-k} \binom{m-k}{r_3} s^{m-k-r_3} - \sum_{r_1=1}^{k} \binom{k}{r_1} (s-i-1)^{k-r_1} s^{m-k}.$$

Therefore,

$$|\mathcal{F}_{i,s}| = ((s-i+1)^k - (s-i)^k)(s+1)^{m-k} - ((s-i)^k - (s-i-1)^k)s^{m-k}. \tag{14}$$

The following lemma will be useful for the calculus of $|\mathcal{S}_{t,s}|$.

LEMMA 3.1. *Let t, q be positive integers such that $t \cdot q = s$. Then the following statements are verified:*

(i) $S_t^o = \{\frac{jq}{s} : 0 \leq j \leq t\}$.

(ii) $\frac{s-i}{s} \in S_t^o$ *if, and only if,* $i = (t-j)q$, *for some* j, $0 \leq j \leq t$.

(iii) $\mathcal{S}_{t,s} \neq \emptyset$ *if, and only if,* $\frac{s-i}{s} \in S_t^o$.

Proof. It is routine. ∎

Let p and h be positive integers such that $p \cdot h = s$ and let $\alpha_{\frac{i}{h}}(p) = |\mathcal{S}_{p,s}|$. In order to calculate $\alpha_{\frac{i}{h}}(p)$, we must make the following considerations:

(a) If $\frac{s-i}{s} \in S_p^o$, then by Lemma 3.1 (ii), the elements $\frac{s-i}{s}$ and $\frac{p-\frac{i}{h}}{p}$ coincide. Besides, as $S_p^o \simeq C_{p+1}^o$, it is clear that $|S_{p,s}| = |S_{p,p}|$. Consequently, from (12) we obtain

$$|S_{p,s}| = |\mathcal{F}_{\frac{i}{h},p}| - \sum_{\substack{t/p \\ t \neq p}} |\mathcal{S}_{t,p}|. \tag{15}$$

(b) If $\frac{s-i}{s} \notin S_p^o$, then by Lemma 3.1 (iii), we can deduce that

$$\alpha_{\frac{i}{h}}(p) = 0. \tag{16}$$

(c) If $\frac{s-i}{s} \in S_p^o$ and, in addition,

(c.1) $s \neq i$, then from (14) and (15) we obtain

$$\alpha_{\frac{i}{h}}(p) = \left(\left(p - \frac{i}{h} + 1\right)^k - \left(p - \frac{i}{h}\right)^k\right)(p+1)^{m-k} - \left(\left(p - \frac{i}{h}\right)^k - \left(p - \frac{i}{h} - 1\right)^k\right)p^{m-k} - \sum_{\substack{t/p \\ t \neq p \\ tq = p}} \alpha_{\frac{i}{q}}(t). \tag{17}$$

(c.2) $s = i$, then $\dfrac{i}{h} = p$. Then, from (13) and (15) we get

$$\alpha_p(p) = (p+1)^{m-k} - \sum_{\substack{t/p \\ t \neq p}} \alpha_t(t). \tag{18}$$

Now we are ready to determine $\beta^n_{i,k}$. Observe that, from (8), it is simple to prove that

$$\mathcal{F}^n_{i,k} = \bigcup_{j=0}^{n-i} \left\{ f \in C^G_{n+1} : f \text{ verifies F1, F2, F3 and } \mu(f(G)) = \frac{j}{n} \right\}$$

and, from (10) we obtain

$$\text{(19)} \quad \left\{ f \in C^G_{n+1} : f \text{ verifies F1, F2, F3 and } \mu(f(G)) = \frac{j}{n} \right\} \simeq \mathcal{S}_{n-j,n-j}.$$

Now, taking into account (16), (17), (18) and (19)

$$\beta^n_{i,k} = \sum_{j=0}^{n-i} |\mathcal{S}_{n-j,n-j}| = \sum_{j=0}^{n-i} \alpha_i(n-j) = \sum_{j=i}^{n} \alpha_i(j),$$

where

$$\alpha_i(s) = \begin{cases} (s+1)^{m-k} - \sum\limits_{\substack{t/s \\ t \neq s}} \alpha_t(t) & \text{if } i = s,\ i \in \mathbb{Z}, \\ ((s-i+1)^k - (s-i)^k)(s+1)^{m-k} - ((s-i)^k & \\ \quad -(s-i-1)^k)s^{m-k} - \sum\limits_{\substack{t/s \\ t \neq s \\ tq = s}} \alpha_{\frac{i}{q}}(t) & \text{if } 0 \leq\ i < s,\ i \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Hence we have shown the main result of this paper which is the following

THEOREM 3.1. *Let $\mathcal{L}(n,m)$ be a free $BCKL_{n+1}$-algebra with m free generators. Then the cardinal $|\mathcal{L}(n,m)|$ can be expressed as the following formula:*

$$|\mathcal{L}(n,m)| = \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} \prod_{i=1}^{n} (i+1)^{\beta^n_{i,k}}$$

where $\beta^n_{i,k} = \sum\limits_{s=i}^{n} \alpha_i(s)$ *and*

$$\alpha_i(s) = \begin{cases} (s+1)^{m-k} - \sum\limits_{\substack{t/s \\ t \neq s}} \alpha_t(t) & \textit{if } i = s,\ i \in \mathbb{Z}, \\ ((s-i+1)^k - (s-i)^k)(s+1)^{m-k} - ((s-i)^k & \\ \quad -(s-i-1)^k)s^{m-k} - \sum\limits_{\substack{t/s \\ t \neq s \\ tq = s}} \alpha_{\frac{i}{q}}(t) & \textit{if } 0 \leq\ i < s,\ i \in \mathbb{Z} \\ 0 & \textit{otherwise.} \end{cases}$$

EXAMPLE 3.1. Now we shall apply the results obtained to determine $|\mathcal{L}(n,m)|$ for certain values of n.

(i) $n = 3$,

$$\beta^3_{1,k} = \sum_{j=1}^{3} \alpha_1(j) = \left(3^k - 2^k\right) 4^{m-k},$$

$$\beta^3_{2,k} = \sum_{j=2}^{3} \alpha_2(j) = (2^k - 1)4^{m-k} - 2^{m-k},$$

$$\beta^3_{3,k} = \alpha_3(3) = 4^{m-k} - 2^{m-k},$$

$$|\mathcal{L}(3,m)| = \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} \cdot 2^{(3^k-2^k)4^{m-k}} \cdot 3^{(2^k-1)4^{m-k}-2^{m-k}} \cdot 4^{4^{m-k}-2^{m-k}}.$$

This formula was obtained by A.V. Figallo in [4].

(ii) $n = 4$,

$$\beta^4_{1,k} = \sum_{j=1}^{4} \alpha_1(j) = \left(4^k - 3^k\right) 5^{m-k},$$

$$\beta^4_{2,k} = \sum_{j=2}^{4} \alpha_2(j) = (3^k - 2^k)5^{m-k} - (2^k - 1)3^{m-k},$$

$$\beta^4_{3,k} = \sum_{j=3}^{4} \alpha_3(j) = (2^k - 1)5^{m-k} - 2^{m-k},$$

$$\beta^4_{4,k} = \alpha_4(4) = 5^{m-k} - 3^{m-k},$$

$$|\mathcal{L}(4,m)| = \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} \cdot 2^{(4^k-3^k)5^{m-k}} \cdot 3^{(3^k-2^k)5^{m-k}-(2^k-1)3^{m-k}}$$
$$4^{(2^k-1)5^{m-k}-2^{m-k}} \cdot 5^{5^{m-k}-3^{m-k}}.$$

(iii) For the sake of brevity, for $n = 5$ we just give the final formula obtained

$$|\mathcal{L}(5,m)| = \sum_{k=1}^{m} (-1)^{k+1} {}^{m}_{k} \cdot 2^{(5^k-4^k)\cdot 6^{m-k}} \cdot 3^{(4^k-3^k)\cdot 6^{m-k}-2^{m-k}}$$
$$4^{(3^k-2^k)\cdot 6^{m-k}-2^{m-k}} \cdot 5^{(2^k-1)\cdot 6^{m-k}-3^{m-k}} \cdot 6^{6^{m-k}-2^{m-k}}.$$

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