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SMOOTHLY PARACOMPACT SIKORSKI CW COMPLEXES

0. Introduction

In this paper, we continue our investigations of the DW complex concept. Since DW complexes are closely related to ordinary CW complexes in many respects, we have found the denomination Sikorski CW complexes or S-CW complexes for short to be rather suggestive and appropriate. See [10] for Sikorski (differential) spaces. For easy reference, let's define the notion of Sikorski space and map between Sikorski spaces.

DEFINITION 0.1. Let X be a set, $\mathcal{F}$ a family of real-valued functions on X, and $\mathcal{T}$ the weakest topology on X for which all functions in $\mathcal{F}$ are continuous. The pair $(X, \mathcal{F})$ is called a Sikorski space if:

(i) For each open covering $\{U_i\}_{i \in I}$ of X and function $g : X \to \mathbb{R}$, if for each $i \in I$, $g|_{U_i} = f_i|_{U_i}$ for some $f_i \in \mathcal{F}$, then $g \in \mathcal{F}$.
(ii) If $f_1, \ldots, f_n$ is a collection of functions in $\mathcal{F}$ and $\omega : \mathbb{R}^n \to \mathbb{R}$ is a smooth real-valued function, then $\omega \circ (f_1, \ldots, f_n)$ is again in $\mathcal{F}$.

$(X, \mathcal{F})$ is said to be Hausdorff if the induced topology $\mathcal{T}$ is Hausdorff. It is easy to show that $(X, \mathcal{F})$ is Hausdorff if and only if for given points $x, y \in X$, there is a function $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.

A map $\phi : (X, \mathcal{F}) \to (Y, \mathcal{G})$ of Sikorski spaces is a set map $\phi : X \to Y$ such that given $f \in \mathcal{G}$, then $f \circ \phi \in \mathcal{F}$. Sikorski spaces and maps between them form a category, which we denote **DIFF**. Maps between Sikorski spaces are called (Sikorski) smooth maps.

Let $(X, \mathcal{F})$ be a Sikorski space and let $\phi : X \to Y$ be a set mapping. The collection $\phi_*(\mathcal{F}) := \{f : Y \to \mathbb{R} |\ f \circ \phi \in \mathcal{F}\}$ is a differential structure. See [6].

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$\phi_*(\mathcal{F})$ is called the differential structure coinduced from $\mathcal{F}$ by ϕ. Next, given a Sikorski space $(X, \mathcal{F})$, we say that $\mathcal{F}$ is the differential structure generated by a family $\mathcal{F}_0$ if the following property holds: $f \to \mathbb{R}$ is in $\mathcal{F}$ if and only if, given a point $p \in X$, there are functions $f_1, \ldots, f_n \in \mathcal{F}_0$, $\omega \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and a neighbourhood $U \in \mathcal{T}$ of p ($\mathcal{T}$ is the topology induced by $\mathcal{F}_0$) such that

$$f|_U = \omega \circ (f_1, \ldots, f_n)|_U.$$

$\mathcal{F}$ is assumed to be the smallest differential structure, containing $\mathcal{F}_0$. Let A be a subset of X; the pair $(A, \mathcal{F}_A)$, where $\mathcal{F}_A$ is the structure generated by $\mathcal{F}|_A$, is called a Sikorski subspace of $(X, \mathcal{F})$. The pair $(\mathbb{R}^n, \varepsilon(\mathbb{R}^n))$, where $\varepsilon(\mathbb{R}^n)$ is the set of (usual) smooth functions $\mathbb{R}^n \to \mathbb{R}$, is a Sikorski space, and is called the n-dimensional Sikorski euclidean space. As an exception to the subspace differential structure notation, we will use $\varepsilon(D^n)$ to denote the differential structure induced on D^n by the inclusion $D^n \hookrightarrow (\mathbb{R}^n, \varepsilon(\mathbb{R}^n))$ instead of the more accurate $\varepsilon(\mathbb{R}^n)_{D^n}$. Likewise, $\varepsilon(\overset{\circ}{D}{}^n)$ and $\varepsilon(S^{n-1})$ will denote the differential structures induced by $\overset{\circ}{D}{}^n \hookrightarrow \mathbb{R}^n$ and $S^{n-1} \hookrightarrow \mathbb{R}^n$ respectively.

LEMMA 0.1. *The category* **DIFF** *has the following properties:*

(1) *Complete and cocomplete.*
(2) *The underlying topological space of a coproduct is a coproduct of underlying topological spaces of cofactors.*

Proof. Easy proof.

LEMMA 0.2. *Let $(X, \mathcal{F})$ be a Sikorski space, and let p be any point in X. For any open neighbourhood U of p, there is a nonnegative function $h \in \mathcal{F}$ such that $p \in h^{-1}(0, \infty) \subset U$ and $\overline{h^{-1}(0, \infty)} \subset U$.*

This is a slightly modified version of Lemma 3 in our paper [7]. We omit the proof thereof.

For easy reference, we now define the notion of Sikorski CW complex. A cell e of dimension k in a Sikorski space $(X, \mathcal{F})$ is a diffeomorphic copy of the open ball $\overset{\circ}{D}{}^k$ via a (Sikorski) smooth map $\Phi_e : D^k \to X$. We proved in [7] that $\Phi_e(D^k) = \overline{e}$, that is the closure of e with respect to the topology $\mathcal{T}$. We let $\mathcal{F}(\overline{e})$ be the differential structure coinduced by Φ_e on $\overline{e}$.

DEFINITION 0.2. A Hausdorff differential space $(X, \mathcal{F})$ is a Sikorski-CW complex (or S-CW complex for short) with respect to a family E of cells and a family Φ of smooth maps provided:

(1) X is a disjoint union of its cells.

(2) If e is a k-cell in X, one assigns a quotient map

$$\Phi_e : (D^k, \varepsilon(D^k)) \to (\overline{e}, \mathcal{F}(\overline{e}))$$

such that

$$\Phi_e|_{\overset{\circ}{D}{}^k} : (\overset{\circ}{D}{}^k, \varepsilon(\overset{\circ}{D}{}^k)) \to (e, \mathcal{F}(e))$$

is a diffeomorphism. $\mathcal{F}(e)$ is the structure induced by the inclusion $e \hookrightarrow (\overline{e}, \mathcal{F}(\overline{e}))$.

(3) If e is a k-cell in X, then $\overline{e}$ is contained in a finite union of some cells of dimension $\leq k$.

(4) The differential structure $\mathcal{F}$ is the coinduced structure corresponding to the inclusions $(\overline{e}, \mathcal{F}(\overline{e})) \hookrightarrow X$.

EXAMPLES. (1) Let $(\mathbf{I}, \varepsilon(\mathbf{I}))$ denote the unit interval as a Sikorski subspace of the usual Sikorski euclidean space $(\mathbb{R}, \varepsilon(\mathbb{R}))$. We call $(\mathbf{I}, \varepsilon(\mathbf{I}))$ the Sikorski unit interval. $(\mathbf{I}, \varepsilon(\mathbf{I}))$ is an S-CW complex with as collection of cells the family $\{\{0\}, \{1\}, (0,1)\}$. The characetristic map Φ^1 of the 1-cell $(0,1)$ is the map $\Phi^1 : [-1,1] \to [0,1] = \overline{e^1}$, defined by $\Phi^1(x) = \frac{x+1}{2}$. Φ^1 is obviously smooth, in the sense of Sikorski spaces, and a diffeomorphism on $(-1,1)$.

(2) Let $\mathcal{F}(S^n)$ be the quotient differential structure on the n-sphere S^n induced by the map $\Phi : (D^n, \varepsilon(D^n)) \to S^n$, defined by

$$\Phi(x_1, \ldots, x_n) = (2\sqrt{1-||x||^2}x_1, \ldots, 2\sqrt{1-||x||^2}x_n, 2||x||^2 - 1).$$

We give S^n an S-CW complex structure by considering $S^n - p \equiv e^n$ and $p \equiv e^0$, where $p = (0, \ldots, 0, 1)$ as its cells. It is obvious that $\overline{e^n} = S^n$, and $\mathcal{F}(\overline{e^n}) = \mathcal{F}(S^n)$. It is clear that $\mathcal{F}(e^0) = \mathbb{R}$. Let $f \in \mathcal{F}(\overline{e^n})$ and $c \in \mathbb{R}$. If $f \vee c$ denotes the map obtained by patching together f and c, then $f \vee c$ is well defined only if $f(p) = c$. Notice that $\mathcal{F}(S^n)$ is not the differential structure induced on S^n by the inclusion $S^n \hookrightarrow (\mathbb{R}^{n+1}, \varepsilon(\mathbb{R}^{n+1}))$. In fact, the map $g(x_1, \ldots, x_{n+1}) = x_1 + \ldots + x_{n+1}$ is not smooth in the sense of Sikorski on the S-CW complex $(S^n, \mathcal{F}(S^n))$, because given a point $q \in S^{n-1} = \partial D^n$, there is no differentiable map $h : \mathbb{R}^n \to \mathbb{R}$ that equals $g \circ \Phi$ in any neighbourhood $U \subset \mathbb{R}^n$ of q.

(3) The decomposition of the n-sphere S^n into two cells of each dimension from 0 to n makes S^n into an S-CW complex with a differential structure different from the one that makes S^n into a subspace of $(\mathbb{R}^{n+1}, \varepsilon(\mathbb{R}^{n+1}))$. In fact, for $n = 1$, the function defined by

$$f(x,y) = \begin{cases} 1 - x^2 & \text{for } (x,y) \in S^1 \text{ with } y = +\sqrt{1-x^2} \\ -1 + x^2 & \text{for } (x,y) \in S^1 \text{ with } y = -\sqrt{1-x^2} \\ 0 & \text{for } (x,y) \in S^1 \text{ with } y = 0 \end{cases}$$

is a structure map on the S-CW complex S^1, since given the characteristic map $\Phi_+ : D^1 \to S^1$, defined by

$$\Phi_+(x) = (x, +\sqrt{1-x^2}),$$

the pullback $f \circ \Phi_+(x) = 1 - x^2$ is smooth on D^1. Similarly, let $\Phi_- : D^1 \to S^1$ be the characteristic map defined by

$$\Phi_-(x) = (x, -\sqrt{1-x^2});$$

the pullback $f \circ \Phi_-(x) = -1 + x^2$ is smooth on D^1. But f is clearly not differentiable in points $(1,0)$ and $(-1,0)$, therefore f is not a map for the subspace structure $\varepsilon(S^1)$.

On the other hand, there are maps in $\varepsilon(S^1)$ that do not belong to the S-CW structure $\mathcal{F}(S^1)$. For instance, consider the differentiable map $g : S^1 \to \mathbb{R}$, defined by $g(x,y) = y$. There is no differentiable map $h : \mathbb{R} \to \mathbb{R}$ that coincides with $g \circ \Phi_+$ near 1. Therefore the map g is not a structure map of the S-CW complex S^1.

§1 discusses the situation regarding quotient S-CW complexes. In fact, X/Y is an S-CW complex provided the S-CW complex X is smoothly paracompact and regular, and the subcomplex Y is compact.

§2 presents the underlying topology of an adjunction of two Sikorski spaces. This leads to an alternative description of S-CW complexes. Much of the importance of Lemma 0.2 is seen in this section.

1. Quotient S-CW Complexes

For the purpose of the paper, we want that every S-CW complex X satisfies the following property: every open covering of the Sikorski space X has a smooth partition of unity. A Sikorski space $(X, \mathcal{F})$ that has this property is called smoothly paracompact with respect to the differential structure $\mathcal{F}$. See [1] for smoothly paracompact spaces.

THEOREM 1.1. *Let $(X, \mathcal{F})$ be a smoothly paracompact S-CW complex whose underlying topological space X is regular, let Y be a compact S-CW subcomplex, and let $\nu : X \to X/Y$ be the natural map. If $\nu^*(\mathcal{F})$ is the coinduced differential structure on X/Y, corresponding to ν, then $(X/Y, \nu^*(\mathcal{F}))$ is a S-CW complex.*

Proof. Let us first prove that the restriction

$$\nu|_{X-Y} : X - Y \to X/Y - \{*\}$$

is a diffeomorphism. By $\mathcal{G}$ and $\mathcal{H}$, we will mean the structures on $X - Y$ and $X/Y - \{*\}$ induced by inclusions $X - Y \hookrightarrow (X, \mathcal{F})$ and $X/Y - \{*\} \hookrightarrow (X/Y, \nu^*(\mathcal{F})$ respectively. It is clear that $\nu|_{X-Y} : (X - Y, \mathcal{G}) \to (X/Y, \mathcal{H})$ is smooth.

Conversely, let p be any point in $X-Y$. Since X is Hausdorff and regular and Y is compact by hypothesis, there exists an open set O containing p and such that $\overline{O} \cap Y = \emptyset$. Using regularity of X again, one can assume that for each $q \in Y$ there exist open neighbourhoods V_q and U_q of p and q respectively such that $\overline{O} \subset V_q$ and $V_q \cap U_q = \emptyset$. Since Y is compact, there are finitely many points $q_1, q_2, \ldots, q_n \in Y$ such that $Y \subset \cup_{i=1}^n U_{q_i}$. But every $U_{q_i} \cap V_{q_i} = \emptyset$, with V_{q_i} an open neighbourhood of $\overline{O}$. Therefore, if we put $U = \cup_{i=1}^n U_{q_i}$ and $V = \cap_{i=1}^n V_{q_i}$, then U and V are open sets such that $Y \subset U$, $p \in \overline{O} \subset V$ and $U \cap V = \emptyset$. Since X is Hausdorff, it follows that the subset $X - Y - \overline{O}$ is open in X and so the family $\mathcal{C} = \{U, V, X - Y - \overline{O}\}$ is an open covering of X. Since X is smoothly paracompact, $\mathcal{C}$ admits a locally finite open refinement, say $\mathcal{W} = \{W_j\}_{j \in J}$. There exist only finitely many $j_1, \ldots, j_n \in J$ such that $p \in W_{j_i}$, $i = 1, \ldots, n$. It is clear that $W_{j_i} \subset V$ for $i = 1, \ldots, n$.

Now let $J_1 = \{j \in J : W_j \subset V\}$ and $J_2 = J - J_1$. Let $f = h|_{X-Y}$, where $h \in \mathcal{F}$. For each $j \in J$, let $g_j : X \to \mathbb{R}$ be a function given as follows:

(1) If $j \in J_1$,

$$g_j(x) = \begin{cases} f(x) & \text{if } x \in V \\ 0 & \text{if } x \notin V \end{cases}$$

(2) If $j \in J_2$,

$$g_j = 0.$$

Let $\{\lambda_j\}_{j \in J}$ be a smooth partition of unity subordinate to $\mathcal{W}$, and let

$$f^* = \sum_{j \in J} \lambda_j g_j.$$

We claim that f^* is smooth. Indeed, let us write f^* this way

$$f^* = \sum_{J_1} \lambda_j g_j + \sum_{J_2} \lambda_j g_j.$$

For each $j \in J_2$, $\lambda_j g_j = 0$. Let $j \in J_1$. To see that $\lambda_j g_j$ is smooth, one need only look at points on the boundary $\mathrm{Fr}V$ of V. But first, observe that $Y \cap \mathrm{Fr}V = \emptyset$. Let $r \in \mathrm{Fr}V$. As Y is closed and $Y \cap \mathrm{Fr}V = \emptyset$, there exists an open neighbourhood $R \subset X - Y$ of r such that for all $j \in J_1$

$$(\lambda_j g_j)(t) = \begin{cases} \lambda_j(t) f(t) & \text{if } t \in R \cap V \\ 0 & \text{if } t \in R - V. \end{cases}$$

Since $\lambda_j(t) f(t) = 0$ for all $t \in R - V$, so $\lambda_j g_j = \lambda_j f$ on R and hence $\lambda_j g_j$ is smooth on R. Thus, f^* is a smooth function. Since $Y \subset X - \cup_{j \in J_1} W_j$, it follows that $f^*|_Y = 0$. Now, let $W^* = \cap_{i=1}^n W_{ji} \cap O$. It is evident that W^* is an open neighbourhood of p and is contained in $X - Y$. Consider,

$$f^*|_{W^*} = \sum_{j \in J_1} \lambda_j|_{W^*} g_j|_{W^*} + \sum_{j \in J_2} \lambda_j|_{W^*} g_j|_{W^*}.$$

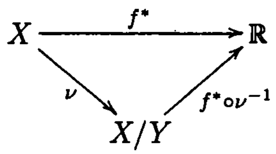
Let $l \in J_2$. Consequently, W_l is either contained in U or in $X - Y - \overline{O}$, but not in V. Thus $\lambda_l|_O = 0$ and hence $\lambda_l|_{W^*} = 0$. Thus

$$f^*|_{W^*} = f|_{W^*}.$$

It follows that

$$f^*|_{W^*} \circ \nu^{-1}|_{\nu(W^*)} = f|_{W^*} \circ \nu^{-1}|_{\nu(W^*)}.$$

We claim that $f^* \circ \nu^{-1} : X/Y \to \mathbb{R}$ is smooth. Indeed, in the diagram



the smooth map $f^* : X \to \mathbb{R}$ is constant on every fiber $\nu^{-1}(u)$, where $u \in X/Y$. Therefore, one applies Corollary 3.1 at once to show that $f^* \circ \nu^{-1} : X/Y \to \mathbb{R}$ is smooth.

Now, since W^* is an open neighbourhood of p then by Lemma 0.2, it follows that there exists a nonnegative function $h^* \in \mathcal{F}$ such that $p \in h^{*-1}(0, \infty) \subset W^*$, and thus $Y \subset (h^*)^{-1}(0)$. Since $h^* : X \to \mathbb{R}$ is constant on every $\nu^{-1}(u)$, where $u \in X/Y$, Corollary 3.1 applies to show that $h^* \circ \nu^{-1} : X/Y \to \mathbb{R}$ is smooth. It is clear that $\nu(p) \in (h^* \circ \nu^{-1})^{-1}(0, \infty) \subset \nu(W^*)$. But p is arbitrary in W^*, therefore $\nu(W^*)$ is an open subset of X/Y. So we have an open subset $\nu(W^*)$, a smooth map $f^* \circ \nu^{-1} : X/Y \to \mathbb{R}$ such that $(f^* \circ \nu^{-1})|_{\nu(W^*)} = (f \circ \nu^{-1})|_{\nu(W^*)}$. Thus, $f \circ \nu^{-1} : X/Y - \{*\} \to \mathbb{R}$ is locally smooth, and consequently $f \circ \nu^{-1}|_{X/Y - \{*\}}$ is smooth. Hence, the mapping $\nu^{-1}|_{X/Y-\{*\}} : X/Y - \{*\} \to X - Y$ is smooth. Thus, $X - Y$ is diffeomorphic to $X/Y - \{*\}$.

Finally, let us prove that $(X/Y, \nu^*(\mathcal{F}))$ is an S-CW complex.

First let us show that X/Y is Hausdorff. Let $\nu(x)$, $\nu(z)$ be distinct points in X/Y. Assume that $\nu(x) \neq *$ and $\nu(z) \neq *$. Therefore $x \notin Y$ and $z \notin Y$; since X is Hausdorff and $X - Y$ is open, there are disjoint open subsets U' and V' in $X - Y$ such that $x \in U'$ and $z \in V'$. But $X - Y$ is diffeomorphic to $X/Y - \{*\}$, therefore $\nu(U')$ and $\nu(V')$ are open neighbourhoods of $\nu(x)$ and $\nu(z)$ respectively and such that $\nu(U') \cap \nu(V') = \emptyset$.

Now assume that $\nu(z) = *$. Therefore $x \notin Y$; there are disjoint open subsets U'' and V'' in X such that $x \in U''$ and $Y \subset V''$. Let

$$\mathcal{R} = \{U'', V'', X - Y\}$$

be an open covering of X. Let $\mathcal{W}' = \{W_i'\}_{i \in I}$ be a locally finite open refinement of $\mathcal{R}$. Let $I_1 = \{i \in I : W_i' \cap Y = \emptyset\}$ and let $I_2 = I - I_1$. Next, for each $i \in I$, let $h_i : X \to \mathbb{R}$ be a function defined by

$$h_i = \begin{cases} 0 & \text{if } i \in I_1 \\ 1 & \text{if } i \in I_2. \end{cases}$$

Let $\{\mu_i\}_{i \in I}$ be a smooth partition of unity subordinate to $\mathcal{W}'$, and let $r = \sum_{i \in I} \mu_i h_i$. It is easy to see that the map $r : X \to \mathbb{R}$ is smooth on X. Put

$$A = \sum_{i \in I_1} \mu_i h_i \qquad \text{and} \qquad B = \sum_{i \in I_2} \mu_i h_i.$$

Clearly, one has

$$A|_{V''} = 0 \qquad \text{and} \qquad B|_{V''} = \sum_{i \in I_2} \mu_i|_{V''}.$$

Thus,

$$r|_{V''} = B|_{V''}.$$

Note that

$$r|_Y = 1$$

and

$$Y \subset r^{-1}(0, \infty) \subset V''.$$

Since r is constant on each $\nu^{-1}(u)$, $u \in X/Y$, Corollary 3.1 says that r induces a smooth map $r' : X/Y \to \mathbb{R}$ such that $* \in (r')^{-1}(0, \infty)$. Finally, since $\nu|_{X-Y} : X - Y \to X/Y - \{*\}$ is a diffeomorphism, it follows that $\nu(U'')$ is an open neighbourhood of $\nu(x)$. Moreover since $U'' \cap r^{-1}(0, \infty) = \emptyset$, it follows that $\nu(U'') \cap (r')^{-1}(0, \infty) = \emptyset$. Hence, X/Y is Hausdorff.

Let $E(X)$ and $E(Y)$ be the families of cells of X and Y respectively. We define cells in X/Y as follows. The 0-cells are given by

$$(X/Y)^0 = \{\nu(e) : e \in E(X) - E(Y) \text{ and } \dim(e) = 0\} \cup \{*\};$$

for $k > 0$, define

$$(X/Y)^k = \{\nu(e) : e \in E(X) - E(Y) \text{ and } \dim(e) = k\}.$$

Let e be a cell in $X - Y$; we define (as usual) the characteristic map of $\nu(e)$ as the composite $\nu\Phi_e$. The characteristic map of the cell $*$ is defined to be the map $\Phi_* : D^0 \to *$.

It is easy to show that X/Y satisfies all the 4 axioms defining S-CW complexes. □

2. Attaching of Cells

Let X be a smoothly paracompact Sikorski space, let Y be any Sikorski space, and let $f : A \to Y$ be a smooth map from a non-empty closed subspace A of X. Consider the coproduct $X \sqcup Y$ and form a quotient space by identifying each set $\{y\} \cup f^{-1}(y)$, for $y \in Y$, to a point. We denote this quotient space by $X \sqcup_f Y$ and call it an adjunction space of X and Y, determined by f.

We have said it above: If $X \sqcup_f Y$ is an adjunction space in **DIFF**, X is assumed to be smoothly paracompact. We will also assume that $\nu : X \sqcup Y \to X \sqcup_f Y$ is the quotient map identifying $a \in A$ with its image $f(a)$. It is clear that $\nu|_Y : Y \to \nu(Y)$ (resp. $\nu|_{X-A} : X - A \to \nu(X - A)$) maps the set Y (resp. $X - A$) bijectively onto $\nu(Y)$ (resp. $\nu(X - A)$). There is more to this. In fact, we now show that $\nu|_Y$ and $\nu|_{X-A}$ are diffeomorphisms.

LEMMA 2.1. *Let $(X \sqcup_f Y, \mathcal{C})$ be an adjunction of spaces $(X, \mathcal{F})$ and $(Y, \mathcal{G})$.*

(i) *Any function $\alpha : \nu(Y) \to \mathbb{R}$ such that $\alpha \circ \nu|_Y \in \mathcal{G}$ is a structure map for the induced differential structure $\mathcal{C}_{\nu(Y)}$.*
(ii) *Given a function $\alpha : \nu(Y) \to \mathbb{R}$, α is smooth on $\nu(Y)$ if and only if $\alpha \circ \nu|_Y : Y \to \mathbb{R}$ is smooth and $\alpha \circ \nu|_A : A \to \mathbb{R}$ is smooth.*

Proof. (i) First note that $\alpha \circ \nu|_A = \alpha \circ \nu|_Y \circ \iota_Y \circ f$, where $\iota_Y : Y \to X \sqcup Y$ is an inclusion map. Since $\alpha \circ \nu|_Y \in \mathcal{G}$, it follows that $\alpha \circ \nu|_Y \circ \iota_Y \circ f \in \mathcal{F}_A$; and thus $\alpha \circ \nu|_A \in \mathcal{F}_A$. As X is smoothly paracompact and A is a closed subspace of X, one applies Theorem 1.1 in [4], which says that every smooth function on a closed subspace A is a restriction of some smooth function on X. Thus, there exists $g \in \mathcal{F}$ such that $g|_A = \alpha \circ \nu|_A$. Now, let $\mu = \nu|_{X-A} : X - A \to X \sqcup_f Y$ and let $\beta : \nu(X - A) \to \mathbb{R}$ be a function given by

$$\beta(p) = g \circ \mu^{-1}(p) \qquad\qquad p \in \nu(X - A).$$

Define a map $h : X \sqcup_f Y \to \mathbb{R}$ by setting

$$h(p) = \begin{cases} \alpha(p) & \text{if } p \in \nu(Y) \\ \beta(p) & \text{if } p \in \nu(X - A). \end{cases}$$

Readily, one has

$$h \circ \nu|_Y = \alpha \circ \nu|_Y,$$
$$h \circ \nu|_A = \alpha \circ \nu|_A,$$
$$h \circ \nu|_{X-A} = g|_{X-A}.$$

Since g is a smooth extension of $\alpha \circ \nu|_A$, it follows that $h \circ \nu|_X = g$; therefore $h \in \mathcal{C}$. Hence $\alpha = h|_{\nu(Y)} \in \mathcal{C}_{\nu(Y)}$.

(ii) For each point $p \in \nu(Y)$ choose a neighbourhood U_p of p in the topology of $\mathcal{C}$ and a structure function $g \in \mathcal{C}$ such that $\alpha|_{U_p \cap \nu(Y)} = g|_{U_p \cap \nu(Y)}$.

Since

$$\nu^{-1}(U_p \cap \nu(Y)) = (\nu^{-1}(U_p) \cap Y) \cup (\nu^{-1}(U_p) \cap A),$$

and $Y \cap A = \emptyset$, it follows that

$$\alpha \circ \nu|_{\nu^{-1}(U_p)\cap Y} = g \circ \nu|_{\nu^{-1}(U_p)\cap Y}$$

and

$$\alpha \circ \nu|_{\nu^{-1}(U_p)\cap A} = g \circ \nu|_{\nu^{-1}(U_p)\cap A}.$$

As $g \circ \nu|_X \in \mathcal{F}$, $g \circ \nu|_Y \in \mathcal{G}$ and $\nu^{-1}(U_p)$ is open in $X \sqcup Y$, it turns out that $\alpha \circ \nu|_{\nu^{-1}(U_p)\cap Y} : \nu^{-1}(U_p) \cap Y \to \mathbb{R}$ and $\alpha \circ \nu|_{\nu^{-1}(U_p)\cap A} : \nu^{-1}(U_p) \cap A \to \mathbb{R}$ are smooth. But p is arbitrary, therefore $\alpha \circ \nu|_Y$ and $\alpha \circ \nu|_A$ are smooth.

Conversely, since $\alpha \circ \nu|_Y \in \mathcal{G}$ it follows that $\alpha : \nu(Y) \to \mathbb{R}$ is smooth. □

LEMMA 2.2. *Let $(X \sqcup_f Y, \mathcal{C})$ be an adjunction of spaces $(X, \mathcal{F})$ and $(Y, \mathcal{G})$. Then the composite*

$$Y \to X \sqcup Y \to X \sqcup_f Y$$

is a diffeomorphism from Y to a subspace of $X \sqcup_f Y$.

Proof. Since $\nu|_Y$ is smooth and bijective, we need only show that $(\nu|_Y)^{-1}$ is smooth. To this end let $\beta \in \mathcal{G}$; in light of Lemma 2.1 $\beta \circ (\nu|_Y)^{-1} : \nu(Y) \to \mathbb{R}$ is smooth if and only if $\beta \circ (\nu|_Y)^{-1} \circ \nu|_Y : Y \to \mathbb{R}$ and $\beta \circ (\nu|_Y)^{-1} \circ \nu|_A : A \to \mathbb{R}$ are smooth. Clearly, if U is any open set in $\nu(Y)$, then

$$\beta \circ (\nu|_Y)^{-1} \circ \nu|_{\nu^{-1}(U)\cap Y} = \beta|_{\nu^{-1}(U)\cap Y}.$$

But $\nu^{-1}(U) \cap Y$ is an open set in Y, therefore $\beta \circ (\nu|_Y)^{-1} \circ \nu|_Y$ is smooth on Y. On the other hand, for all $a \in \nu^{-1}(U) \cap A$, one has $\beta \circ (\nu|_Y)^{-1} \circ \nu(a) = \beta \circ f(a)$. Therefore,

$$\beta \circ (\nu|_Y)^{-1} \circ \nu|_{\nu^{-1}(U)\cap A} = \beta \circ f|_{\nu^{-1}(U)\cap A} \tag{1}$$

Since $\nu^{-1}(U) \cap A$ is an open set in A and $\beta \circ f$ is smooth on A, it follows that $\beta \circ (\nu|_Y)^{-1} \circ \nu|_A : A \to \mathbb{R}$ is smooth. Thus, $(\nu|_Y)^{-1}$ is smooth. □

The next lemma is of pivotal importance when it is necessary to show that the underlying topological space of an adjunction of differential spaces is in fact an adjunction of the corresponding underlying topological spaces.

LEMMA 2.3. *Let $(X \sqcup_f Y, \mathcal{C})$ be an adjunction space of $(X, \mathcal{F})$ and $(Y, \mathcal{G})$, obtained via the attaching map $f : A \to Y$, where A is closed in X. Then the composite*

$$X - A \to X \sqcup Y \to X \sqcup_f Y$$

maps $X - A$ diffeomorphically onto an open subset of $X \sqcup_f Y$.

Proof. To show that μ^{-1}, where $\mu := (\nu_{X-A})$, is smooth we need to prove that for every smooth $g : X - A \to \mathbb{R}$, the composite $g \circ \mu^{-1} : \nu(X - A) \to \mathbb{R}$ is smooth. For every point $p \in X - A$, there is, by virtue of Lemma 0.2 , an open neighbourhood U of p contained in $X - A$ such that $U = \alpha^{-1}(0, \infty)$, where $\alpha \in \mathcal{F}$ is nonnegative for all $x \in X$, and $g|_U = h|_U$ for some $h \in \mathcal{F}$. Suppose that $\alpha(p) = k > 0$; choose a nonnegative smooth increasing function $\beta : \mathbb{R} \to \mathbb{R}$ such that

$$\beta(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } k - \varepsilon \leq t \leq k + \varepsilon, \end{cases}$$

where $\varepsilon > 0$ is chosen in such a way that $p \in \alpha^{-1}(k - \varepsilon, k + \varepsilon) \subset U$. The composite $\beta \circ \alpha$ is smooth and, on another hand, the open neighbourhood $V := \alpha^{-1}(k - \varepsilon, k + \varepsilon)$ of p is such that $\beta \circ \alpha|_V = 1$. It follows that $g|_V = h \cdot (\beta \circ \alpha)|_V$. Define a smooth map $H : X \sqcup Y \to \mathbb{R}$ by setting

$$H(x) = \begin{cases} h(x) \cdot (\beta \circ \alpha)(x) & \text{if } x \in X \\ 0 & \text{if } x \in Y. \end{cases}$$

Since $(\beta \circ \alpha)(a) = 0$ for all $a \in A$, it follows that H is identically 0 on A. By its definition, H induces a smooth map $\overline{H} : X \sqcup_f Y \to \mathbb{R}$ such that

$$\overline{H}(\nu(x)) = \begin{cases} H(x) & \text{if } x \in X \\ 0 & \text{if } x \in Y. \end{cases}$$

It is evident that $\overline{H} \in \mathcal{C}$. In turn, since V is an open set containing p, it follows that there is an open interval (a, b) such that $p \in H^{-1}(a, b) \subset V$. But V is contained in $X - A$, so $\nu(p) \in \overline{H}^{-1}(a, b) \subset \nu(V)$. Therefore, $\nu(V)$ is an open neighbourhood of $\nu(p)$, and it is easily seen that $g \circ \mu^{-1}|_{\nu(V)} = \overline{H}|_{\nu(V)}$. □

Lemmas 2.2 and 2.3 lead to a theorem of particular importance, with the help of which one may prove that the underlying topological of a finite smoothly paracompact S-CW complex is a CW complex. This attempt is restrictive because it only works for finite smoothy paracompact S-CW complexes.

THEOREM 2.1. *Let X be a Hausdorff smoothly paracompact differential space, let A be a compact closed subspace of X, and let $f : A \to Y$ be a smooth map from A into a Hausdorff differential space Y. If $\mathcal{C}$ is the structure on the adjunction $X \sqcup_f Y$, determined by the quotient map $\nu : X \sqcup Y \to X \sqcup_f Y$, then the underlying topology of $\mathcal{C}$ is the quotient topology corresponding to the map ν.*

In other words, every subset U of $X \sqcup_f Y$ such that $\nu^{-1}(U)$ is open in Y is open in the topology of $\mathcal{C}$.

Proof. Let U be a subset of $X \sqcup_f Y$ such that $\nu^{-1}(U)$ is open in $X \sqcup Y$. $\nu^{-1}(U)$ is open in $X \sqcup Y$ if and only if $\nu^{-1}(U) \cap X$ is open in X and $\nu^{-1}(U) \cap Y$ is open in Y. Let p be a point in $U \cap \nu(Y)$; there is $q \in \nu^{-1}(U) \cap Y$ such that $\nu(q) = p$. Suppose that $q \in \nu^{-1}(U) \cap (Y - f(A))$. Since $\nu^{-1}(U) \cap (Y - f(A))$ is an open subset of Y, there is a nonnegative structure map $g_1 : Y \to \mathbb{R}$ such that $q \in g_1^{-1}(0, \infty) \subset \nu^{-1}(U) \cap (Y - f(A))$. Thus, $g_1|_{f(A)} = 0$. We now let $g_2 : X \to \mathbb{R}$ be the identically zero map, and define $h : X \sqcup Y \to \mathbb{R}$ by $h = g_1 \sqcup g_2$. Let $h^* : X \sqcup_f Y \to \mathbb{R}$ be such that

$$h^* \circ \nu|_Y = g_1$$
$$h^* \circ \nu|_X = 0;$$

h^* is well defined since $h^* \circ \nu|_{f(A)}$. Thus, $h^* \in \mathcal{C}$; and since $q \in g_1^{-1}(0, \infty) \subset \nu^{-1}(U) \cap Y$, it follows that $p \in (h^*)^{-1}(0, \infty) \subset U \cap \nu(Y)$.

Now suppose that the point p is such that there is $q \in f(A)$ with $p = \nu(q)$. Since $q \in \nu^{-1}(U) \cap Y$ and $\nu^{-1}(U) \cap Y$ is open in Y, by Lemma 0.2 there exists a nonnegative smooth map $F : Y \to \mathbb{R}$ such that $q \in F^{-1}(0, \infty) \subset \nu^{-1}(U) \cap Y$ and $\overline{F^{-1}(0, \infty)} \subset \nu^{-1}(U) \cap Y$. It follows that $F \circ f : A \to \mathbb{R}$ is smooth and if $\overline{q}$ is a point in A such that $f(\overline{q}) = q$ then $\overline{q} \in (F \circ f)^{-1}(0, \infty) \subset \nu^{-1}(U) \cap A$. But f is smooth, so $\overline{q} \in \overline{(F \circ f)^{-1}(0, \infty)} \subset \nu^{-1}(U) \cap A$. Since $\overline{(F \circ f)^{-1}(0, \infty)}$ and $X - \nu^{-1}(U)$ are disjoint closed subsets of X, and X is smoothly paracompact, it follows from [1], pages 165, 166 that there exists a function $g : X \to \mathbb{R}$ in $\mathcal{F}$ with

$$g|_{\overline{(F \circ f)^{-1}(0,\infty)}} = 1$$

and

$$g|_{X - \nu^{-1}(U)} = 0.$$

Now, as A is a closed subset of the smoothly paracompact differential space X, let $h : X \to \mathbb{R}$ be a smooth extension of $F \circ f$, (h exists by virtue of Theorem 1.1 [4]). Next, we define $H : X \to \mathbb{R}$ by $H(x) = g(x)h(x)$. It is clear that H is smooth on account that $\mathcal{F}$ is an algebra. Moreover $H|_A = F \circ f|_A$ and $H|_{X - \nu^{-1}(U)} = 0$; so $H^{-1}(0, \infty) = (F \circ f)^{-1}(0, \infty) \subset \nu^{-1}(U) \cap X$. Now consider the function $H^* : X \sqcup_f Y \to \mathbb{R}$ such that $H^* \circ \nu = H \sqcup F$, where $H \sqcup F : X \sqcup Y \to \mathbb{R}$ is a smooth map defined by

$$H \sqcup F(x) = \begin{cases} H(x) & \text{if } x \in X \\ F(x) & \text{if } x \in Y. \end{cases}$$

It is clear that H^* is smooth and $p \in (H^*)^{-1}(0, \infty) \subset U$.

Now suppose that p is a point in $U \cap \nu(X - A)$ and suppose that its preimage by ν is a point r in $\nu^{-1}(U) \cap (X - A)$. Since $\nu^{-1}(U) \cap (X - A)$ is open, there is a nonnegative smooth function $l \in \mathcal{F}$ such that $r \in l^{-1}(0, \infty) \subset$

$\nu^{-1} \cap (X - A)$. Therefore, $A \subset l^{-1}(0)$. Define $d : X \sqcup Y \to \mathbb{R}$ by setting

$$d(x) = \begin{cases} l(x) & \text{if } x \in X \\ 0 & \text{if } x \in Y. \end{cases}$$

Since $l(x) = 0$ for all $x \in A$, it follows that d induces a map $\overline{d} : X \sqcup_f Y \to \mathbb{R}$ such that

$$\overline{d}(\nu(x)) = \begin{cases} d(x) & \text{if } x \in X \\ 0 & \text{if } x \in Y. \end{cases}$$

Clearly $\overline{d} \in \mathcal{C}$ and $p \in \overline{d}^{-1}(0, \infty) \subset U \cap \nu(X - A)$.

We conclude that for all $p \in U$ there is a smooth map $h \in \mathcal{C}$ such that $p \in h^{-1}(0, \infty) \subset U$. Therefore U is open for the underlying topology of $\mathcal{C}$. □

Now, suppose $n \in \mathbb{N}$ is fixed; let $(\mathbf{D}^n, \varepsilon(\mathbf{D}^n))$ be the finite coproduct of the Sikorski spaces $\{(D^n_\alpha, \varepsilon(D^n_\alpha));\ \alpha \in \Lambda_n\}$, where D^n_α is the α-th copy of the euclidean ball D^n_α. Next, let $(\mathbf{S}^{n-1}, \varepsilon(\mathbf{S}^{n-1}))$ be the coproduct of the finite family $\{(S^{n-1}_\alpha, \varepsilon(S^{n-1}_\alpha));\ \alpha \in \Lambda_n\}$, where S^{n-1}_α is the boundary of D^n_α. For each $\alpha \in \Lambda_n$, let $\Phi^n_\alpha : S^{n-1}_\alpha \to Y$ be a smooth map carrying S^{n-1}_α into a Hausdorff Sikorski space Y and let $f : \mathbf{S}^{n-1} \to Y$ be the map $f = \sqcup_{\alpha \in \Lambda_n} \Phi^n_\alpha$. The space $\mathbf{D}^n \sqcup_f Y$ is called the space obatined by attaching finitely many cells D^n_α to Y.

LEMMA 2.4. *$\mathbf{D}^n \sqcup_f Y$ is Hausdorff.*

Proof. $\mathbf{D}^n$ as a coproduct of smoothly paracompact spaces D^n_α, $n \in \Lambda_n$, is also smoothly paracompact. In light of Lemma 0.1, the underlying topological space of $\mathbf{D}^n \sqcup_f Y$ is an adjunction of the underlying topological spaces. Since $\mathbf{S}^{n-1}$ is a compact subset of $\mathbf{D}^n$, Theorem 8.5 in [2] implies that $\mathbf{D}^n \sqcup_f Y$ is Hausdorff. □

One more useful lemma is

LEMMA 2.5. *Let Y be an S-CW complex of dimension n, and whose collection of cells is finite. Then*

(i) *$\mathbf{D}^n \sqcup_f Y^{(n-1)}$ is an S-CW complex, diffeomorphic to Y.*
(ii) *$X := \mathbf{D}^{n+1} \sqcup_f Y$ is an S-CW complex, and Y is its n-skeleton.*

The proof of the analogue results in the category of CW complexes can be easily adapted. An easy reference is [2].

By mimicking the proof that every CW complex is a cellular space, and every cellular space is a CW complex [3], we now prove that every S-CW complex is a cellular space (in the sense of Sikorski) and every cellular space (in the sense of Sikorski) is an S-CW complex.

THEOREM 2.2. *A space X is an S-CW complex with finitely many cells if and only if there is a sequence $X_0 \subset X_1 \subset \ldots \subset X_m = X$ such that $X = \cup_{n=0}^{m} X_n$ and the following properties hold:*

(1) *X_0 is a discrete space.*
(2) *For each $n > 0$, there is an indexing set Λ_n and a family of smooth maps $\{\Phi_\alpha^n : S_\alpha^{n-1} \to X_{n-1} | \alpha \in \Lambda_n\}$ so that*
$$X_n = \mathbf{D}^n \sqcup_f X_{n-1},$$
where $f = \sqcup \Phi_\alpha^n$.
(3) *If $\mathcal{F}$ is the differential structure on X and $\mathcal{F}_n$ is the structure induced on X_n by the inclusion $X_n \hookrightarrow X$, then a function $\sigma : X \to \mathbb{R}$ is in $\mathcal{F}$ if and only if $\sigma|_{X_n} \in \mathcal{F}_n$ for each $n \geq 0$.*

Proof. Suppose that X is an S-CW complex with finitely many cells and $X_n = X^{(n)}$, where $X^{(n)}$ is the n-skeleton. We showed in [7] that skeletons $X^{(n)}$ are S-CW subcomplexes of X, so all real-valued functions $\sigma : X^{(0)} \to \mathbb{R}$ are $\mathcal{F}_0$-structure functions. It follows that each subset of $X^{(0)}$ is open in $X^{(0)}$. Thus $X^{(0)}$ is discrete, and hence (1) holds.

Conditions (2) and (3) are immediate.

Assume the condition. X_0 is clearly an S-CW complex. By means of Lemma 2.5(ii), each X_n is an S-CW complex. If $\nu^n : \mathbf{D}^n \sqcup X_{n-1} \to \mathbf{D}^n \sqcup_f X_{n-1} = X_n$ is the quotient map defining the adjunction X_n, and $\iota_n : X_n \hookrightarrow X$ is an inclusion carrying X_n into X, we let
$$\iota_n \circ \nu^n|_{D_\alpha^n} : D_\alpha^n \to \mathbf{D}^n \to \mathbf{D}^n \sqcup X_{n-1} \to \mathbf{D}^n \sqcup_f X_{n-1} = X_n \to X$$
be the characteristic map of an n-cell e_α. Now suppose that E_n is the family of cells of the S-CW complex X_n, $n \geq 0$. Define $E = \cup\{E_n : \ n \geq 0\}$ and
$$\Phi = \{\text{constant maps to } X_0\} \cup \cup_{n \geq 1}\{\iota_n \circ \nu^n|_{D_\alpha^n} : \ \alpha \in \Lambda_n\}.$$
The pair (E, Φ) defines an S-CW structure on X. □

Theorem 2.2 serves as a stepping stone to proving

THEOREM 2.3. *The underlying topological space of a finite S-CW complex is an S-CW complex.*

Proof. Let X be a finite S-CW complex. Its skeletons $X^{(k)}$, $k = 0, 1, \ldots, n$ are closed subspaces of X (see [7]) such that $X = \cup_{k=0}^{n} X^{(k)}$ and satisfy conditions (1), (2) and (3) of Theorem 2.2. Under the notation of Theorem 2.2, we have
$$X = X^{(n)} = \mathbf{D}^n \sqcup_{f^n} \mathbf{D}^{n-1} \sqcup_{f^{n-1}} \ldots \sqcup_{f^1} X^{(0)}$$
where $f^k = \sqcup_{\alpha \in \Lambda_k} \Phi_\alpha^k$, $k = 1, \ldots, n$. Since every $\mathbf{D}^k$ is smoothly paracompact and $X^{(0)}$ is Hausdorff, it follows from Theorem 2.2 that X has the quotient

topology determined by

$$\nu : \mathbf{D}^n \sqcup (\mathbf{D}^{n-1} \sqcup_{f^{n-1}} \dots \sqcup_{f^1} X^{(0)}) \to X.$$

But the topology coinduced by ν defines a CW complex structure on X (see [8]), therefore X, as a topological space, is a CW complex. □

3. Appendix

In this section, we elaborate on quotient maps and quotient spaces in the category **DIFF** of Sikorski spaces; quotient maps and quotient spaces are very useful tools that one needs when dealing with adjunctions of Sikorski spaces.

DEFINITION 3.1. Let X and Y be Sikorski spaces. A smooth surjection $q : X \to Y$ is called a quotient map provided, given a real-valued function $f : Y \to \mathbb{R}$, then f is smooth on Y if and only if $f \circ q$ is smooth on X.

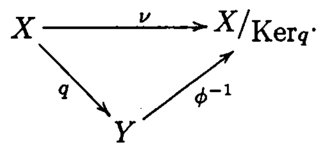
LEMMA 3.1. *Let X, Y, Z be Sikorski spaces, and let $q : X \to Y$ be a quotient map. Then every set map $\Phi : Y \to Z$ such that $\Phi \circ q : X \to Z$ is smooth is a smooth map.*

Proof. Easy to see.

LEMMA 3.2. *Let X, Y be Sikorski spaces and let $q : X \to Y$ be a smooth surjective map. Then q is a quotient map if and only if, for all Sikorski spaces Z, and all functions $g : Y \to Z$, one has g smooth if and only if $g \circ q$ is smooth.*

Proof. Assume that q is a quotient map. If g is smooth, then $g \circ q$ is smooth. Conversely let $g \circ q$ be smooth and let $h : Z \to \mathbb{R}$ be a smooth map. Then $h \circ g \circ q : X \to \mathbb{R}$ is smooth; since q is a quotient map, it follows that $h \circ g : Y \to \mathbb{R}$ is smoot. Hence g is smooth.

Now assume that it is true that for all Sikorski spaces Z, and all functions $g : Y \to Z$, one has g smooth if and only if $g \circ q$ is smooth. We claim that this condition implies that q is a quotient map. Let Ker$q$ be the equivalence relation on X, defined by $x \sim x'$ if $q(x) = q(x')$. See [8]. Let $X|_{\mathrm{Ker}q}$ denote the quotient set of X by Ker$q$. We let abusively $X|_{\mathrm{Ker}q}$ denote the quotient Sikorski space, determined by $\nu \equiv$ Ker$q$. It is easy to see that the map $\phi : X|_{\mathrm{Ker}q} \to Y$, defined by $\phi([x]) = q(x)$ is one-to-one. But since q is onto, it follows that ϕ is also onto; therefore ϕ is a bijection. Consider the commutative diagram



Since $\phi^{-1} \circ q = \nu$ is smooth, it follows by hypothesis that ϕ^{-1} is smooth. Moreover, since ν is a quotient and $\phi \circ \nu = q$ is a smooth function, it follows that ϕ is smooth. Thus, ϕ is a diffeomorphism, and hence q is a quotient map. □

COROLLARY 3.1. *Let $q : X \to Y$ be a quotient map of Sikorski spaces and, for some Sikorski space Z let $h : X \to Z$ be a smooth map that is constant on every $q^{-1}(y)$, where $y \in Y$. Then $h \circ q^{-1} : Y \to Z$ is smooth.*

Proof. That h is constant on each $q^{-1}(y)$, where $y \in Y$, implies that $h \circ q^{-1} : Y \to Z$ is a well defined function; $h \circ q^{-1}$ is smooth because $(h \circ q^{-1}) \circ q = h$ is smooth, and Lemma 3.2 applies. □

COROLLARY 3.2. *Let X and Y be Sikorski spaces, and let $q : X \to Y$ be a quotient map. Then the map $\phi : X|_{Kerq} \to Y$, defined by $[x] \mapsto q(x)$, is a diffeomorphism.*

Proof. That $\phi : X|_{\mathrm{Ker}q} \to Y$ is a bijection is clear. Let $\nu : X \to X|_{\mathrm{Ker}q}$ be the natural map. It is easily seen that the smooth map $q : X \to Y$ is constant on every $\nu^{-1}([x])$, where $x \in X$. Since $\phi = q \circ \nu^{-1}$, Corollary 3.1 applies to show that ϕ is smooth. Now, by Lemma 3.2, since q is a quotient map and ν is smooth, it follows that ϕ^{-1} is smooth. □

By means of Corollary 3.2, one can easily prove

COROLLARY 3.3. *Let W be a Sikorski space for which there exists a smooth surjective map $h : X \sqcup Y \to W$ such that, for u, $v \in X \sqcup Y$, one has $\nu(u) = \nu(v)$ if and only if $h(u) = h(v)$. Then $[u] \mapsto h(u)$ is a diffeomorphism $X \sqcup_f Y \to W$. Thus, $X \sqcup_f Y$ is unique up to a diffeomorphism.*

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