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SOME PROPERTIES OF ORTHOCENTRIC SIMPLEXES

1. Introduction

It is known that any triangle is orthocentric. For a tetrahedron to be orthocentric an additional condition must be satisfied. The question arises: what is the necessary and sufficient condition for an n-simplex ($n \geq 4$) to be orthocentric, and next, which properties of triangles and tetrahedrons can be extended to higher dimensions? Some answers are given in this paper.

2. Conditions for the orthocentrity of a simplex

Let $A_1,\dots,A_{n+1}$ be vertices of a nondegenerate simplex S in the n-dimensional Euclidean space E^n. By $H_{i_1\dots i_k}$ we denote the join of points $A_{i_{k+1}},\dots,A_{i_{n+1}}$, where $(i_1,\dots,i_{n+1})$ is a permutation of numbers $1,2,\dots,$ $n+1$.

THEOREM 1. *Simplex S is orthocentric if and only if line A_iA_j is perpendicular to subspace H_{ij} for all $i,j = 1,\dots,n+1, i \neq j$.*

Proof. Suppose A_iA_j is perpendicular to H_{ij} for all distinct i,j. Let P be the orthogonal projection of A_1 onto H_1. Since A_1P is perpendicular to H_1 and $A_1A_2 \perp H_{12}$, the plane A_1A_2P is normal (see [2]) to H_{12}. This implies $A_2P \perp H_{12}$. It means that P lies on the line projecting orthogonally point A_2 onto H_{12} in H_1. Taking into account the fact $A_1A_3 \perp H_{13}$ we see that plane A_1A_3P is normal to H_{13} and, consequently, P lies on the line projecting A_3 onto H_{13} in H_1, and so on. Hence P is orthocenter of H_1. Denoting by Q the orthogonal projection of A_2 onto H_2, we state that Q is the orthocenter of H_2. Analogously, since $PA_2 \perp H_{12}$ and $A_1Q\perp \cap H_{12}$ we infer that the intersection point of A_2P and H_{12} is the orthocenter of H_{12}, as well as the point $A_1Q\perp \cap H_{12}$ is the orthocenter of the same subspace H_{12}. Hence $A_2P\perp \cap H_{12} = A_1Q\perp \cap H_{12}$. Thus every two lines l_i, l_j projecting orthogonally A_i,A_j onto H_i,H_j, respectively, $i,j = 1,\dots,n+1$, have a common point. Suppose there is, among the lines l_i, a line, say l_3, not passing through point $A_2P\perp \cap H_{12}$. Since l_3 cuts l_1 and l_2, lines l_1,l_2,l_3 are coplanar, and since each of lines l_i, $i = 4, \ldots, n+1$, cuts l_1, l_2 and l_3, all of them are coplanar. This contradicts the assumption that S is nondegenerate. Assume now that S is orthocentric. Then its faces of arbitrary dimension are orhocentric too. In particular, it concerns 3-dimensional faces. It is known in such a case that $A_iA_j \perp A_kA_l$, distinct i, j, k, l. It implies that $A_iA_j \perp H_{ij}$ for distinct i, j.

REMARK 1. It is known that in the 3-dimensional space if $AB\perp CD$ and $AC\perp BD$ in the tetrahedron ABCD, then $AD\perp BC$ and the tetrahedron is orthocentric. An analogous property holds in higher dimensions.

LEMMA 1. *If* $A_1A_2\perp H_{1i}$, $i = 2, \ldots, n+1$, *then* $A_1A_j\perp H_{ij}$ *for all* $i \neq j$.

Proof. Obviously, we may assume that $i, j \neq 1$. From $A_1A_k\perp H_{1k}$, it follows that $A_iA_j\perp A_1A_k$, k=2,...,n+1, $k \neq i, j$. This implies that $A_iA_j\perp A_kA_m$ for distinct i, j, k, m. Hence $A_iA_j\perp H_{ij}$.

REMARK 2. In an identical way we prove that if for fixed k, $A_kA_i\perp H_{ki}$ i=1,...,n+1, $i \neq k$, then $A_iA_j\perp H_{ij}$, all $i \neq j$.

LEMMA 2. *If* $A_1A_i\perp H_{1i}$, $i = 2, \ldots, n$, *then* $A_1A_{n+1}\perp H_{1n+1}$.

Proof. From the assumption we obtain the relations $A_1A_i\perp A_nA_{n+1}$ for i=2,...,n-1. This is equivalent to the condition $A_nA_{n+1}\perp H_{nn+1}$. Taking also into account relation $A_1A_n\perp H_{1n}$, we infer that the plane $A_1A_nA_{n+1}$ is normal to H_{1n+1}. Hence $A_1A_{n+1}\perp H_{1n+1}$. ∎

REMARK 3. The above remains true in the more general version: if $A_1A_i\perp H_{1i}$ $i = 2, \ldots, n+1$, $i \neq k$, then $A_1A_k\perp H_{1k}$ $(k \in 2, \ldots, n+1)$.

LEMMA 3. *If* $A_iA_{i+1}\perp H_{ii+1}$ *(adding of indices modulo* $n+1$*) for* $i = 1, \ldots, n+1$ *and* $k \in 1, \ldots, n+1$, $i \neq k$, $k+1$, *then*

$$A_kA_{k+1}\perp H_{kk+1}, \quad A_{k+1}A_{k+2}\perp H_{k+1k+2}$$

and next $A_1A_i\perp H_{1i}$ *for* $i = 2, \ldots, n+1$.

Proof. We have, from the assumption, $A_iA_{i+1}\perp A_kA_{k+1}$ for $i \neq k-1$, $k, k+1$, i.e. $A_kA_{k+1}\perp H_{kk+1}$. In a similar way we state that

$$A_{k+1}A_{k+2}\perp H_{k+1k+2}.$$

Notice that we have, by the assumption,

(1) $$A_1A_i\perp A_mA_{m+1} \quad \text{for } m \neq 1, i-1, i.$$

Take into account the tetrahedron $A_1A_{i-1}A_iA_{i+1}$. We have

$$A_iA_{i+1}\perp A_1A_{i-1} \quad \text{and} \quad A_{i-1}A_i\perp A_1A_{i+1}.$$

Hence $A_1A_i \perp A_{i-1}A_{i+1}$. From the last relation and (1) it follows that $A_1A_i \perp H_{1i}$ for $3 \leq i \leq n$. Since also $A_1A_{n+1} \perp H_{1n+1}$, the thesis holds. ■

As a corollary from the above lemmas we obtain

THEOREM 2. *If, for fixed k and m,* $A_kA_i \perp H_{ki}$ *for* $i \neq k, m$ *or* $A_iA_{i+1} \perp H_{ii+1}$ *(adding of indices modulo* $n+1$*)* $i \neq m, m+1$, $i = 1, \dots, n+1$, *then* $A_pA_q \perp H_{pq}$ *all* $p \neq q$, *i.e. S is orthocentric.*

Thus we see that only n-1 perpendicularities of type $A_iA_j \perp H_{ij}$ are sufficient to S be orthocentric. In particular, when n=3 we obtain the well-known property quoted above. However it should be noticed that these n-1 perpendicularities cannot be assumed arbitrarily. The following counterexample shows it:

Let A_1(0,0,0,0), A_2(1,0,0,0), A_3(2,-2,0,0), A_4(2,1,2,0), A_5(3,2,-3,1) be five points in the four- dimensional space. It is easy to check that $A_1A_3 \perp H_{13}$, $A_3A_4 \perp H_{34}$ and $A_2A_5 \perp H_{25}$, but A_1A_2 is not perpendicular to A_3A_5.

3. The orthocentric structure of simplexes

In this section let S^{n+2} be an orthocentric, nondegenerate simplex with the vertices $A_1, \dots, A_{n+1}$ and the orthocenter A_{n+2} in the space E^n. Then by Theorem 1 we get the following

LEMMA 4. *The point* A_i *is the orthocenter of the simplex* S^i *with the vertices* $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_{n+2}$, $i = 1, \dots, n+2$.

The above suggests us to introduce a definition.

DEFINITION 1. Let A_{n+2} be the orthocenter of the simplex S^{n+2}. The $n+2$ points $A_1, \dots, A_{n+1}, A_{n+2}$ are such that the 1-dimensional edge joining any two of them is perpendicular to the (n-1)- dimensional subspace determined by the remaining n points. n+2 such points will be said to form an orthocentric structure. Each point of it is the orthocenter of the simplex determined by the remaining n+1 points. Hence the n+2 points determine also an orthocentric structure of n+2 simplexes.

Using some properties of the Euler line of an orthocentric simplex (see [1] and [3]) we prove

THEOREM 3. *The n+2 Euler lines of an orthocentric structure of simplexes are concurrent.*

Proof. Let $\varepsilon_{1\dots n}$ be the Euler line of the (n-1) - dimensional face $A_1 \dots A_n$. Obviously $A_1 \dots A_n$ is the common face of the simplexes S^{n+1} and S^{n+2}. Let G^{n+2} be the centroid and O^{n+2} the circumcenter of S^{n+2} (Fig. 1). The point A_{n+2} is the orthocenter of S^{n+2} hence the line $A_{n+2}G^{n+2}O^{n+2}$ is the Euler line of S^{n+2}. By some theorems from [3], we know that $Q_{1\dots n}G' : G'G_{1\dots n} = n : 1$, where $Q_{1\dots n}$ is the orthocenter, $G_{1\dots n}$ the centroid of the face $A_1 \dots A_n$

and G' is the projection of G^{n+2} onto $\varepsilon_{1\dots n}$. Let $A_{n+2}G_{1\dots n}$ be the median of S^{n+1}. This line meets $G'G_{n+2}$ in some point. It must be the centroid G^{n+1} of S^{n+1} because the projection of this point cuts $\varepsilon_{1\dots n}$ at the same ratio n:1. The line $A_{n+1}G^{n+1}O^{n+1}$ is the Euler line of S^{n+1}. Let N be the point of intersection of the two lines $A_{n+2}G^{n+2}O^{n+2}$ and $A_{n+1}G^{n+1}O^{n+1}$. Applying Menelaus' theorem to the triangle $A_{n+2}G^{n+2}G_{1\dots n}$ and the transversal $A_{n+1}G^{n+1}O^{n+1}$ we find that

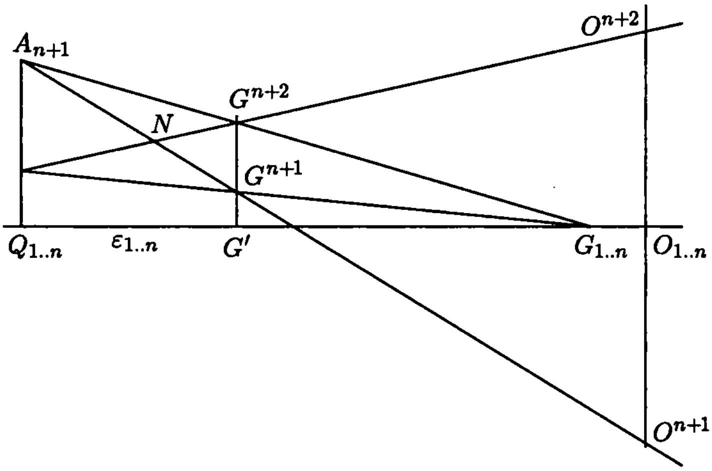

Fig. 1

$$(A_{n+2}G^{n+1} : G^{n+1}G_{1\dots n}) \cdot (G_{1\dots n}A_{n+1} : A_{n+1}G^{n+2}) \cdot (G^{n+2}N : NA_{n+2}) = 1.$$

The first two ratios (by the Area Principle, see [4]) are $n : 1$ and $(n+1) : n$, respectively then

$$G^{n+2}N : NA_{n+2} = 1 : (n+1).$$

Certainly $G^{n+1}N : NA_{n+1} = 1 : (n+1)$. Thus the Euler line of S^{n+2} is met by the Euler line of another simplex of the orthocentric structure in N, hence the proposition. ■

DEFINITION 2. The point N is called the orthic point of the orthocentric structure of simplexes.

As the corollaries from Theorem 3 we get

THEOREM 4. (a) *The centroids of an orthocentric structure of simplexes form an orthocentric structure of points.*

(b) *The circumcenters of an orthocentric structure of simplexes form an orthocentric structure of points.*

The $(n+2)$ centroids form a figure homothetic to the points of the given orthocentric structure, the homothetic center being the orthic point N, and the homothetic ratio being $-1 : (n+1)$. The $(n+2)$ circumcenters also form a figure homothetic to the points of the given orthocentric structure, the homothetic center being the orthic point N, and the homothetic ratio being $-n : 2$. ■

REMARK 4. The preceding theorems are analogous to the corresponding propositions dealing with the orthocentric structure of four points in the plane. The orthic point corresponds to the nine- point center in the plane. But the analogy can not be pursued much further. Contrary to what happens on the plane, on the Euler line of a simplex in n-dimensional Euclidean space we have

$$GO' : O'Q = (n-1) : (n+1)$$

where G is the centroid, Q is the orthocenter and O' the center of the $3(n+1)$-points sphere of an orthocentric simplex (see [3]). Thus accordingly to Theorem 3

$$GN : NQ = 1 : (n+1).$$

Hence $(n-1) : (n+1) = 1 : (n+1)$ (i.e. the orthic point N equals the $3(n+1)$-points center) only for $n = 2$. Similarly, the simplexes of an orthocentric structure do not have the same orthic simplex as it is in E^2 (see [1] for this and others examples for E^2 and E^3).

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