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PROJECTIONS OF CYLINDERS AND GENERALIZATION OF THE PARABELN MODEL OF AFFINE GEOMETRY

Abstract. We generalize the known construction of "parabeln model" of the affine plane geometry to arbitrary finite dimension and study some transformations characteristic for the obtained structures.

1. Introduction

It is known that in R^2 the class of all parabolas with equation $y = \alpha \cdot x^2 + b \cdot x + c$ with fixed parameter $\alpha \neq 0$, together with the lines $x = c$, $c \in R$ is an affine plane. At the same time the class of all parabolas (i.e. with α arbitrary) is a kind of Galileo plane – a projection of the real Laguerre plane (see [4]).

It is seen that similar procedure can be applied in the case of three dimensional space R^3, in which we can consider paraboloids as planes of an affine space. In the paper we generalize this construction to the n-dimensional real space and projection of a (projective) cylinder onto this space. We establish main formulas describing analytically the obtained surfaces and the geometry determined by these surfaces.

2. Projection of a cylinder

Let us consider the space R^{n+1} and its subspace $H: x_{n+1} = 0$, the elements of H will be identified with points of R^n under the map

$$H \ni (x_1, \ldots, x_n, 0) \mapsto (x_1, \ldots, x_n).$$

In the space R^{n+1} we consider the cylinder W defined by

$$(1) \qquad W: \sum_{i=1}^{n-1} \varepsilon_i a_i^2 x_i^2 + (x_{n+1} - 1)^2 = 1,$$

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where $a_i \neq 0$ are arbitrary, and $\varepsilon_i^2 = 1$. The vertex of W is the direction of the x_n-axis. Let $b = (0, \dots, 0, 2) \in W$.

Let ξ be the projection of W onto H with center b. We shall investigate the image of geometry of W under the map ξ. As usual, this geometry is defined on W by the family of all the intersections $W \cap P$, where P is a hyperplane of R^{n+1}.

Let P be an arbitrary hyperplane of R^{n+1}:

$$P\colon \quad \sum_{i=1}^{n+1} A_i x_i + E = 0. \tag{2}$$

We are going to determine $\xi(W \cap P)$.

FACT 2.1. *If $b \in P$ and $P \not\parallel H$ then $\xi(W \cap P) = H \cap P$ is a hyperplane of H.*

Proof. The projection of P is a hyperplane $P \cap H$ of H, so it remains to prove that each line in P which is not parallel to H and goes through b – crosses W in some other point. But this is clear, since all the lines through b tangent to W lie in the hyperplane tangent to W at b, and this hyperplane is parallel to H. ∎

Now, let $b \notin P$, this yields $2A_{n+1} + E \neq 0$, so we can assume

$$E + 2A_{n+1} = 1, \text{ i.e. } E = 1 - 2A_{n+1}. \tag{3}$$

FACT 2.2. *Let P be a hyperplane of R^{n+1} with equation* (2) *such that $b \notin P$.*

(i) *If P is not parallel to x_n axis then $T = \xi(W \cap P)$ is a quadric in R^n and it can represented as the graph of a function:*

$$T\colon \quad x_n = \sum_{i=1}^{n-1} \frac{-\varepsilon_i a_i^2}{\mu}(x_i + p_i)^2 + p_n, \tag{4}$$

where $\mu = 4A_n$, $p_i = \frac{2\varepsilon_i A_i}{a_i^2}$ for $i = 1, \dots, n-1$, and

$$p_n = \frac{1}{4A_n}\Big(\sum_{i=1}^{n-1} \varepsilon_i a_i^2 p_i^2 + 8A_{n+1} - 4\Big).$$

Conversely, for every set T with equation (4) *there exists a hyperplane P such that $\xi(W \cap P) = T$.*

(ii) *If P is parallel to x_n axis then $T = \xi(W \cap P)$ is a cylinder in R^n with equation*

$$T\colon \quad \sum_{i-1}^{n-1} \varepsilon_i a_i^2 (x_i + p_i)^2 = -p_n, \tag{5}$$

where p_i are as in (i). *The direction of the x_n-axis is the vertex of T.*

Proof. Let $x = (x_1, \dots, x_n, 0) \in H$. Points of the line which joins x and b are of the form $p_\alpha = (\alpha x_1, \dots, \alpha x_n, 2 - 2\alpha)$ with $\alpha \in R$. The condition $p_\alpha \in P$ leads with (3) to

$$\alpha \cdot \Big(\sum_{i=1}^{n} A_i x_i - 2A_{n+1}\Big) = -1, \tag{6}$$

and the condition $p_\alpha \in W$ yields

$$\alpha \cdot \Big(\sum_{i=1}^{n-1} \varepsilon_i a_i^2 x_i^2 + 4\Big) = 4. \tag{7}$$

Combining (6) and (7) we obtain an equation of the set $T = \xi(W \cap P)$:

$$T\colon \quad \sum_{i=1}^{n-1} (\varepsilon_i a_i^2 x_i^2 + 4A_i x_i) + 4A_n x_n - 8A_{n+1} + 4 = 0. \tag{8}$$

For $p_i = \frac{2\varepsilon_i A_i}{a_i^2}$ we get $\varepsilon_i a_i^2 x_i^2 + 4A_i x_i = \varepsilon_i a_i^2 (x_i + p_i)^2 - \varepsilon_i p_i^2 a_i^2$ for $i = 1, \dots, n-1$. Thus (8) is equivalent to

$$T\colon \quad \sum_{i=1}^{n-1} \varepsilon_i a_i^2 (x_i + p_i)^2 + 4A_n x_n - \sum_{i=1}^{n-1} \varepsilon_i p_i^2 a_i^2 - 8A_{n+1} + 4 = 0. \tag{9}$$

Clearly, P is not parallel to x_n axis iff $A_n \neq 0$. In this case the equation (9) is equivalent to (4), and if P is parallel to x_n axis then (9) is equivalent to (5). ∎

3. Generalized paraboloids

Let $\mathcal{T}$ be the class of all sets of the form (4). Elements of $\mathcal{T}$ will be called *generalized paraboloids*. Let $T_0 \in \mathcal{T}$ be defined by

$$T_0\colon \quad x_n = \sum_{i=1}^{n-1} \varepsilon_i a_i^2 x_i^2. \tag{10}$$

FACT 3.1. *The family $\mathcal{T}$ consists of all the images of T_0, given by* (10), *under dilatations of R^n.*

Proof. Consider a dilatation f of R^n such that f^{-1} is defined by the formula

$$f^{-1}(x) = \beta \cdot x + t, \quad t = (t_1, \dots, t_n). \tag{11}$$

Then $f(T_0)$ has equation

$$f(T_0)\colon \quad \beta x_n + t_n = \sum_{i=1}^{n-1} \varepsilon_i a_i^2 (\beta x_i + t_i)^2 = \beta \sum_{i=1}^{n-1} \varepsilon_i a_i^2 \Big(x_i + \frac{t_i}{\beta}\Big)^2 \beta. \tag{12}$$

This equation can be rewritten in the form

$$(13)\qquad f(T_0):\quad x_n = \sum_{i=1}^{n-1} \varepsilon_i a_i^2 \beta\Big(x_i + \frac{t_i}{\beta}\Big)^2 - \frac{1}{\beta} t_n.$$

It is seen that $f(T_0) \in \mathcal{T}$. But conversely, for every $T \in \mathcal{T}$ with equation (4) we set $\beta = \frac{1}{\mu}$, $t_i = \beta p_i$ for $i = 1, \ldots, n-1$, and $t_n = -\beta p_n$ and then the dilatation f defined by (11) maps T_0 onto T. ∎

Let us fix a point $b' \in W$ on the affine generator passing through b, then $b' = (0, \ldots, 0, c, 2)$ for some $c \neq 0$. If b' belongs to a hyperplane P with equation (2) such that (3) holds then we get $A_n = \frac{-1}{c}$. Consequently, the general form of the projections of such hyperplanes P is the following

$$(14)\qquad T:\quad \lambda x_n = \sum_{i=1}^{n-1} \varepsilon_i a_i^2 (x_i - q_i)^2 + q_n,$$

where $\lambda = \frac{4}{c}$ is fixed and $q_i = -p_i$, $q_n = \frac{4}{c} p_n$ are arbitrary. Let $\mathcal{T}_\lambda$ be the class of all the sets with equations of the form (14). We have evident

FACT 3.2. *For every two paraboloids $T_1, T_2 \in \mathcal{T}_\lambda$ there is a translation of R^n which maps T_1 onto T_2, and every image of a paraboloid $T \in \mathcal{T}_\lambda$ under a translation of R^n is in $\mathcal{T}_\lambda$ as well.* ∎

4. Reflections in paraboloids

For every $T \in \mathcal{T}$ we define the symmetry σ_T by the condition

$\sigma_T(x) = x'$ iff $x \in T$ and $x' = x$, or

$x \oplus x' \in T$ and the line xx' has direction of x_n axis.

Clearly, σ_T is a well defined involution with $\mathrm{Fix}(\sigma_T) = T$. Let $T_0 \in \mathcal{T}$ be the paraboloid with equation

$$(15)\qquad T_0:\quad 2\mu x_n = \sum_{i=1}^{n-1} \varepsilon_i a_i^2 x_i^2.$$

We write $\mathcal{T}_0$ for the class of all the hyperplanes of R^n which are parallel to the x_n axis. Clearly, $\sigma_T(S) = S$ for every $T \in \mathcal{T}$ and $S \in \mathcal{T}_0$. Reflections in paraboloids are automorphisms of the family $\mathcal{T} \cup \mathcal{P}$, where $\mathcal{P}$ is the set of all the $(n-1)$-hyperplanes of R^n.

FACT 4.1. *Let $T' \in \mathcal{T}_{2\mu}$. The function $\sigma_{T'}$ maps the family $\mathcal{P} \setminus \mathcal{T}_0$ onto $\mathcal{T}_\mu$. Consequently, $\sigma_{T'}$ maps $\mathcal{T}_\mu$ onto $\mathcal{P} \setminus \mathcal{T}_0$.*

Proof. Let $T' = T_0$ be defined by (15). Right from definition we get

$$(16)\qquad \sigma_{T_0}(x_1, \ldots, x_n) = \Big(x_1, \ldots, x_{n-1}, \frac{1}{\mu} \sum_{i=1}^{n-1} \varepsilon_i a_i^2 x_i^2 - x_n\Big).$$

Let S be a hyperplane of R^n such that $S \notin \mathcal{T}_0$. Then S has equation

$$(17) \qquad S:\quad x_n = \sum_{i=1}^{n-1} D_i x_i + D_0$$

and its image $\sigma_{T_0}(S)$ has equation

$$(18) \qquad \sigma_{T_0}(S):\quad \frac{1}{\mu}\sum_{i=1}^{n-1} \varepsilon_i a_i^2 x_i^2 - x_n = \sum_{i=1}^{n-1} D_i x_i + D_0,$$

which is equivalent to

$$(19) \qquad \sigma_{T_0}(S):\quad \mu x_n = \sum_{i=1}^{n-1} \varepsilon_i a_i^2 x_i^2 - \sum_{i=1}^{n-1} \mu D_i x_i - \mu D_0.$$

Note that $\varepsilon_i a_i^2 x_i^2 - \mu D_i x_i = \varepsilon_i a_i^2 (x_i - p_i)^2 + p_i^2$, where $p_i = \frac{\varepsilon_i \mu D_i}{2a_i^2}$ for $i = 1, \ldots, n-1$. With $p_n = \mu(\mu \sum_{i=1}^{n-1} \frac{D_i^2}{4a_i^4} - D_0)$ we obtain the equation

$$(20) \qquad \sigma_{T_0}(S):\quad \mu x_n = \sum_{i=1}^{n-1} \varepsilon_i a_i^2 (x_i - p_i)^2 - p_n.$$

This yields $\sigma_{T_0}(S) \in \mathcal{T}_\mu$.

Conversely, if $T \in \mathcal{T}_\mu$ has equation (14) then we set $\mu = \lambda$, $D_i = -\frac{\varepsilon_i a_i^2 q_i}{c}$ for $i = 1, \ldots, n-1$, and $D_0 = \frac{1}{c}(\sum_{i=1}^{n-1} q_i^2 + q_n)$ and then for the hyperplane S with the above coefficients in the equation (17) we obtain $\sigma_{T_0}(S) = T$.

Now, let $T' \in \mathcal{T}_{2\mu}$ be arbitrary. By 3.2 there is a translation τ of R^n with $\tau(T_0) = T'$. It is seen that $\sigma_{T'} = (\sigma_{T_0})^\tau = \tau \circ \sigma_{T_0} \circ \tau^{-1}$. This yields $\sigma_{T'}(S) \in \mathcal{T}_\mu$ for every $S \in \mathcal{P}$. ∎

FACT 4.2. *Let $T' \in \mathcal{T}_{2\mu}$ and $\mu \neq \lambda$. Then the function $\sigma_{T'}$ maps $\mathcal{T}_\lambda$ onto $\mathcal{T}_{\frac{\mu\cdot\lambda}{\lambda-\mu}}$.*

Proof. Let $T' = T_0$ be given by the equation (15); then the symmetry σ_{T_0} is defined by (16). Let $T \in \mathcal{T}_\lambda$ be defined by the formula (14). Then we obtain the following equation of $\sigma_{T_0}(T)$:

$$(21) \qquad \sigma_{T_0}(T):\quad \lambda\Big(\frac{1}{\mu}\sum_{i=1}^{n-1} \varepsilon_i a_i^2 x_i^2 - x_n\Big) = \sum_{i=1}^{n-1} \varepsilon_i a_i^2 (x_i - q_i)^2 + q_n,$$

which is equivalent to

$$(22) \qquad \lambda\mu x_n = \sum_{i=1}^{n-1} (\lambda \varepsilon_i a_i^2 x_i^2 - \mu \varepsilon_i a_i^2 (x_i - q_i)^2) - \mu q_n$$

$$= \sum_{i=1}^{n-1} (\varepsilon_i(\lambda - \mu) a_i^2 x_i^2 + 2\mu \varepsilon_i a_i^2 q_i x_i - \mu \varepsilon_i a_i^2 q_i^2) - \mu q_n.$$

Set $r_i = -\frac{\mu q_i}{\lambda-\mu}$ and $r_0 = \frac{\mu}{\lambda-\mu} q_n + \frac{\mu\lambda}{(\lambda-\mu)^2} \sum_{i=1}^{n-1} \varepsilon_i a_i^2 q_i^2$. Then (22) is equivalent to

$$\frac{\lambda \cdot \mu}{\lambda - \mu} x_n = \sum_{i=1}^{n-1} \varepsilon_i a_i^2 (x_i - r_i)^2 - r_0. \tag{23}$$

Therefore, $\sigma_{T_0}(T) \in \mathcal{T}_{\frac{\mu \cdot \lambda}{\lambda-\mu}}$. Now, let $T' \in \mathcal{T}_{2\mu}$ be arbitrary. As in the proof of 4.1, by 3.2 we consider a translation τ of R^n with $\sigma_{T'} = \sigma_{T_0}^{\tau}$ and then, by 3.2 we get $\sigma_{T'}(T) = \tau(\sigma_{T_0}(\tau^{-1}(T))) \in \mathcal{T}_{\frac{\mu \cdot \lambda}{\lambda-\mu}}$ for every $T \in \mathcal{T}_\lambda$. ∎

5. Paraboloid model of affine space

As an immediate consequence of 4.1 we have

COROLLARY 5.1. *If T_0 is defined by* (15) *then the map σ_{T_0} is an isomorphism between the affine geometry of R^n and the structure $\mathsf{G}_\mu^n = \langle R^n, \mathcal{T}_0 \cup \mathcal{T}_\mu \rangle$, where elements of the set $\mathcal{T}_0 \cup \mathcal{T}_\mu$ are considered as hyperplanes.*

Clearly, the symmetry σ_{T_0} maps ordinary dilatations of R^n onto the dilatations of G_μ^n. Calculating we come to the following

FACT 5.2. *Let T_0 be given by the equation* (15)*, so the symmetry σ_{T_0} is defined by the formula* (16).

(i) *If f is a homothety defined by the formula $f(x) = \alpha x$ then the map $f^{\sigma_{T_0}} = \sigma_{T_0} \circ f \circ \sigma_{T_0}$ is defined by*

$$f^{\sigma_{T_0}}(x_1, \ldots, x_n) = \left(\alpha x_1, \ldots, \alpha x_{n-1}, \alpha\left(\frac{1}{\mu} \sum_{i=1}^{n-1} \varepsilon_i a_i^2 x_i^2 (\alpha - 1) + x_n\right)\right). \tag{24}$$

(ii) *If f is a translation of R^n, $f(x) = x + \tau$, $\tau = (\tau_1, \ldots, \tau_n)$ then the map $f^{\sigma_{T_0}}$ is defined by*

$$\begin{aligned} f^{\sigma_{T_0}}&(x_1, \ldots, x_n) \\ &= \left(x_1 + \tau_1, \ldots, x_{n-1} + \tau_{n-1}, \frac{2}{\mu} \sum_{i=1}^{n-1} \varepsilon_i a_i^2 \tau_i x_i + x_n + \tau_n'\right), \end{aligned} \tag{25}$$

where $\tau_n' = \frac{1}{\mu} \sum_{i=1}^{n-1} \varepsilon_i a_i^2 \tau_i^2 - \tau_n$. Consequently, $f^{\sigma_{T_0}}$ is a translation of R^n iff f has the x_n axis as its direction. Otherwise, $f^{\sigma_{T_0}}$ is a composition of a translation

$$(x_1 \ldots, x_n) \mapsto (x_1 + \tau_1, \ldots, x_{n-1} + \tau_{n-1}, x_n + \tau_n')$$

and a shear of R^n,

$$(x_1, \ldots, x_n) \mapsto \left(x_1, \ldots, x_{n-1}, \frac{2}{\mu} \sum_{i=1}^{n-1} \varepsilon_i a_i^2 \tau_i x_i + x_n\right)$$

which has the hyperplane with equation $\frac{2}{\mu} \sum_{i=1}^{n-1} \varepsilon_i a_i^2 \tau_i x_i = 0$ as an axis. ∎

As a consequence we obtain

FACT 5.3. *If $T_1, T_2 \in \mathcal{T}_\mu$ then T_1 and T_2 are parallel in G^n_μ iff there is a translation along the x_n axis which maps T_1 onto T_2.* ■

Now we shall give a characterization of the (affine) lines of such models. Clearly, every line L can be determined as the intersection of a family of hyperplanes. We can choose them in such a way, that $L = D \cap T$, where D is a plane parallel to the x_n-axis, and $T \in \mathcal{T}_0 \cup \mathcal{T}_\mu$. Let $o = (o_1, \ldots, o_n) \in D$, then

(26) $$D = \{(o_1 + \gamma e_1, \ldots, o_{n-1} + \gamma e_{n-1}, o_n + \delta) \colon \gamma, \delta \in R\},$$

where $e = (e_1, \ldots, e_{n-1}, 0)$ and $w = (0, \ldots, 0, 1)$ are two vectors which span D. Indeed, let a, b be any two points of L, and let L' be the affine line on these two points. We choose D as the affine plane parallel to the x_n-axis, which contains L'. Clearly, $a, b \in D$, so $L \subset D$. Then, either, L' is parallel to x_n as well, or e is the vector of the line in which D crosses the hyperplane with equation $x_n = 0$. Let T be given by the formula (14). To find the intersection $D \cap T$ we must solve the equation

(27) $$\lambda(o_n + \delta) = \sum_{i=1}^{n-1} \varepsilon_i a_i^2 (\gamma e_i + o_i - q_i)^2 + q_n.$$

Then we get a parametric definition of $D \cap T$:

(28) $$D \cap T\colon \quad o + \gamma e + p(\gamma) w,$$
$$\text{where } p(\gamma) = \sum_{i=1}^{n-1} \frac{\varepsilon_i a_i^2}{\lambda} (\gamma e_i + o_i - q_i)^2 + \frac{q_n}{\lambda} - o_n.$$

We see that $p(\gamma) = p_2\gamma^2 + p_1\gamma + p_0$ is, either, a linear map, or a quadratic map. Consequently, $D \cap T$ is either a line (ordinary affine), or a parabola. Note that the coefficients of $p(\gamma)$ are

(29) $$p_2 = \frac{1}{\lambda} \sum_{i=1}^{n-1} \varepsilon_i a_i^2 e_i^2, \; p_1 = \frac{1}{\lambda} \sum_{i=1}^{n-1} 2\varepsilon_i a_i^2 e_i (o_i - q_i),$$
$$\text{and} \quad p_0 = \frac{1}{\lambda} \sum_{i=1}^{n-1} \varepsilon_i a_i^2 (o_i - q_i)^2 + \frac{q_n}{\lambda} - o_n.$$

The vector e is determined by D uniquely – up to proportionality, and coefficients p_i depend on this vector, $p_i = p_i(e)$. Note that

(30) $$p_2(\alpha e) = \alpha^2 p_2(e), \; p_1(\alpha e) = \alpha p_1(e), \text{ and } p_0(\alpha e) = p_0(e)$$

for every scalar α.

Let us fix a plane D. Therefore, either, $p_2(e) = 0$ for every suitable vector e – in this case $D \cap T$ is always a line and the set of lines in D determined by G^n_μ coincides with the set of lines of R^n which lie on D, or $p_2(e) \neq 0$, and we can choose e so as $|p_2(e)| = \frac{1}{\lambda}$. In this case the set of "lines" in D determined by G^n_μ yields the ordinary parabeln model (see [1]).

6. Geometry on a paraboloid

Clearly, for every paraboloid $T \in \mathcal{T}_\lambda$ with equation (14) the map

$$\pi^{-1}\colon \quad (x_1,\dots,x_{n-1}) \mapsto \Big(x_1,\dots,x_{n-1}, \frac{1}{\lambda}\Big(\sum_{i=1}^{n-1} \varepsilon_i a_1^2 (x_i - q_i)^2 + q_n\Big)\Big) \tag{31}$$

establishes a one-to-one correspondence between the hyperplane $M\colon \quad x_n = 0$ of R^n and the set T; its inverse π is just a projection along the x_n-axis.

There is a natural structure of a metric affine space defined on M determined by the form

$$\zeta(x,y) = \sum_{i=1}^{n-1} \varepsilon_i a_i^2 x_i y_i. \tag{32}$$

The map π^{-1} defines on T an isomorphic copy of the geometry on M, but, as we can see, this geometry is also induced on T just by its "own nature".

FACT 6.1. *Let $S \in \mathcal{P}$ be a $(n-1)$-hyperplane in R^n.*

(i) *If $S \in \mathcal{T}_0$ then $\pi(S \cap T)$ is a hyperplane in M.*

(ii) *If $S \notin \mathcal{T}_0$ then, either $\pi(S \cap T)$ is empty, or $\pi(S \cap T)$ is a (possibly degenerated) hypersphere in M, with respect to the form defined by* (32).

Every hyperplane in M and every hypersphere in M can be obtained by (i) *and* (ii).

Proof. The statement (i) is evident.

(ii) Without loss of generality we can assume that $T \in \mathcal{T}_\lambda$ is defined by (14) with $q_i = 0$ for $i = 1,\dots,n$. Let $S \in \mathcal{P} \setminus \mathcal{T}_0$ be defined by (17). Then $S \cap T$ is characterized on S by the equation

$$\sum_{i=1}^{n-1} (\varepsilon_i a_i^2 x_i^2 - \lambda D_i x_i) - \lambda D_0 = 0. \tag{33}$$

Consequently, $\pi(S \cap T)$ is characterized by the equation

$$\sum_{i=1}^{n-1} \varepsilon_i a_i^2 (x_i - s_i)^2 = \lambda^2 \Big(\sum_{i=1}^{n-1} \varepsilon_i \frac{D_i^2}{4a_i^2}\Big) + \lambda D_0, \tag{34}$$

where $s_i = \frac{\lambda D_i}{2\varepsilon_i a_i^2}$. But (34) is a general equation of a hypersphere in M. Conversely, given a hypersphere S_0 with equation $\sum_{i=1}^{n-1} \varepsilon_i a_i^2 (x_i - s_i)^2 = \rho$ we take the hyperplane S with equation (17), where $D_i = \frac{2\varepsilon_i a_i^2 s_i}{\lambda}$ for $i = 1, \ldots, n-1$ and $D_0 = \frac{\rho - \sum_{i=1}^{n-1} \varepsilon_i a_i^2 s_i^2}{\lambda}$; by the above, $\pi(S \cap T) = S_0$. ∎

Analogously, for every symmetry σ_K of M, where K is an admitting reflection hyperplane, the map $(\sigma_K)^{\pi^{-1}} = \pi^{-1} \circ \sigma_K \circ \pi$ is a symmetry of T. This symmetry is, in fact, a suitable symmetry of R^n.

FACT 6.2. *Let $T \in \mathcal{T}$, $S \in \mathcal{T}_0$, $S' = \pi(S \cap T)$, and $t = [t_1, \ldots, t_n]$ be a direction (a vector) in R^n.*

(i) *If the skew symmetry σ_S^t of R^n with axis S and direction t preserves T then in M the map $(\sigma_S^t)^\pi$ is the reflection in S'.*

(ii) *Every symmetry $\sigma_{S'}$ of M is of the form given in (i).*

Proof. Without loss of generality we can assume that $T \in \mathcal{T}_{2\mu}$ is defined by the formula (15) and S is defined by the formula $\sum_{i=1}^{n-1} D_i x_i + D_0 = 0$ (cf. (17)). Then the projective closures: $\overline{T}$ of T and $\overline{S}$ of S have equations

$$(35) \quad \overline{T}\colon \quad 2\mu x_0 x_n - \sum_{i=1}^{n-1} \varepsilon_i a_i^2 x_i^2 = 0, \text{ and } \overline{S}\colon \quad \sum_{i=1}^{n-1} D_i x_i + D_0 x_0 = 0.$$

Thus the hyperplane polar to the projective point $d = \{d_0, d_1, \ldots, d_n\}$ has equation

$$(36) \qquad \mu d_0 x_n + \mu d_n x_0 - \sum_{i=1}^{n-1} \varepsilon_i a_i^2 d_i x_i = 0 \text{ (cf. [2])}.$$

Combining (35) and (36) we get the pole $d = \{0, d_1, \ldots, d_n\}$ of $\overline{S}$ with respect to $\overline{T}$, where $d_i = \frac{D_i}{\varepsilon_i a_i^2}$ for $i = 1, \ldots, n-1$ and $d_n = -\frac{D_0}{\mu}$. Therefore, $\sigma_S^t(T) = T$ iff the vector t is parallel to the vector $[\frac{D_1}{\varepsilon_1 a_1^2}, \ldots, \frac{D_{n-1}}{\varepsilon_{n-1} a_{n-1}^2}, -\frac{D_0}{\mu}]$. Set $t' = [\frac{D_1}{\varepsilon_1 a_1^2}, \ldots, \frac{D_{n-1}}{\varepsilon_{n-1} a_{n-1}^2}]$. It is seen that $(\sigma_S^t)^\pi$ coincides with the skew symmetry $\sigma_{S'}^{t'}$ of M. To finish the proof we note that, evidently, S' and t' are orthogonal with respect to the form defined by (32). ∎

The above result can be strengthened, establishing affine automorphisms of the considered paraboloid. For convenience, let T be determined by the equation (10). Recall that the form ζ is defined on $M\colon \; x_n = 0$ by the formula (32).

Let f be an affine transformation of R^n; therefore there is a matrix $A = [a_{i,j}]_{i,j=1,\ldots,n}$ and a vector $u \in R^n$ such that $\det A \neq 0$ and

$$(37) \qquad f(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$$

where $y_i = \sum_{j=1}^n a_{i,j} x_j + u_i$ for $i = 1, \ldots, n$.

LEMMA 6.3. *The following conditions are equivalent:*

(i) *The function f leaves T invariant.*

(ii) *The matrix A can be written in the form*

$$\begin{bmatrix} B & \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \end{bmatrix} \\ \begin{bmatrix} w_1\ w_2 \ldots w_{n-1} \end{bmatrix} & D \end{bmatrix}$$

such that: B is a $(n-1) \times (n-1)$ matrix with columns denoted by $B_1, \ldots,$ B_{n-1}; let us write $v = [u_1, \ldots, u_{n-1}]$ and $V = u_n$, then we have

a) $V = \zeta(v, v)$;

b) $\zeta(B_j, B_r) = 0$ *for* $j, r = 1, \ldots, n-1$, $j \neq r$;

c) $d = [0, \ldots, 0]$;

d) $w_j = 2\zeta(B_j, v)$ *for* $j = 1, \ldots, n-1$;

e) $\zeta(B_j, B_j) = D\varepsilon_j a_j^2$ *for* $j = 1, \ldots, n-1$, *and* $D \neq 0$.

Proof. Evidently, $f^{-1}(T)$ is determined by the equation $\sum_{i=1}^{n-1} \varepsilon_i a_i^2 y_i^2 - y_n = 0$, where y_i, y_n are given by (37). Substituting we come to

$$(38) \quad \sum_{i=1}^{n-1} \varepsilon_i a_i^2 \Big(\sum_{j,r=1}^{n} a_{i,j} a_{i,r} x_j x_r + u_i^2 + 2 \sum_{j=1}^{n} a_{i,j} u_i x_j \Big) - \sum_{j=1}^{n} a_{n,j} x_j - u_n = 0.$$

Then we find that the left hand side of (38) can be written as the sum of the following items

$$\sum_{i=1}^{n-1} \varepsilon_i a_i^2 u_i^2 - u_n \quad [= \zeta(v, v) - V]$$

$$\Big(2 \sum_{i=1}^{n-1} \varepsilon_i a_i^2 u_i a_{i,j} - a_{n,j}\Big) \cdot x_j \quad [= (2\zeta(v, B_j) - w_j) \cdot x_j] : j = 1, \ldots, n-1;$$

$$\Big(2 \sum_{i=1}^{n-1} \varepsilon_i a_i^2 u_i a_{i,n} - a_{n,n}\Big) \cdot x_n \quad [= (2\zeta(v, d) - D) \cdot x_n]$$

$$\Big(\sum_{i=1}^{n-1} \varepsilon_i a_i^2 a_{i,n}^2\Big) \cdot x_n^2 \quad [= \zeta(d, d) \cdot x_n^2]$$

$$\Big(\sum_{i=1}^{n-1} \varepsilon_i a_i^2 a_{i,j}^2\Big) \cdot x_j^2 \quad [= \zeta(B_j, B_j) \cdot x_j^2] : j = 1, \ldots, n-1;$$

$$\Big(2\sum_{i=1}^{n-1}\varepsilon_i a_i^2 a_{i,j} a_{i,r}\Big)\cdot x_j x_r \quad [=2\zeta(B_j,B_r)\cdot x_j x_r] : 1\le j<r\le n-1;$$

$$\Big(2\sum_{i=1}^{n-1}\varepsilon_i a_i^2 a_{i,j} a_{i,n}\Big)\cdot x_j x_n \quad [=2\zeta(B_j,d)\cdot x_j x_n] : j=1,\dots,n-1.$$

Taking into account that the equation (38) must be proportional with some coefficient $\lambda \neq 0$ to the equation

$$\sum_{j=1}^{n-1}(\varepsilon_j a_j^2)\cdot x_j^2 + (-1)\cdot x_n = 0$$

we get (b) and $\zeta(B_j,B_j)=\lambda\varepsilon_j a_j^2$; therefore columns B_j form an orthogonal basis of R^{n-1}. Then we get $\zeta(B_j,d)=0$ for $j=1,\dots,n-1$, which gives (c). In particular, it follows $\zeta(d,d)=0$ and $\zeta(v,d)=0$. The remaining conditions are now evident. ∎

As a consequence we obtain

PROPOSITION 6.4. *The two classes coincide: one consisting of the restrictions $f{\restriction}T$ where f is an affine transformations of R^n which preserves T, and the second consisting of all the maps $\pi\circ g\circ\pi^{-1}$ where g is a similarity of R^{n-1} equipped with the form ζ.*

Proof. It suffices to note that (in notation of 6.3) for a given f the map $g\colon R^{n-1}\longrightarrow R^{n-1}$ defined by

$$g(x_1,\dots,x_{n-1})=(z_1,\dots,z_{n-1}) \tag{39}$$

with

$$z_i=\sum_{j=1}^{n-1}b_{i,j}x_j+v_i \text{ for } i=1,\dots,n-1$$

has the required properties. Conversely, given g (i.e. given B and v) uniquely determines f – i.e. it determines A and u. ∎

7. Inversions

The formula (32) defines a form on R^n as well, though now, the form ζ is degenerate. A hypersphere S' determined by this form has equation

$$S'\colon\ \sum_{i=1}^{n-1}\varepsilon_i a_i^2(x_i-s_i)^2=\rho, \tag{40}$$

which is, in fact, equivalent to (5). One can define *inversion* $\sigma_{s,S'}$ in S' with center $s=(s_1,\dots,s_{n-1},s_n)$ (s_n arbitrary - S' does not determine its center uniquely!) by the following requirements:

a) $\sigma_{s,S'}(P) \subseteq P$ for every hyperplane P of R^n through s;

b) if $s \in P$ and ζ is nondegerate on P then $\sigma_{s,S'}|P$ is an ordinary (metric) inversion.

FACT 7.1. *Let S' be a ζ-hypersphere in R^n with a center s, and S be a $(n-1)$-hyperplane in R^n. If $s \notin S$ then, either*

(i) *S is not parallel to the x_n-axis, and $\sigma_{s,S'}(S)$ is a paraboloid, or*

(ii) *$S \in \mathcal{T}_0$ and then $\sigma_{s,S'}(S)$ is a hypersphere.*

Proof. Without loss of generality we can assume that S' is defined by (40) with $s_i = 0$ for $i = 1, \ldots, n$. One can see that the inversion $\sigma_\rho = \sigma_{s,S'}$ is defined by the formula

$$\sigma_\rho(x) = y, \text{ where } y_i = \frac{\rho}{\sum_{j=1}^{n-1} \varepsilon_j a_j^2 x_j^2} \cdot x_i \text{ for } i = 1, \ldots, n. \tag{41}$$

Note that σ_ρ is an involution.

Let $S \in \mathcal{P} \setminus \mathcal{T}_0$. Then S is characterized by the equation of the form (17), so $\sigma_\rho(S)$ is defined by

$$\sigma_\rho(S):\quad \rho x_n = \sum_{i=1}^{n-1} \rho D_i x_i + D_0 \cdot \sum_{i=1}^{n-1} \varepsilon_i a_i^2 x_i^2, \text{ where } \sum_{j=1}^{n-1} \varepsilon_j a_j^2 x_j^2 \neq 0. \tag{42}$$

Assumption $s \notin S$ yields $D_0 \neq 0$, so (42) is equivalent to

$$\sigma_\rho(S):\quad \frac{\rho}{D_0} x_n = \sum_{i=1}^{n-1} \varepsilon_i a_i^2 (x_i - q_i)^2 + q_n, \tag{43}$$

where $q_i = -\frac{\rho D_i}{2D_0 \varepsilon_i a_i^2}$ for $i = 1, \ldots, n-1$, and $q_n = \frac{-\rho^2}{4D_0^2} \sum_{i=1}^{n-1} \varepsilon_i \frac{D_i^2}{a_i^2}$. Consequently, $\sigma_\rho(S) \in \mathcal{T}$.

If S is parallel to the x_n-axis then its equation is of the form $\sum_{i=1}^{n-1} D_i x_i + D_0 = 0$, so its image under σ_ρ is defined by the equation

$$\sigma_\rho(S):\quad 0 = \sum_{i=1}^{n-1} \rho D_i x_i + D_0 \cdot \sum_{i=1}^{n-1} \varepsilon_i a_i^2 x_i^2. \tag{44}$$

Clearly, (44) is equivalent to

$$\sigma_\rho(S):\quad \sum_{i=1}^{n-1} \varepsilon_i a_i^2 (x_i - q_i)^2 = -q_n, \tag{45}$$

where q_i are as before. Therefore, $\sigma_\rho(S)$ is a hypersphere. ∎

8. Final remarks

Most of the results of the paper can be stated and proved in a more general setting, for spaces over an arbitrary field $\mathfrak{F}$ with $\mathrm{char}(\mathfrak{F}) \neq 2$ and symmetric bilinear form ζ_0 with $\dim(\mathrm{rad}(\zeta_0)) = 1$. Indeed, let $\mathbb{V}$ be a $(n+1)$-dimensional vector space over $\mathfrak{F}$ and ζ_0 be a form on $\mathbb{V}$. Let $\zeta_0(p,p) \neq 0$. One can consider the projection ξ of the cylinder $W = \{u \in \mathbb{V}: \zeta_0(u-p, u-p) = \zeta_0(p,p)\}$ from the point $2p \in W$ onto the subspace Y of $\mathbb{V}$ tangent to W at the zero vector $\theta \in W$. The form ζ_0 determines on Y a form ζ such that W is the inversive closure of (Y, ζ) (cf. [3]); in fact, $\zeta = \zeta_0|Y$ and $\mathrm{rad}(\zeta) = \mathrm{rad}(\zeta_0) \subset Y$. The family of all the sets $W \cap P$, where P is a hyperplane of $\mathbb{V}$ with $P \not\parallel \mathrm{rad}(\zeta_0)$, $p \notin P$ are mapped by ξ onto a family $\mathcal{T}$ of affine quadrics in Y such that the group of dilatations of Y acts transitively on $\mathcal{T}$. Each line in Y parallel to $\mathrm{rad}(\zeta)$ crosses $T \in \mathcal{T}$ in exactly one point – this enables us to define reflection σ_T for every $T \in \mathcal{T}$. Then σ_T maps $\mathcal{T} \cup \mathcal{P}$ onto itself, where $\mathcal{P}$ is the set of all the hyperplanes in Y. On the other hand, one can define in (Y, ζ) reflections in hyperspheres (suitable cylinders). Such a reflection maps a hyperplane of Y either, on itself, or on a hypersphere, or on an element of $\mathcal{T}$.

To get the above statements it suffices to choose in $\mathbb{V}$ a suitable orthogonal coordinate system so as $\mathrm{rad}(\zeta_0)$ is the x_n-axis, Y is spanned by the first n vectors of the basis, and $p = (0, \ldots, 0, 1)$. Then the form ζ_0 is defined by the formula $\zeta_0(x,y) = \sum_{i=1}^{n-1} \alpha_i x_i y_i + \alpha_{n+1} x_{n+1} y_{n+1}$ for some parameters α_j, and we can repeat (slightly more complicated) calculations similar to those presented in the previous sections. However, such an approach, though more general, seems to be less intuitive.

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