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SUPERPOSITION OPERATORS ON SEQUENCE SPACES DEFINED BY φ-FUNCTIONS

Abstract. Let λ and μ be solid sequence spaces. For a sequence of φ-functions $\Phi = (\phi_k)$ let $\lambda(\Phi) = \{x = (x_k) : (\phi_k(|x_k|)) \in \lambda\}$. Provided an another sequence of φ-functions $\Psi = (\psi_k)$, we present a method for the characterization of superposition operators $P_f : \lambda(\Phi) \to \mu(\Psi)$ by the assumption that acting conditions for $P_f : \lambda \to \mu$ are known. As applications we subscribe superposition operators on sequence spaces of Maddox and on multiplier spaces.

1. Introduction

Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{N} = \{1, 2, \ldots\}$. Let ω be the vector space of all real sequences $x = (x_k) = (x_k)_{k\in\mathbb{N}}$. By the term *sequence space*, we shall mean any linear subspace of ω. A sequence space (or a set of sequences) λ is called *solid* if $(x_k) \in \lambda$ and $|y_k| \leq |x_k| \quad (k \in \mathbb{N})$ yield $(y_k) \in \lambda$. Well known examples of solid sequence spaces are the space ℓ_∞ of all bounded sequences and the space c_0 of all convergent to zero sequences, also the spaces

$$\ell_p = \left\{x = (x_k) \in \omega : \sum_k |x_k|^p < \infty\right\}$$

and

$$(w_0)_p = \left\{x = (x_k) \in \omega : \lim_n \frac{1}{n} \sum_{k=1}^n |x_k|^p = 0\right\}$$

for $0 < p < \infty$. The sequences from $(w_0)_p$ are called strongly convergent (with index p) to zero. For $p = 1$ we write ℓ and w_0 instead of ℓ_1 and $(w_0)_1$, respectively.

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Let $p = (p_k)$ be a sequence of strictly positive numbers. As the generalizations of spaces ℓ_∞, c_0, ℓ_p and $(w_0)_p$ we consider the following solid sets of sequences (cf., for example, [17]):

$$\ell_\infty(p) = \{x = (x_k) \in \omega : \sup_k |x_k|^{p_k} < \infty\},$$

$$c_0(p) = \{x = (x_k) \in \omega : \lim_k |x_k|^{p_k} = 0\},$$

$$\ell(p) = \Big\{x = (x_k) \in \omega : \sum_k |x_k|^{p_k} < \infty\Big\},$$

$$w_0(p) = \Big\{x = (x_k) \in \omega : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_k} = 0\Big\}.$$

It is known that these sets are linear if the sequence $p = (p_k)$ is bounded, they are called also the sequence spaces of Maddox (see, for example, [11]). We note that sequence spaces of type $\ell(p)$ were introduced much earlier by Orlicz [21].

These and some other generalizations of classical sequence spaces may be given by means of moduli and Orlicz functions or, more generally, by means of φ-functions. Recall that a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *modulus* if

(i) $\phi(t) = 0 \Leftrightarrow t = 0$,
(ii) $\phi(t + u) \le \phi(t) + \phi(u) \quad (t, u \ge 0)$,
(iii) ϕ is nondecreasing,
(iv) ϕ is continuous.

In this definition, because of (ii), we may replace (iv) with

(iv′) ϕ is continuous from the right at 0.

We remark also that the moduli are the same as the moduli of continuity: a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a modulus of continuity of a continuous function if and only if the conditions (i)–(iii) and (iv') are satisfied (see [8], p. 866).

If in the definition of a modulus the condition (ii) is replaced by the condition of convexity

(v) $\phi(\alpha t + (1 - \alpha)u) \le \alpha\phi(t) + (1 - \alpha)\phi(u) \quad (t, u \ge 0,\ 0 \le \alpha \le 1)$,

ϕ is called an *Orlicz function.*

Provided a modulus ϕ, Ruckle [27] defined and studied the space

$$\ell(\phi) = \Big\{x = (x_k) \in \omega : \sum_k \phi(|x_k|) < \infty\} = \{x = (x_k) \in \omega : (\phi(|x_k|)) \in \ell\Big\}.$$

For an Orlicz function ϕ, the *Orlicz sequence space* is determined by (see, [16], p. 137)

$$\ell^{\exists}(\phi) = \Big\{x = (x_k) \in \omega : \exists \rho > 0 \ \ \sum_k \phi(\rho|x_k|) < \infty\Big\}.$$

If $\Phi = (\phi_k)$ is a sequence of Orlicz functions, the space

$$\ell^{\exists}(\Phi) = \Big\{x = (x_k) \in \omega : \exists \rho > 0 \ \ \sum_k \phi_k(\rho|x_k|) < \infty\Big\}$$

is called a *modular* or *Musielak–Orlicz sequence space* (see [20], p. 173). Together with $\ell^{\exists}(\phi)$ and $\ell^{\exists}(\Phi)$ there are examined also the sets

$$\ell^{\forall}(\phi) = \{x = (x_k) \in \omega : \forall \rho > 0 \ \ \sum_k \phi(\rho|x_k|) < \infty \},$$

$$\ell^{\forall}(\Phi) = \{x = (x_k) \in \omega : \forall \rho > 0 \ \ \sum_k \phi_k(\rho|x_k|) < \infty\}.$$

In the mathematical literature there exist various modifications of these definitions, where ℓ is replaced by an another solid sequence space (see, for example, [2, 6, 9, 10, 12–15, 18, 19, 22, 23, 28]). To investigate all such spaces from a more general point of view, we use the following notion (cf. [20], p. 4).

DEFINITION 1. A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *φ-function* if the conditions (i), (iii) and (iv) are satisfied.

Let $\Phi = (\phi_k)$ be a sequence of φ-functions and let $\Phi(x) = (\phi_k(|x_k|))$. For a solid sequence space λ we define the solid sets

$$\lambda^{\rho}(\Phi) = \{x = (x_k) \in \omega : \Phi(\rho x) \in \lambda\} \quad (\rho > 0),$$

$$\lambda^{\exists}(\Phi) = \{x = (x_k) \in \omega : \exists \rho > 0 \ \ \Phi(\rho x) \in \lambda\} = \bigcup_{\rho>0} \lambda^{\rho}(\Phi),$$

$$\lambda^{\forall}(\Phi) = \{x = (x_k) \in \omega : \forall \rho > 0 \ \ \Phi(\rho x) \in \lambda\} = \bigcap_{\rho>0} \lambda^{\rho}(\Phi).$$

We write $\lambda(\Phi)$ instead of $\lambda^1(\Phi)$.

For example, defining $r = max\{1, \sup_k p_k\}$, it is easy to see that the sequence spaces of Maddox $\ell_\infty(p)$, $c_0(p)$, $\ell(p)$ and $w_0(p)$ we may consider as the spaces $\ell_\infty(\mathcal{M})$, $c_0(\mathcal{M})$, $\ell_r(\mathcal{M})$ and $(w_0)_r(\mathcal{M})$, respectively, where $\mathcal{M} = (\mu_k)$ is the sequence of moduli defined by $\mu_k(t) = t^{p_k/r}$.

For an arbitrary sequence of φ-functions $\Phi = (\phi_k)$ the sets $\lambda^{\rho}(\Phi)$, $\lambda^{\exists}(\Phi)$ and $\lambda^{\forall}(\Phi)$ are different in general, and

$$\lambda^{\forall}(\Phi) \subset \lambda^{\rho}(\Phi) \subset \lambda^{\exists}(\Phi). \tag{1}$$

At the same time, the sets $\lambda^{\rho}(\Phi)$ $(\rho > 0)$ may not be linear, i.e., they may not be sequence spaces. However, a routine verification shows that, provided λ be a solid sequence space, the sets $\lambda^{\rho}(\Phi)$, $\lambda^{\exists}(\Phi)$ and $\lambda^{\forall}(\Phi)$ are solid sequence spaces whenever all ϕ_k satisfy either (ii) or (v). Moreover, the equalities

$$\lambda^{\forall}(\Phi) = \lambda^{\rho}(\Phi) = \lambda^{\exists}(\Phi), \tag{2}$$

hold if the sequence of φ-functions Φ satisfies so-called *uniform* Δ_2*-condition*: there exists a constant $K > 0$ such that $\phi_k(2t) \leq K\phi_k(t) \quad (k \in \mathbb{N},\ t > 0)$ (cf. [16], p. 167).

In particular, for a solid sequence space λ, the sets $\lambda^{\rho}(\Phi)$, $\lambda^{\exists}(\Phi)$ and $\lambda^{\forall}(\Phi)$ are sequence spaces whenever $\phi_k \ (k \in \mathbb{N})$ are either moduli or Orlicz functions. Since uniform Δ_2-condition holds (with $K = 2$) for every sequence of moduli $\Phi = (\phi_k)$, we also conclude that (2) is true whenever all ϕ_k are either moduli or Orlicz functions such that Φ satisfies uniform Δ_2-condition. The exact conditions for the equalities (2) in the case $\lambda = \ell$ are given by Šragin [29 , Proposition 6].

Let λ and μ be two sets of sequences and let $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be a function such that

(S1) $f(k, 0) = 0 \qquad (k \in \mathbb{N})$.

A *superposition operator* $P_f : \lambda \to \mu$ is defined by

$$P_f(x) = (f(k, x_k)) \in \mu \quad (x = (x_k) \in \lambda).$$

We say that a superposition operator P_f is *even* if the functions $f(k, \cdot)$ $(k \in \mathbb{N})$ are even, i. e. $f(k, -t) = f(k, t)$ $(k \in \mathbb{N},\ t \in \mathbb{R})$ or, equivalently,

(S2) $f(k, t) = f(k, |t|) \qquad (k \in \mathbb{N},\ t \in \mathbb{R})$.

It should be noted that an even superposition operator $P_f : \lambda \to \mu$ is characterized by

$$P_f(-x) = P_f(x) \quad (x \in \lambda)$$

whenever λ and μ contain the space φ of all finite sequences, i. e. of sequences having finitely many non-zero elements. For example, the spaces ℓ_∞, c_0, ℓ_p, $(w_0)_p$ and also $\ell_\infty(p)$, $c_0(p)$, $\ell(p)$, $w_0(p)$ contain φ.

Superposition operators on sequence spaces are not studied so intensiv as on spaces of functions (see, for example, [1]). The complete investigation of superposition operators on sequence spaces ℓ_∞, c_0 and ℓ^p for $1 \leq p < \infty$ was given by Dedagich and Zabreĭko [7] (see also [4, 24]). The acting conditions for $P_f : w_0 \to \ell$ are proved in [3] by the assumption that the functions $f(k, \cdot)$ are continuous. The results of Šragin [29] contain characterizations of superposition operators on $\ell^{\rho}(\Phi)$ and $\ell^{\exists}(\Phi)$, where $\Phi = (\phi_k)$ is a sequence of φ-functions. Some authors [5, 25, 26, 30, 31] have been studied continuity and boundedness of superposition operators in various sequence spaces.

Throughout this paper let λ and μ be solid sequence spaces, and let $\Phi = (\phi_k)$ and $\Psi = (\psi_k)$ be sequences of φ-functions such that $\lambda(\Phi)$ and $\mu(\Psi)$ are sequence spaces. It is not difficult to see that superposition operators $P_f : \lambda(\Phi) \to \mu(\Psi)$ play a fundamental role by the investigation of superposition operators $P_f : \lambda^\rho(\Phi) \to \mu^\sigma(\Psi)$ and $P_f : \lambda^\exists(\Phi) \to \mu^\exists(\Psi)$. We present a method which permits to characterize superposition operators $P_f : \lambda(\Phi) \to \mu(\Psi)$ by the assumption that the φ-functions ϕ_k are unbounded and the acting conditions for $P_f : \lambda \to \mu$ are known. This method of reduction bases on the possibility to describe a given superposition operator by means of two even superposition operators. As applications we extend the characterizations of $P_f : \lambda \to \mu$ with $\lambda = \ell_p$ or $\mu = \ell_p$ from [7, 24] and of $P_f : w_0 \to \ell$ from [3] to the case $0 < p < \infty$ and describe superposition operators on sequence spaces of Maddox and on multiplier spaces.

2. Superposition operators from $\lambda(\Phi)$ into $\mu(\Psi)$

For a given sequence space λ let

$$\lambda^+ = \{x = (x_k) \in \lambda : x_k \geq 0 \quad (k \in \mathbb{N})\}.$$

The characterizations of superposition operators $P_f : \lambda^\rho(\Phi) \to \mu^\sigma(\Psi)$ and $P_f : \lambda^\exists(\Phi) \to \mu^\exists(\Psi)$ essentially reduce to the examination of operators $P_f : \lambda(\Phi) \to \mu(\Psi)$. Indeed, in view of $\lambda^\exists(\Phi) = \bigcup_{\rho>0} \lambda^\rho(\Phi)$ it is clear that $P_f : \lambda^\exists(\Phi) \to \mu^\exists(\Psi)$ if and only if for every $\rho > 0$ there exists $\sigma > 0$ such that $P_f : \lambda^\rho(\Phi) \to \mu^\sigma(\Psi)$. But a superposition operator $P_f : \lambda^\rho(\Phi) \to \mu^\sigma(\Psi)$ we may interpret as superposition operator $P_f : \lambda(\Phi\rho) \to \mu(\Psi\sigma)$ with respect the φ-function sequences $\Phi\rho = (\phi_k^\rho)$ and $\Psi\sigma = (\psi_k^\sigma)$ with $\phi_k^\rho(t) = \phi_k(\rho t)$ and $\psi_k^\sigma(t) = \psi_k(\sigma t)$, respectively.

Therefore, in the sequel we consider only superposition operators of the type $P_f : \lambda(\Phi) \to \mu(\Psi)$, the formulation of the characterizations of corresponding operators $P_f : \lambda^\rho(\Phi) \to \mu^\sigma(\Psi)$ and $P_f : \lambda^\exists(\Phi) \to \mu^\exists(\Psi)$ we leave to readers.

The characterization of superposition operators on $\ell(\Phi)$ is contained in Proposition 13 of Šragin [29].

THEOREM 1. *$P_f : \ell(\Phi) \to \ell(\Psi)$ if and only if there exist a sequence $(a_k) \in \ell^+$ and numbers $\gamma \geq 0$, $\delta > 0$ and $k_0 \in \mathbb{N}$ such that*

$$\psi_k(|f(k,t)|) \leq a_k + \gamma \phi_k(|t|)$$

whenever $\phi_k(|t|) \leq \delta$ and $k \geq k_0$.

Let λ and μ be solid sequence spaces such that $P_f : \lambda \to \mu$. Defining the functions $f^{(+)} : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ and $f^{(-)} : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ by

$$f^{(+)}(k,t) = \begin{cases} f(k,-t) & \text{if } t<0 \\ f(k,t) & \text{if } t \geq 0, \end{cases} \qquad f^{(-)}(k,t) = \begin{cases} f(k,t) & \text{if } t<0 \\ f(k,-t) & \text{if } t \geq 0, \end{cases}$$

or, equivalently, by

$$f^{(+)}(k,t) = f(k,|t|), \quad f^{(-)}(k,t) = f(k,-|t|), \tag{3}$$

we get two even superposition operators $P_{f^{(+)}}$ and $P_{f^{(-)}}$ on λ. Using that for every $x = (x_k) \in \lambda$ we have

$$x = x^+ + x^-,$$

where $2x^+ = (x_k + |x_k|)$ and $2x^- = (x_k - |x_k|)$, by (S1) we may write

$$P_f(x) = P_{f^{(+)}}(x^+) + P_{f^{(-)}}(x^-),$$

$$P_{f^{(+)}}(x) = P_f(x^+) + P_f(-x^-), \quad P_{f^{(-)}}(x) = P_f(x^-) + P_f(-x^+).$$

Since λ contains together with an element x also the elements x^+, x^-, $-x^+$ and $-x^-$, we have proved the following statement.

LEMMA. *Superposition operator P_f maps λ into μ if and only if the even superposition operators $P_{f^{(+)}}$ and $P_{f^{(-)}}$ map λ into μ.*

Now we consider the superposition operator $P_f : \lambda(\Phi) \to \mu(\Psi)$, where $\Phi = (\phi_k)$ is a sequence of unbounded φ-functions. By Lemma we get $P_{f^{(+)}} : \lambda(\Phi) \to \mu(\Psi)$ and $P_{f^{(-)}} : \lambda(\Phi) \to \mu(\Psi)$. To describe these even superposition operators, for any ϕ_k we define a new function ϕ_k^{-1} by

$$\phi_k^{-1}(t) = \sup\{u : \phi_k(u) = t\}.$$

Then $\phi_k(\phi_k^{-1}(t)) = t$ and since λ is solid, for every $x = (x_k) \in \lambda(\Phi)$ there exists a sequence $y = (y_k) \in \lambda$ with $|x_k| = \phi_k^{-1}(|y_k|)$, and conversely, for every $y = (y_k) \in \lambda$ any sequence $x = (x_k)$ with $|x_k| = \phi_k^{-1}(|y_k|)$ belongs to $\lambda(\Phi)$. Since $f^{(+)}$ satisfies (S2), we have

$$P_{f^{(+)}}(x) \in \mu(\Psi) \iff (\psi_k(|f^{(+)}(k,|x_k|)|)) \in \mu,$$

and so $P_{f^{(+)}}$ maps $\lambda(\Phi)$ into $\mu(\Psi)$ if and only if

$$(\psi_k(|f^{(+)}(k,\phi_k^{-1}(|y_k|))|)) \in \mu \quad ((y_k) \in \lambda).$$

Therefore, defining the function $f^{(+)}_{\Psi\Phi^{-1}} : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ by

$$f^{(+)}_{\Psi\Phi^{-1}}(k,t) = \psi_k(|f(k,\phi_k^{-1}(|t|))|) \quad (k \in \mathbb{N},\ t \in \mathbb{R}),$$

because of (3) we get

$$P_{f^{(+)}} : \lambda(\Phi) \to \mu(\Psi) \iff P_{f^{(+)}_{\Psi\Phi^{-1}}} : \lambda \to \mu.$$

Similarly we find

$$P_{f^{(-)}} : \lambda(\Phi) \to \mu(\Psi) \iff P_{f^{(-)}_{\Psi\Phi^{-1}}} : \lambda \to \mu,$$

where the function $f^{(-)}_{\Psi\Phi^{-1}} : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is defined by the equality

$$f^{(-)}_{\Psi\Phi^{-1}}(k,t) = \psi_k(|f(k, -\phi_k^{-1}(|t|))|) \quad (k \in \mathbb{N},\ t \in \mathbb{R}).$$

Consequently, we have proved our key result.

THEOREM 2. *If the φ-functions ϕ_k $(k \in \mathbb{N})$ are unbounded, then $P_f : \lambda(\Phi) \to \mu(\Psi)$ if and only if the even superposition operators $P_{f^{(+)}_{\Psi\Phi^{-1}}}$ and $P_{f^{(-)}_{\Psi\Phi^{-1}}}$ map λ into μ.*

First we apply Theorem 1 to obtain extensions of two known results about the superposition operators $P_f : \ell_p \to \ell_q$.

PROPOSITION 1. (A) *Let $0 < p, q < \infty$. Then $P_f : \ell_p \to \ell_q$ if and only if there exist a sequence $(a_k) \in \ell^+$ and numbers $\gamma \geq 0$, $\delta > 0$, $k_0 \in \mathbb{N}$ such that*

$$|f(k,t)|^q \leq a_k + \gamma|t|^p \quad (|t| \leq \delta,\ k \geq k_0). \tag{4}$$

(B) *Let $0 < p < \infty$ and $1 \leq q < \infty$. Then $P_f : \ell_p \to \ell_q$ if and only if there exist a sequence $(b_k) \in \ell_q^+$ and numbers $\gamma \geq 0$, $\delta > 0$, $k_0 \in \mathbb{N}$ such that*

$$|f(k,t)| \leq b_k + \gamma|t|^{p/q} \quad (|t| \leq \delta,\ k \geq k_0). \tag{5}$$

Proof. For $0 < p, q < \infty$ let $\phi_k(t) = t^p$ and $\psi_k(t) = t^q$. From Theorem 1 it follows that $P_f : \ell_p \to \ell_q$ or, equivalently, $P_f : \ell(\Phi) \to \ell(\Psi)$ if and only if there exist a sequence $(a_k) \in \ell^+$ and numbers $\gamma \geq 0$, $\delta > 0$, $k_0 \in \mathbb{N}$ such that

$$|f(k,t)|^q \leq a_k + \gamma|t|^p \quad (|t|^p \leq \delta,\ k \geq k_0).$$

But this is (4) with $\delta^{1/p}$ instead of δ.

If $q \geq 1$ then (4) yields

$$|f(k,t)| \leq (a_k)^{1/q} + \gamma^{1/q}|t|^{p/q} \quad (|t| < \delta,\ k \geq k_0).$$

So, denoting $b_k = (a_k)^{1/q}$, we get (5). Since the implication (5) $\Longrightarrow P_f : \ell_p \to \ell_q$ is obvious, (B) is also proved. ∎

For $1 \leq p < \infty$, Proposition 1 was proved by Petranuarat and Kemprasit [24, Theorem 2.2] (statement (A) with $1 \leq q < \infty$) and by Dedagich and Zabreĭko [7, Theorem 1] (statement (B)).

Subsequently we apply Theorem 2 to prove extensions of some known characterizations of superposition operators on (or into) ℓ_p and on w_0.

In [7, Theorem 7] the authors assert (without proof) that a superposition operator P_f maps ℓ_p $(1 \leq p < \infty)$ into ℓ_∞ if and only if $\lim_{k\to\infty, t\to 0} |f(k,t)| < \infty$. It seems that this is not true in general. Defining, for example, $f(k,0) = 0$ and $f(k,t) = 1 - (-1)^k$ if $t > 0$, we clearly have $P_f : \ell_p \to \ell_\infty$

but the limit $\lim_{k\to\infty,t\to 0}|h(k,t)|$ does not exist. We show that the characterization of $P_f : c_0 \to \ell_\infty$ from [7, Theorem 8] is true also for $P_f : \ell_p \to \ell_\infty$. In addition, we consider the case where $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous on $\mathbb{R}$.

PROPOSITION 2. *Let* $0 < p < \infty$. *Then the following are equivalent:*

(a) $P_f : c_0 \to \ell_\infty$;
(b) $P_f : \ell_p \to \ell_\infty$;
(c) $\exists (a_k) \in \ell_\infty^+ \ \exists \delta > 0 \ \exists k_0 \in \mathbb{N} \quad |f(k,t)| \le a_k \quad (|t| \le \delta,\ k \ge k_0)$;
(d) $\exists \delta > 0 \ \exists k_0 \in \mathbb{N} \quad \sup_{|t|\le\delta, k\ge k_0} |f(k,t)| < \infty$.

If $f(k,\cdot)$ $(k \in \mathbb{N})$ *are continuous, then each of* (a)–(d) *is equivalent to*

(e) $\exists \delta > 0 \quad \sup_{|t|\le\delta, k\in\mathbb{N}} |f(k,t)| < \infty$.

Proof. (a)$\Longrightarrow$(b) and (c)$\Longleftrightarrow$(d) are obvious.

(b)$\Longrightarrow$(d). Let $x = (x_k) \in \ell_p$. If (d) is not satisfied, then there exist a subsequence (y_i) of x and indices l_i, $l_i < l_{i+1}$ $(i \in \mathbb{N})$ such that

$$|f(l_i, y_i)| \ge i \qquad (i \in \mathbb{N}). \tag{6}$$

Defining

$$z_k = \begin{cases} y_i & \text{if } k = l_i \quad (i \in \mathbb{N}) \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

we obtain a sequence $z = (z_k) \in \ell_p$. But $P_f(z) \notin \ell_\infty$ since by (6) we have

$$|f(l_i, z_{l_i})| \ge i \qquad (i \in \mathbb{N}).$$

(d)$\Longrightarrow$(a). If $x = (x_k) \in c_0$, then we can choose an index l_0 such that $|x_k| \le \delta$ $(k \ge l_0)$. Denoting $k_1 = \max\{k_0, l_0\}$, by (d) we get

$$\sup_{k\ge k_1} |f(k, x_k)| < \infty$$

which yields $P_f(x) \in \ell_\infty$.

Now let $f(k,\cdot)$ be continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$. Then (d)$\Longrightarrow$(e) since, by continuity of $f(k,\cdot)$,

$$\sup_{1\le k<k_0, |t|\le\delta} |f(k,t)| < \infty.$$

Using that (e)$\Longrightarrow$(d) is obvious, we have (d)$\Longleftrightarrow$(e). ∎

PROPOSITION 3. *Let* $0 < p < \infty$. *Then the following are equivalent:*

(a) $P_f : c_0 \to c_0$;
(b) $P_f : \ell_p \to c_0$;
(c) $\lim_{k\to\infty, t\to 0} |f(k,t)| = 0$;
(d) $\exists (a_k) \in c_0^+ \ \exists \delta > 0 \ \exists k_0 \in \mathbb{N} \quad |f(k,t)| \le a_k \quad (|t| \le \delta,\ k \ge k_0)$;
(e) $\exists k_0 \in \mathbb{N} \quad \lim_{t\to 0} \sup_{k\ge k_0} |f(k,t)| = 0$.

If $f(k,\cdot)$ $(k \in \mathbb{N})$ *are continuous, then each of* (a)–(e) *is equivalent to*

(f) $\lim_{t\to 0}\sup_k |f(k,t)| = 0$.

Proof. For $1 \le p < \infty$ the equivalences (a)$\Longleftrightarrow$(b)$\Longleftrightarrow$(c) $\Longleftrightarrow$(d) follow from Theorems 7 and 8 of [7].

(e)$\Longrightarrow$(c) is trivial.

(a)$\Longrightarrow$(e). Suppose that (a) is satisfied but (e) fails to hold. We can choose a sequence $(y_i) \in c_0$, a number $\varepsilon_0 > 0$ and a sequence (l_i) of indices such that

$$|f(l_i, y_i)| \ge \varepsilon_0 \qquad (i \in \mathbb{N}). \tag{8}$$

Then c_0 contains the sequence $z = (z_k)$ defined as in (7). But $P_f(z) \notin c_0$ since by (8) we have

$$|f(l_i, z_{l_i})| \ge \varepsilon_0 > 0 \qquad (i \in \mathbb{N}).$$

Thus (e) must hold.

Now let $0 < p < 1$. It is sufficient to prove that (b) is equivalent, for example, to (c). Defining for all $n \in \mathbb{N}$, $\phi_k(t) = t^p$ and $\psi_k(t) = t$, we may write

$$P_f : \ell_p \to c_0 \iff P_f : \ell(\Phi) \to c_0(\Psi).$$

Therefore, since (b)$\Longleftrightarrow$(c) holds for $p = 1$ and $\phi_k^{-1}(t) = t^{1/p}$, by Theorem 2 we have that $P_f : \ell_p \to c_0$ if and only if

$$\lim_{k\to\infty, t\to 0} |f(k, \pm|t|^{1/p})| = 0.$$

But this is equivalent to (c), because $\pm|t|^{1/p} \to 0 \iff t \to 0$.

Finally, if all functions f_k are continuous and (e) holds, by

$$\lim_{t\to 0}\sup_{k<k_0} |f(k,t)| = 0$$

we get (f). Since (f)$\Longrightarrow$(e) is obvious, the proof is completed. ∎

PROPOSITION 4. *Let* $0 < p < \infty$. *Then* $P_f : c_0 \to \ell_p$ *if and only if*

(a) $\exists \delta > 0\ \exists k_0 \in \mathbb{N}\quad \sum_{k\ge k_0} \sup_{|t|\le\delta} |f(k,t)|^p < \infty$.

If $f(k,\cdot)$ *is continuous for all* $k \in \mathbb{N}$ *then* (a) *is equivalent to*

(b) $\exists \delta > 0\quad \sum_k \sup_{|t|\le\delta} |f(k,t)|^p < \infty$.

Proof. It is known that (a) is necessary and sufficient for $P_f : c_0 \to \ell_p$ in the case $1 \le p < \infty$ [7, Theorem 8].

For $0 < p < 1$ we define $\phi_k(t) = t$ and $\psi_k(t) = t^p$. Then $P_f : c_0 \to \ell_p \iff P_f : c_0(\Phi) \to \ell(\Psi)$. So by Theorem 2 we conclude that $P_f : c_0 \to \ell_p$

if and only if there exist $\delta > 0$ and $k_0 \in \mathbb{N}$ such that

$$\sum_{k \geq k_0} \sup_{|t| \leq \delta} |f(k, \pm|t|)|^p < \infty,$$

which clearly coincides with (a).

The last part we can prove in the same way as in Proposition 2. ∎

PROPOSITION 5. *Let* $0 < p < \infty$. *Then* $P_f : \ell_\infty \to \ell_p$ *if and only if*

$$\forall \eta > 0 \quad \sum_k \sup_{|t| \leq \eta} |f(k, t)|^p < \infty.$$

Proof. The case $1 \leq p < \infty$ is considered in [7, Theorem 8]. For $0 < p < 1$ we apply Theorem 2 in the same way as in Proposition 4. ∎

For completeness we formulate also characterizations of superposition operators of the remainder two types, connected with the spaces ℓ_∞ and c_0 (see [7, Theorem 8]).

PROPOSITION 6. 1) $P_f : \ell_\infty \to c_0$ *if and only if*

$$\forall \eta > 0 \quad \lim_k \sup_{|t| \leq \eta} |f(k, t)| = 0.$$

2) $P_f : \ell_\infty \to \ell_\infty$ *if and only if*

$$\forall \eta > 0 \quad \sup_{k \in \mathbb{N}} \sup_{|t| \leq \eta} |f(k, t)| < \infty.$$

Chew [3] characterized superposition operators from w_0 into ℓ. We consider the operators $P_f : (w_0)_p \to \ell_q$ with $0 < p, q < \infty$.

PROPOSITION 7. *Let* $0 < p, q < \infty$. *If* $f(k, \cdot)$ *is continuous for every* $k \in \mathbb{N}$ *then* $P_f : (w_0)_p \to \ell_q$ *if and only if there exist a number* $\delta > 0$ *and sequences* $(c_k)_{k=0}^\infty \in \ell^+$ *and* $(d_k) \in \ell^+$ *such that*

$$|f(k, t)|^q \leq d_k + c_r 2^{-r} |t|^p \tag{9}$$

whenever $|t|^p \leq 2^r \delta,\ 2^r \leq k < 2^{r+1}$ $(r = 0, 1, 2, \ldots)$.

Proof. For $0 < p < \infty$ let $\phi_k(t) = t^p, \psi_k(t) = t^q$ $(k \in \mathbb{N})$. Then $P_f : (w_0)_p \to \ell_q$ we may interpret as $P_f : w_0(F) \to \ell(\Psi)$. Therefore, since (9) with $p = q = 1$ is necessary and sufficient for $P_f : w_0 \to \ell$ (see [3]), by Theorem 2 we have that $P_f : (w_0)_p \to \ell_q$ if and only if there exist a number $\delta > 0$ and sequences $(c_k)_{k=0}^\infty \in \ell^+$ and $(d_k) \in \ell^+$ such that

$$|f(k, \pm|t|^{1/p})|^q \leq d_k + c_r 2^{-r} |t| \quad (|t| \leq 2^r \delta,\ 2^r \leq k < 2^{r+1},\ r = 0, 1, 2, \ldots).$$

But this is (9) with $\pm|t|^{1/p}$ instead of t. The proof is finished. ∎

At the end of this section we characterize superposition operators $P_f : \lambda(\Phi) \to \mu(\Psi)$ for the same pairs of sequence spaces $\lambda, \mu \in \{\ell_\infty, c_0, \ell_p, (w_0)_p\}$ as in Propositions 1–7.

THEOREM 3. *Let $0 < p, q < \infty$. Then $P_f : \ell_p(\Phi) \to \ell_q(\Psi)$ if and only if there exist a sequence $(a_k) \in \ell^+$ and numbers $\gamma \geq 0$, $\delta > 0$ and $k_0 \in \mathbb{N}$ such that*

$$(\psi_k(|f(k,t)|))^q \leq a_k + \gamma(\phi_k(|t|))^p \quad (\phi_k(|t|) \leq \delta,\ k \geq k_0).$$

Proof. The statement follows immediately from Theorem 1 since operator $P_f : \ell_p(\Phi) \to \ell_q(\Psi)$ we may consider as operator $P_f : \ell(\Phi^p) \to \ell(\Psi^q)$, where $\Phi^p = (\phi_k^p)$ and $\Psi^q = (\psi_k^q)$ with $\phi_k^p(t) = (\phi_k(t))^p$ and $\psi_k^q(t) = (\psi_k(t))^q$, respectively. ■

THEOREM 4. *Let $0 < p < \infty$. If the φ-functions ϕ_k $(k \in \mathbb{N})$ are unbounded then:*

(A) *The following are equivalent:*

(a) $P_f : c_0(\Phi) \to \ell_\infty(\Psi)$;
(b) $P_f : \ell_p(\Phi) \to \ell_\infty(\Psi)$;
(c) $\exists (a_k) \in \ell_\infty^+\ \exists \delta > 0\ \exists k_0 \in \mathbb{N}\ \psi_k(|f(k,t)|) \leq a_k\ (\phi_k(|t|) \leq \delta,\ k \geq k_0)$;
(d) $\exists \delta > 0\ \exists k_0 \in \mathbb{N}\quad \sup_{\phi_k(|t|) \leq \delta, k \geq k_0} \psi_k(|f(k,t)|) < \infty$.

If, in addition, $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous and φ-functions ϕ_k $(k \in \mathbb{N})$ are strictly increasing, then each of (a)–(d) *is equivalent to*

(e) $\exists \delta > 0\quad \sup_{\phi_k(|t|) \leq \delta, k \in \mathbb{N}} \psi_k(|f(k,t)|) < \infty$.

(B) *The following are equivalent:*

(a) $P_f : c_0(\Phi) \to c_0(\Psi)$;
(b) $P_f : \ell_p(\Phi) \to c_0(\Psi)$;
(c) $\lim_{t \to 0, k \to \infty} \psi_k(|f(k, \pm\phi_k^{-1}(|t|))|) = 0$;
(d) $\exists (a_k) \in c_0^+\ \exists \delta > 0\ \exists k_0 \in \mathbb{N}\ \psi_k(|f(k,t)|) \leq a_k\ (\phi_k(|t|) \leq \delta,\ k \geq k_0)$;
(e) $\exists k_0 \in \mathbb{N}\quad \lim_{t \to 0} \sup_{k \geq k_0} \psi_k(|f(k, \pm\phi_k^{-1}(|t|))|) = 0$.

If, in addition, $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous and φ-functions ϕ_k $(k \in \mathbb{N})$ are strictly increasing, then each of (a)–(e) *is equivalent to*

(f) $\lim_{t \to 0} \sup_k \psi_k(|f(k, \pm\phi_k^{-1}(|t|))|) = 0$.

(C) $P_f : c_0(\Phi) \to \ell_p(\Psi)$ *if and only if*

(a) $\exists \delta > 0\ \exists k_0 \in \mathbb{N}\quad \sum_{k \geq k_0} \sup_{\phi_k(|t|) \leq \delta} (\psi_k(|f(k,t)|))^p < \infty$.

If, in addition, $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous and φ-functions ϕ_k $(k \in \mathbb{N})$ are strictly increasing, then (a) *is equivalent to*

(b) $\exists \delta > 0\quad \sum_k \sup_{\phi_k(|t|) \leq \delta} (\psi_k(|f(k,t)|))^p < \infty$.

Proof. (A). The equivalence (c) $\Longleftrightarrow$ (d) is obvious.

Basing on Proposition 2, by Theorem 2 we have that both conditions (a) and (b) are true if and only if there exist a sequence $(a_k) \in \ell_\infty^+$ and numbers $\delta > 0$, $k_0 \in \mathbb{N}$ such that

$$\psi_k(|f(k, \pm\phi_k^{-1}(|t|))|) \leq a_k \quad (|t| \leq \delta,\ k \geq k_0).$$

Since $\phi_k(\phi_k^{-1}(|t|)) = |t|$ then, writing t instead of $\pm\phi_k^{-1}(|t|)$, we get the equivalent condition (c).

If the φ-functions ϕ_k $(k \in \mathbb{N})$ are strictly increasing, the functions ϕ_k^{-1} are continuous. Since ϕ_k and $f(k, \cdot)$ are assumed continuous, we conclude that the even functions $f^{(+)}_{\Psi\Phi^{-1}}(k, \cdot)$ and $f^{(-)}_{\Psi\Phi^{-1}}(k, \cdot)$ are continuous too. Therefore, the equivalence (d) $\iff$ (e) follows by Theorem 2 because of corresponding equivalence in Proposition 2.

Statements (B) and (C) we can prove similarly, using Propositions 3 and 4, respectively. ∎

Analogously, using Propositions 5 and 6, we get

THEOREM 5. *Let* $0 < p < \infty$. *If the* φ*-functions* ϕ_k $(k \in \mathbb{N})$ *are unbounded then:*

(A) $P_f : \ell_\infty(\Phi) \to \ell_p(\Psi)$ *if and only if*

$$\forall \eta > 0 \quad \sum_k \sup_{\phi_k(|t|) \leq \eta} (\psi_k(|f(k, t)|))^p < \infty.$$

(B) $P_f : \ell_\infty(\Phi) \to c_0(\Psi)$ *if and only if*

$$\forall \eta > 0 \quad \lim_k \sup_{\phi_k(|t|) \leq \eta} \psi_k(|f(k, t)|) = 0.$$

(C) $P_f : \ell_\infty(\Phi) \to \ell_\infty(\Psi)$ *if and only if*

$$\forall \eta > 0 \quad \sup_{k \in \mathbb{N}} \sup_{\phi_k(|t|) \leq \eta} \psi_k(|f(k, t)|) < \infty.$$

By Proposition 7 we can formulate

THEOREM 6. *Let* $0 < p, q < \infty$. *If* $f(k, \cdot)$ $(k \in \mathbb{N})$ *are continuous and* φ*-functions* ϕ_k $(k \in \mathbb{N})$ *are unbounded and strictly increasing then* $P_f : (w_0)_p(\Phi) \to \ell_q(\Psi)$ *if and only if there exist a number* $\delta > 0$ *and sequences* $(c_k)_{k=0}^\infty \in \ell^+$ *and* $(d_k) \in \ell^+$ *such that*

$$(\psi_k(|f(k, t)|))^q \leq d_k + c_r 2^{-r} (\phi_k(|t|))^p$$

whenever $\phi_k(|t|))^p \leq 2^r \delta,\ 2^r \leq k < 2^{r+1}$ $(r = 0, 1, 2, \ldots)$.

3. Superposition operators on sequence spaces of Maddox and on multiplier spaces

Let $p = (p_k)$ and $q = (q_k)$ be two bounded sequences of strictly positive numbers. The sequence spaces of Maddox $\ell_\infty(p)$, $c_0(p)$, $\ell(p)$ and $w_0(p)$ we can consider as the spaces $\ell_\infty(\Phi)$, $c_0(\Phi)$, $\ell(\Phi)$ and $w_0(\Phi)$, where $\Phi = (\phi_k)$ with $\phi_k(t) = t^{p_k}$ $(k \in \mathbb{N})$. So, defining $\Psi = (\psi_k)$ by $\psi_k(t) = t^{q_k}$ $(k \in \mathbb{N})$ and taking into account that our φ-functions ϕ_k $(k \in \mathbb{N})$ are unbounded and strictly increasing, from Theorems 3–6 we get the following characterizations of superposition operators on sequence spaces of Maddox.

COROLLARY 1. *$P_f : \ell(p) \to \ell(q)$ if and only if there exist a sequence $(a_k) \in \ell^+$ and numbers $\gamma \geq 0$, $\delta > 0$ and $k_0 \in \mathbb{N}$ such that*

$$|f(k,t)|^{q_k} \leq a_k + \gamma |t|^{p_k} \quad (|t|^{p_k} \leq \delta,\ k \geq k_0).$$

COROLLARY 2. (A) *The following are equivalent:*

(a) $P_f : c_0(p) \to \ell_\infty(q)$;
(b) $P_f : \ell(p) \to \ell_\infty(q)$;
(c) $\exists (a_k) \in \ell_\infty^+\ \exists \delta > 0\ \exists k_0 \in \mathbb{N}\quad |f(k,t)|^{q_k} \leq a_k \quad (|t|^{p_k} \leq \delta,\ k \geq k_0)$;
(d) $\exists \delta > 0\ \exists k_0 \in \mathbb{N}\quad \sup_{|t|^{p_k} \leq \delta, k \geq k_0} |f(k,t)|^{q_k} < \infty$.

If $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous then each of (a)–(d) *is equivalent to*

(e) $\exists \delta > 0 \quad \sup_{|t|^{p_k} \leq \delta, k \in \mathbb{N}} |f(k,t)|^{q_k} < \infty$.

(B) *The following are equivalent:*

(a) $P_f : c_0(p) \to c_0(q)$;
(b) $P_f : \ell(p) \to c_0(q)$;
(c) $\lim_{t \to 0, k \to \infty} |f(k, \pm |t|^{1/p_k})|^{q_k} = 0$;
(d) $\exists (a_k) \in c_0^+\ \exists \delta > 0\ \exists k_0 \in \mathbb{N}\quad |f(k,t)|^{q_k} \leq a_k \quad (|t|^{p_k} \leq \delta,\ k \geq k_0)$;
(e) $\exists k_0 \in \mathbb{N}\quad \lim_{t \to 0} \sup_{k \geq k_0} |f(k, \pm |t|^{1/p_k})|^{q_k} = 0$.

If $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous then each of (a)–(e) *is equivalent to*

(f) $\lim_{t \to 0} \sup_k |f(k, \pm |t|^{1/p_k})|^{q_k} = 0$.

(C) *$P_f : c_0(p) \to \ell(q)$ if and only if*

(a) $\exists \delta > 0\ \exists k_0 \in \mathbb{N}\quad \sum_{k \geq k_0} \sup_{|t|^{p_k} \leq \delta} |f(k,t)|^{q_k} < \infty$.

If $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous then (a) *is equivalent to*

(b) $\exists \delta > 0 \quad \sum_k \sup_{|t|^{p_k} \leq \delta} |f(k,t)|^{q_k} < \infty$.

COROLLARY 3. (A) *$P_f : \ell_\infty(p) \to \ell(q)$ if and only if*

$$\forall \eta > 0 \quad \sum_k \sup_{|t|^{p_k} \leq \eta} |f(k,t)|^{q_k} < \infty.$$

(B) *$P_f : \ell_\infty(p) \to c_0(q)$ if and only if*

$$\forall \eta > 0 \quad \lim_k \sup_{|t|^{p_k} \leq \eta} |f(k,t)|^{q_k} = 0.$$

(C) $P_f : \ell_\infty(p) \to \ell_\infty(q)$ *if and only if*

$$\forall \eta > 0 \quad \sup_{k\in\mathbb{N}} \sup_{|t|^{p_k}\le\eta} |f(k,t)|^{q_k} < \infty.$$

COROLLARY 4. *If $f(k,\cdot)$ is continuous for every $k \in \mathbb{N}$ then $P_f : w_0(p) \to \ell(q)$ if and only if there exist a number $\delta > 0$ and sequences $(c_k)_{k=0}^{\infty} \in \ell^+$ and $(d_k) \in \ell^+$ such that*

$$|f(k,t)|^{q_k} \le d_k + c_r 2^{-r}|t|^{p_k}$$

whenever $|t|^{p_k} \le 2^r\delta,\ 2^r \le k < 2^{r+1} \quad (r = 0,1,2,\ldots)$.

Finally, let λ, μ be solid sequence spaces and let $u = (u_k)$, $v = (v_k)$ be two sequences such that $u_k \neq 0$, $v_k \neq 0 \quad (k \in \mathbb{N})$. We consider multiplier spaces

$$M(u,\lambda) = \{x = (x_k) \in \omega : (u_k x_k) \in \lambda\} = \{x = (x_k) \in \omega : (|u_k x_k|) \in \lambda\},$$
$$M(v,\mu) = \{x = (x_k) \in \omega : (v_k x_k) \in \mu\} = \{x = (x_k) \in \omega : (|v_k x_k|) \in \mu\}.$$

Defining for all $k \in \mathbb{N}$ and $t \in \mathbb{R}^+$,

$$\phi_k(t) = |u_k|t, \quad \psi_k(t) = |v_k|t,$$

we get two sequences $\Phi = (\phi_k)$ and $\Psi = (\psi_k)$ of unbounded and strictly increasing φ-functions. Since $M(u,\lambda) = \lambda(\Phi)$, $M(v,\mu) = \mu(\Psi)$, from Theorems 3–6 we get the following characterizations of superposition operators on multiplier spaces.

COROLLARY 5. *Let $0 < p,q < \infty$. $P_f : M(u,\ell_p) \to M(v,\ell_q)$ if and only if there exist a sequence $(a_k) \in \ell^+$ and numbers $\gamma \ge 0$, $\delta > 0$ and $k_0 \in \mathbb{N}$ such that*

$$|v_k f(k,t)|^q \le a_k + \gamma|u_k t|^p \quad (|u_k t|^p \le \delta,\ k \ge k_0).$$

COROLLARY 6. *Let $0 < p < \infty$. Then:*

(A) *The following are equivalent:*

- (a) $P_f : M(u,c_0) \to M(v,\ell_\infty)$;
- (b) $P_f : M(u,\ell_p) \to M(v,\ell_\infty)$;
- (c) $\exists (a_k) \in \ell_\infty^+\ \exists \delta > 0\ \exists k_0 \in \mathbb{N} \quad |v_k f(k,t)| \le a_k \quad |u_k t| \le \delta,\ k \ge k_0)$;
- (d) $\exists \delta > 0\ \exists k_0 \in \mathbb{N} \quad \sup_{|u_k t|\le\delta, k\ge k_0} |v_k f(k,t)| < \infty$.

If $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous then each of (a)–(d) *is equivalent to*

- (e) $\exists \delta > 0 \quad \sup_{|u_k t|\le\delta, k\in\mathbb{N}} |v_k f(k,t)| < \infty$.

(B) *The following are equivalent:*

- (a) $P_f : M(u,c_0) \to M(v,c_0)$;
- (b) $P_f : M(u,\ell_p) \to M(v,c_0)$;
- (c) $\lim_{t\to 0, k\to\infty} |v_k f(k, \pm|u_k t|)| = 0$;

(d) $\exists (a_k) \in c_0^+ \ \exists \delta > 0 \ \exists k_0 \in \mathbb{N} \quad |v_k f(k,t)| \le a_k \quad (|u_k t| \le \delta, \ k \ge k_0)$;
(e) $\exists k_0 \in \mathbb{N} \quad \lim_{t\to 0} \sup_{k \ge k_0} |v_k f(k, \pm|u_k t|)| = 0$.

If $f(k,\cdot)$ $(k \in \mathbb{N})$ *are continuous then each of* (a)–(e) *is equivalent to*

(f) $\lim_{t\to 0} \sup_k |v_k f(k, \pm|u_k t|)| = 0$.

(C) $P_f : M(u, c_0) \to M(v, \ell_p)$ *if and only if*
(a) $\exists \delta > 0 \ \exists k_0 \in \mathbb{N} \quad \sum_{k \ge k_0} \sup_{|u_k t| \le \delta} |v_k f(k,t)| < \infty$.

If $f(k,\cdot)$ $(k \in \mathbb{N})$ *are continuous then* (a) *is equivalent to*

(b) $\exists \delta > 0 \quad \sum_k \sup_{|u_k t| \le \delta} (|v_k f(k,t)| < \infty$.

COROLLARY 7. *Let* $0 < p < \infty$. *Then:*
(A) $P_f : M(u, \ell_\infty) \to M(v, \ell_p)$ *if and only if*

$$\forall \eta > 0 \quad \sum_k \sup_{|u_k t| \le \eta} |v_k f(k,t)|^p < \infty.$$

(B) $P_f : M(u, \ell_\infty) \to M(v, c_0)$ *if and only if*

$$\forall \eta > 0 \quad \lim_k \sup_{|u_k t| \le \eta} |v_k f(k,t)| = 0.$$

(C) $P_f : M(u, \ell_\infty) \to M(v, \ell_\infty)$ *if and only if*

$$\forall \eta > 0 \quad \sup_{k \in \mathbb{N}} \sup_{|u_k t| \le \eta} |v_k f(k,t)| < \infty.$$

COROLLARY 8. *Let* $0 < p, q < \infty$. *If* $f(k,\cdot)$ *is continuous for every* $k \in \mathbb{N}$ *then* $P_f : M(u, (w_0)_p) \to M(v, \ell_q)$ *if and only if there exist a number* $\delta > 0$ *and sequences* $(c_k)_{k=0}^\infty \in \ell^+$ *and* $(d_k) \in \ell^+$ *such that*

$$|v_k f(k,t)|^q \le d_k + c_r 2^{-r} |u_k t|^p$$

whenever $|u_k t|^p \le 2^r \delta, \ 2^r \le k < 2^{r+1} \quad (r = 0, 1, 2, \ldots)$.

Suantai [30] described superposition operators $P_f : M(u, \ell) \to \ell$ for $u_k(t) = t^r$ and $P_f : M(u, \ell_\infty) \to \ell$ for $u_k(t) = t^{-r}$ (with $r > 0$) by additional assumption that the functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are bounded on every bounded subset of $\mathbb{R}$.

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