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INTERPOLATED SUBSPACES OF EXPONENTIAL VECTORS OF THE UNBOUNDED OPERATORS IN BANACH SPACES

Abstract. It is shown that the spectral subspaces of the unbounded operators in Banach spaces and also their integer degrees can be described with help of interpolation. The spectral subspaces of operators are described on the basis of abstract Bernstein inequality. The results are applied to research of the root subspaces of regular elliptic operators in a bounded domains.

1. The given work is devoted to the problem of construction of the spectral decomposition for the unbounded linear operators in Banach spaces. The difficulties in constructing such decomposition with help of the spectral measure are known [6]. We use for this purpose the Lions-Peetre interpolation method [11].

For the given unbounded operator A in Banach space $\mathfrak{X}$ the element x in its domain $\mathcal{C}^1 \subset \mathfrak{X}$ is named as a vector of the exponential type $\nu > 0$ if x satisfies the inequality

$$(1) \qquad \|A^k x\| \leq \nu^k c, \qquad k \in \mathbb{Z}_+,$$

where $c > 0$ is some constant independent from k [9]. Subspace of such vectors is further denoted by $\mathcal{E}^{\nu}_{\infty}(\mathcal{C}^1)$. Such subspace in case of the operator A with a discrete spectrum (see [8]) is equal to the linear hull of its root vectors corresponding to eigenvalues in the circle of radius ν. In [1] it is shown that in case of the operators with a separated spectrum entered in [6] the union $\mathcal{E}(\mathcal{C}^1) := \bigcup_{\nu} \mathcal{E}^{\nu}_{\infty}(\mathcal{C}^1)$ is precisely equal to the set of all spectral subspaces. In case of the operator of differentiation $A = D$ in space $L_p(\mathbb{R})$ the subspace $\mathcal{E}^{\nu}_{\infty}(\mathcal{C}^1)$ consists of the restrictions on $\mathbb{R}$ of the entire analytic functions of the exponential type ν and the condition (1) is reduced to the classical Bernstein inequality.

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Thus in Banach spaces with help of the vectors of exponential type which are determined by the abstract Bernstein inequality (1) it is possible to describe the spectral subspaces of unbounded operators. In the given work the technique of research is based essentially on properties of the vectors of exponential type of the unbounded operators established in [8], [1].

We describe the scales of spaces of the exponential type vectors of the unbounded operator and also their integer degrees determined on intermediate spaces (Theorem 1). We show existence of decomposition of the given space in a series of spectral subspaces (Theorem 2). The received results are applied to the description of spectral subspaces of integer degrees of the regular elliptic operators given in bounded domains of space $\mathbb{R}^n$ (Theorem 3). In case of the elliptic operators with constant coefficients the existence of analytical expansion of root functions on the complex space $\mathbb{C}^n$ and also their completeness is established (Theorem 4, Corollary 6).

Among other works which are connected with the given research we shall note to the following [4], [10], [7]. We use a terminology from the book [11].

2. Let in the Banach complex space $\mathfrak{X}$ with the norm $\|\cdot\|$ a closed unbounded linear operator A with the dense domain $\mathcal{C}^1$ is given and such that $A^{-1} \in \mathfrak{L}(\mathfrak{X})$. We denote by $\mathcal{C}^m$, $(m \in \mathbb{Z}_+)$ the domain of A^m with the norm $\|x\|_{\mathcal{C}^m} = \|A^m x\|$, $(x \in \mathcal{C}^m)$. We put $\mathcal{C}^0 = \mathfrak{X}$ for the unit operator $A^0 = I$. By Theorem VII.9.7 [3] and equality $\overline{\bigcap_{m=0}^{\infty} \mathcal{C}^m} = \mathfrak{X}$ [5] there follows that operator A^m in $\mathfrak{X}$ is closed.

For numbers $\nu > 0$ we define the spaces

$$\mathcal{E}_p^{\nu}(\mathcal{C}^m) = \left\{x \in \mathcal{C}^{\infty} : \|x\|_{\mathcal{E}_p^{\nu}(\mathcal{C}^m)} < \infty\right\}, \qquad \mathcal{C}^{\infty} := \bigcap_{k=0}^{\infty} \mathcal{C}^k$$

normed by

$$\|x\|_{\mathcal{E}_p^{\nu}(\mathcal{C}^m)} = \begin{cases} \left(\sum\limits_{k=0}^{\infty} \dfrac{\|A^k x\|_{\mathcal{C}^m}^p}{\nu^{kp}}\right)^{1/p} & : 1 \le p < \infty, \\ \sup\limits_{k \in \mathbb{Z}_+} \dfrac{\|A^k x\|_{\mathcal{C}^m}}{\nu^k} & : p = \infty. \end{cases}$$

Using the closure of A^m, by analogy with [9], the completeness of spaces $\mathcal{E}_p^{\nu}(\mathcal{C}^m)$ can be established. Further we suppose $\mathcal{E}_p^{\nu}(\mathcal{C}^m) \neq \{0\}$.

PROPOSITION 1. *The following inclusions*

$$\mathcal{E}_1^{\nu}(\mathcal{C}^m) \subset \mathcal{E}_{\infty}^{\nu}(\mathcal{C}^m) \subset \mathcal{E}_1^{\nu+\varepsilon}(\mathcal{C}^m), \qquad (\forall \varepsilon > 0)$$

are valid.

Proof. If $x \in \mathcal{E}_1^{\nu}(\mathcal{C}^m)$, than $\|x\|_{\mathcal{E}_{\infty}^{\nu}(\mathcal{C}^m)} \le \|x\|_{\mathcal{E}_1^{\nu}(\mathcal{C}^m)}$. Therefore, $x \in \mathcal{E}_{\infty}^{\nu}(\mathcal{C}^m)$. If $x \in \mathcal{E}_{\infty}^{\nu}(\mathcal{C}^m)$, than there exists the independent of k constant α, that

$\|A^k x\|_{\mathcal{C}^m}^{1/k} \le \nu\, \alpha^{1/k}$. From this $\limsup\limits_{k\to\infty} \|A^k x\|_{\mathcal{C}^m}^{1/k} \le \nu$ and $\|x\|_{\mathcal{E}_\infty^{\nu+\varepsilon}(\mathcal{C}^m)} < \infty$ for any $\varepsilon > 0$. Thus, $x \in \mathcal{E}_1^{\nu+\varepsilon}(\mathcal{C}^m)$. ∎

Let $0 < t$, ν_0, $\nu_1 < \infty$ and $1 \le p_0$, $p_1 \le \infty$. For numbers θ, $(0 < \theta < 1)$ we define the interpolation spaces

$$(\mathcal{E}_{p0}^{\nu_0}(\mathcal{C}^m), \mathcal{E}_{p1}^{\nu_1}(\mathcal{C}^m))_{\theta,p} = \Big\{x \in \mathcal{E}_{p0}^{\nu_0}(\mathcal{C}^m) + \mathcal{E}_{p1}^{\nu_1}(\mathcal{C}^m) :\ \|x\|_{(\mathcal{E}_{p0}^{\nu_0}(\mathcal{C}^m), \mathcal{E}_{p1}^{\nu_1}(\mathcal{C}^m))_{\theta,p}} < \infty\Big\}$$

with the norm

$$\|x\|_{(\mathcal{E}_{p0}^{\nu_0}(\mathcal{C}^m), \mathcal{E}_{p1}^{\nu_1}(\mathcal{C}^m))_{\theta,p}} = \begin{cases} \Big(\int\limits_0^\infty [t^{-\theta}\mathcal{K}(t,x)]^p \dfrac{dt}{t}\Big)^{1/p} & : p < \infty \\ \sup\limits_{0<t<\infty} t^{-\theta}\mathcal{K}(t,x) & : p = \infty, \end{cases}$$

where is designated $\mathcal{K}(t,x) = \inf\limits_{x=x_0+x_1} (\|x_0\|_{\mathcal{E}_{p0}^{\nu_0}(\mathcal{C}^m)} + t\,\|x_1\|_{\mathcal{E}_{p1}^{\nu_1}(\mathcal{C}^m)})$, $x_0 \in \mathcal{E}_{p0}^{\nu_0}(\mathcal{C}^m)$, $x_1 \in \mathcal{E}_{p1}^{\nu_1}(\mathcal{C}^m)$.

PROPOSITION 2. *For all* $1 \le p$, p_0, $p_1 \le \infty$ *and* $\nu_0 \ne \nu_1$ *the equalities*

$$\mathcal{E}_p^{\nu}(\mathcal{C}^m) = (\mathcal{E}_{p0}^{\nu_0}(\mathcal{C}^m), \mathcal{E}_{p1}^{\nu_1}(\mathcal{C}^m))_{\theta,p}, \quad \text{where} \quad \nu = \nu_0^{1-\theta}\nu_1^{\theta},$$

are valid (with an accuracy to the equivalence of norms).

Proof. The space $\mathcal{E}_p^{\nu}(\mathcal{C}^m)$ is isometric to the subspace of sequences in $\mathfrak{X}$ of the form

$$l_{p,m}^{\sigma} = \{(A^k x)_{k=0}^{\infty} :\ \|(A^k x)\|_{l_{p,m}^{\sigma}} < \infty\},$$

where $\sigma = \log_2 \nu^{-1}$ and $\|(A^k x)\|_{l_{p,m}^{\sigma}} = \|x\|_{\mathcal{E}_p^{\nu}(\mathcal{C}^m)}$ for $x \in \mathcal{E}_p^{\nu}(\mathcal{C}^m)$. From Theorem 1.18.2 [11] for $\sigma_0 \ne \sigma_1$ follows the next equality $(l_{p0,m}^{\sigma_0}, l_{p1,m}^{\sigma_1})_{\theta,q} = l_{p,m}^{\sigma}$, where $\sigma = (1-\theta)\sigma_0 + \theta\sigma_1$. From this we obtain the needed equality. ∎

COROLLARY 1. *For all* $1 \le p \le \infty$ *the equalities*

$$\mathcal{E}(\mathcal{C}^m) := \bigcup_{\nu>0} \mathcal{E}_\infty^{\nu}(\mathcal{C}^m) = \bigcup_{\nu>0} \mathcal{E}_p^{\nu}(\mathcal{C}^m), \qquad \bigcap_{\nu>0} \mathcal{E}_\infty^{\nu}(\mathcal{C}^m) = \bigcap_{\nu>0} \mathcal{E}_p^{\nu}(\mathcal{C}^m)$$

are valid.

In particular, the space $\mathcal{E}(\mathfrak{X})$ consists of exponential type vectors of A, introduced in [9].

COROLLARY 2. *In conditions of proposition we have the inequalities* $M_\nu \le M_{\nu_0}^{1-\theta} M_{\nu_1}^{\theta}$, *where* M_μ *is the norm* A *in* $\mathfrak{L}(\mathcal{E}_p^{\mu}(\mathcal{C}^m))$.

3. Let $0 < t < \infty$ and $1 \le p$, $q \le \infty$. For $0 < \theta < 1$ and $r \in \mathbb{N}$ we put

$$\Big(\mathcal{C}^m, \mathcal{C}^{m+r}\Big)_{\theta,q} = \Big\{x \in \mathcal{C}^m + \mathcal{C}^{m+r} :\ \|x\|_{(\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,q}} < \infty\Big\},$$

where

$$\|x\|_{(\mathcal{C}^m,\mathcal{C}^{m+r})_{\theta,q}} = \begin{cases} \Big(\int\limits_0^\infty [t^{-\theta}\mathcal{K}_{m,r}(t,x)]^q \frac{dt}{t}\Big)^{1/q} & : q<\infty, \\ \sup\limits_{0<t<\infty} t^{-\theta}\mathcal{K}_{m,r}(t,x) & : q=\infty, \end{cases}$$

$$\mathcal{K}_{m,r}(t,x) = \inf_{x=x_0+x_1} (\|x_0\|_{\mathcal{C}^m} + t\,\|x_1\|_{\mathcal{C}^{m+r}}), \quad x_0\in\mathcal{C}^m, \quad x_1\in\mathcal{C}^{m+r}.$$

THEOREM 1. *The restriction of unbounded operator A^r on space $(\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,q}$ is the closed operator with domain $(\mathcal{C}^{m+r}, \mathcal{C}^{m+2r})_{\theta,q}$. In particular for arbitrary $\nu>0$ the linear space*

$$\mathcal{E}_p^\nu((\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,q}) = \Big\{x\in(\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,q} : \|x\|_{(\mathcal{C}^m,\mathcal{C}^{m+r})_{\theta,q}} < \infty\Big\}$$

relatively the norm

$$\|x\|_{\mathcal{E}_p^\nu((\mathcal{C}^m,\mathcal{C}^{m+r})_{\theta,q})} = \begin{cases} \Big(\sum\limits_{k=0}^\infty \frac{\|A^{rk}x\|^p_{(\mathcal{C}^m,\mathcal{C}^{m+r})_{\theta,q}}}{\nu^{kp}}\Big)^{1/p} & : p<\infty, \\ \sup\limits_{k\in\mathbb{Z}_+} \frac{\|A^{rk}x\|_{(\mathcal{C}^m,\mathcal{C}^{m+r})_{\theta,q}}}{\nu^k} & : p=\infty \end{cases}$$

is complete.

For $1\le p_0,\, p_1<\infty$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ the equality

$$\mathcal{E}_p^\nu\Big((\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,p}\Big) = (\mathcal{E}_{p_0}^\nu(\mathcal{C}^m), \mathcal{E}_{p_1}^\nu(\mathcal{C}^{m+r}))_{\theta,p},$$

where

$$\begin{aligned}&(\mathcal{E}_{p_0}^\nu(\mathcal{C}^m), \mathcal{E}_{p_1}^\nu(\mathcal{C}^{m+r}))_{\theta,p} \\ &= \Big\{x\in\mathcal{E}_{p_0}^\nu(\mathcal{C}^m)+\mathcal{E}_{p_1}^\nu(\mathcal{C}^{m+r}) : \|x\|_{(\mathcal{E}_{p_0}^\nu(\mathcal{C}^m),\mathcal{E}_{p_1}^\nu(\mathcal{C}^{m+r}))_{\theta,p}} \\ &= \Big(\int\limits_0^\infty [t^{-\theta}\mathcal{K}_{m,r}^{\nu,p_0,p_1}(t,x)]^p\,\frac{dt}{t}\Big)^{1/p} < \infty\Big\},\end{aligned}$$

and

$$\begin{aligned}&\mathcal{K}_{m,r}^{\nu,p_0,p_1}(t,x) = \inf_{x=x_0+x_1}\Big(\|x_0\|_{\mathcal{E}_{p_0}^\nu(\mathcal{C}^m)} + t\,\|x_1\|_{\mathcal{E}_{p_1}^\nu(\mathcal{C}^{m+r})}\Big), \\ &x_0\in\mathcal{E}_{p_0}^\nu(\mathcal{C}^m), \quad x_1\in\mathcal{E}_{p_1}^\nu(\mathcal{C}^{m+r}),\end{aligned}$$

is valid.

Proof. For elements $x=x_0+x_1\in\mathcal{C}^m$ we have

$$\|x\|_{\mathcal{C}^m} = \|x_0+x_1\|_{\mathcal{C}^m} \le \|x_0\|_{\mathcal{C}^m} + \|x_1\|_{\mathcal{C}^m} \le \|x_0\|_{\mathcal{C}^m} + \alpha_{m,r}\|x_1\|_{\mathcal{C}^{m+r}},$$

where $\alpha_{m,r}$ is norm of the imbedding $\mathcal{C}^{m+r}\hookrightarrow\mathcal{C}^m$. From this $\|x\|_{\mathcal{C}^m}\le\mathcal{K}_{m,r}(\alpha_{m,r},x)$. Defining the norm $\mathcal{K}_{m,r}(\alpha_{m,r},x)$ on space $\mathcal{C}^m+\mathcal{C}^{m+r}$, we arrive at continuous imbedding $\mathcal{C}^m + \mathcal{C}^{m+r} \hookrightarrow \mathcal{C}^m$. From surjection we have the isomorphism $\mathcal{C}^m + \mathcal{C}^{m+r} = \mathcal{C}^m$. Moreover,

$$\|x\|_{\mathcal{C}^m} \le \|x_0\|_{\mathcal{C}^m} + \alpha_{m,r}\|x_1\|_{\mathcal{C}^{m+r}} \le 2\max\{\|x_0\|_{\mathcal{C}^m},\, \alpha_{m,r}\|x_1\|_{\mathcal{C}^{m+r}}\},$$

where $\max\{\|x_0\|_{\mathcal{C}^m},\, \alpha_{m,r}\|x_1\|_{\mathcal{C}^{m+r}}\}$ is norm on $\mathcal{C}^m \bigcap \mathcal{C}^{m+r}$. Thus we have the isomorphism $\mathcal{C}^m \bigcap \mathcal{C}^{m+r} = \mathcal{C}^{m+r}$. At last $A^r \in \mathfrak{L}(\mathcal{C}^{m+r}, \mathcal{C}^m)$ for all $m \in \mathbb{Z}_+$. Hence, the conditions of Theorem 3.1.2 [2] are valid and we have $A^r \in \mathfrak{L}((\mathcal{C}^{m+r},\, \mathcal{C}^{m+r+1})_{\theta,q},\, (\mathcal{C}^m,\, \mathcal{C}^{m+1})_{\theta,q})$. Moreover, for some constant $\gamma_{\theta,q,m,r}$ we obtain the inequality

$$\mathcal{K}_{m,r}(\alpha_{m,r}, x) \le \gamma_{\theta,q,m,r}\, \alpha^{\theta}_{m,r}\, \|x\|_{(\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,q}}, \qquad x \in (\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q}.$$

In particular, for $m = 0$ the imbedding $(\mathfrak{X},\, \mathcal{C}^r)_{\theta,q} \hookrightarrow \mathfrak{X}$ is continuous. If now $x_n \in (\mathcal{C}^r,\, \mathcal{C}^{2r})_{\theta,q}$ and $x_n \to x$, $A^r x_n \to y$ in norm $(\mathfrak{X},\, \mathcal{C}^r)_{\theta,q}$, that $x_n \to x$ and $A^r x_n \to y$ in norm $\mathfrak{X}$. From closure A^r in $\mathfrak{X}$ we have $A^r x = y$. Therefore, A^r is closed in $(\mathfrak{X},\, \mathcal{C}^r)_{\theta,q}$. By induction the operator A^r is closed in $(\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q}$ for any $m \in \mathbb{Z}_+$. Thus for any $\nu > 0$ can be determined the Banach space $\mathcal{E}^{\nu}_p\, ((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,p})$.

For arbitrary $\sigma \in \mathbb{R}$ we form the following Banach spaces of sequences

$$l_p(2^{k\sigma}\mathcal{C}^m) = \Big\{(y_k)_{k=0}^{\infty} : y_k \in \mathcal{C}^m;\, \|(y_k)\|_{l_p} = \Big(\sum_{k=0}^{\infty} 2^{kp\sigma}\|y_k\|^p_{\mathcal{C}^m}\Big)^{1/p} < \infty\Big\},$$

$$l_p(2^{k\sigma}(\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,p}) = \Big\{(y_k)_{k=0}^{\infty} : y_k \in (\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,p};$$
$$\|(y_k)\|_{\theta,l_p} = \Big(\sum_{k=0}^{\infty} 2^{kp\sigma}\|y_k\|^p_{(\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,p}}\Big)^{1/p} < \infty\Big\}.$$

From Theorem 1.18.1 [11] there follows

$$l_p(2^{k\sigma}(\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,p}) = (l_{p_0}(2^{k\sigma}\mathcal{C}^m),\, l_{p_1}(2^{k\sigma}\mathcal{C}^{m+r}))_{\theta,p},$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$.

For $x \in \mathcal{E}^{\nu}_p(\mathcal{C}^m)$ and $\nu^{-1} = 2^{\sigma}$ we have $\|x\|_{\mathcal{E}^{\nu}_p(\mathcal{C}^m)} = \|(A^{rk}x)\|_{l_p}$. Therefore, the isometric imbedding

$$\mathcal{E}^{\nu}_p(\mathcal{C}^m) \hookrightarrow l_p(2^{k\sigma}\mathcal{C}^m)$$

holds. Similarity we receive

$$\mathcal{E}^{\nu}_p((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,p}) \hookrightarrow l_p(2^{k\sigma}(\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,p}),$$

where $\|x\|_{\mathcal{E}^{\nu}_p((\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,p})} = \|(A^{rk}x)\|_{\theta,l_p}$ for all $x \in \mathcal{E}^{\nu}_p((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,p})$. From this we obtain $\|x\|_{\mathcal{E}^{\nu}_p((\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,p})} \sim \|x\|_{(\mathcal{E}^{\nu}_{p_0}(\mathcal{C}^m), \mathcal{E}^{\nu}_{p_1}(\mathcal{C}^{m+r}))_{\theta,p}}$. ■

COROLLARY 3. *For numbers $1 \le p, q \le \infty$, $0 < \theta < 1$ and $r \in \mathbb{N}$ the imbedding*

$$\mathcal{E}^{\nu}_p((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q}) \subset \mathcal{E}^{\nu}_p(\mathcal{C}^m)$$

is valid.

It follows from continuous imbedding $(\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q} \hookrightarrow \mathcal{C}^m$, which established above.

COROLLARY 4. *For numbers $1 \le p,\ q \le \infty$, $\varepsilon > 0$, $0 < \theta < 1$ and $r \in \mathbb{N}$ we have*

$$\mathcal{E}_1^{\nu}((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q}) \subset \mathcal{E}_p^{\nu}((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q}) \subset \mathcal{E}_1^{\nu+\varepsilon}((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q}),$$

$$\mathcal{E}((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q}) := \bigcup_{\nu>0} \mathcal{E}_1^{\nu}((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q}) = \bigcup_{\nu>0} \mathcal{E}_p^{\nu}((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q}),$$

$$\bigcap_{\nu>0} \mathcal{E}_1^{\nu}((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q}) = \bigcap_{\nu>0} \mathcal{E}_p^{\nu}((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q}).$$

For proof is sufficiently to use the closure of A^r in $(\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q}$ and to apply Proposition 1, 2 and Corollary 1.

4. Let $1 \le q < \infty$, $1 \le p \le \infty$ and $\{\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)\}$ be a sequence of Banach spaces, corresponding to non decreasing sequence of positive numbers $\{\nu(n)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \nu(n) = \infty$. Let's fix the number $1 \le \rho < \infty$ and define the space

$$\ell_\rho[\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)] = \left\{ \sum_{n=1}^{\infty} x_n = x \in \mathcal{C}^m : \, x_n \in \mathcal{E}_p^{\nu(n)}(\mathcal{C}^m); \;\; \|x\|_{\ell_\rho} < \infty \right\}$$

with the norm

$$\|x\|_{\ell_\rho} = \inf_{x=\sum x_n} \Big(\sum_{n=1}^{\infty} 2^{n(\rho-1)} \|x_n\|^{\rho}_{\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)} \Big)^{1/\rho},$$

where inf over all convergent series by norm of $\mathcal{C}^m$ is taken. Consider a sequence of spaces $\{\mathcal{E}_p^{\nu(n)}((\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,q})\}$ too. For every vector $x \in (\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,q}$ we define the convergent in $(\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,q}$ series $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in \mathcal{E}_p^{\nu(n)}((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q})$. For $1 \le \rho < \infty$ we define the Banach space

$$\ell_\rho[\mathcal{E}_p^{\nu(n)}((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q})] = \{x \in (\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q} : \|x\|_{\ell_\rho[\mathcal{E}_p^{\nu(n)}((\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,q})]} < \infty\}$$

where

$$\|x\|_{\ell_\rho[\mathcal{E}_p^{\nu(n)}((\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,q})]} = \inf_{x=\sum x_n} \Big(\sum_{n=1}^{\infty} 2^{n(\rho-1)} \|x_n\|^{\rho}_{\mathcal{E}_p^{\nu(n)}((\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,q})} \Big)^{1/\rho}.$$

THEOREM 2. *Let $1 \le q,\ \rho < \infty$ and $1 \le p \le \infty$. The following equalities*

$$\ell_\rho[\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)] = \overline{\mathcal{E}(\mathcal{C}^m)},$$

$$\ell_\rho[\mathcal{E}_p^{\nu(n)}((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q})] = \overline{\mathcal{E}((\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q})},$$

where the closure are by norms $\mathcal{C}^m$ and $(\mathcal{C}^m,\, \mathcal{C}^{m+r})_{\theta,q}$ respectively, are valid.

Proof. For $y\in\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)$ we have $\|y\|_{\ell_\rho}\le 2^{\rho-1}\|y\|_{\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)}$. Since $\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)$ $\subset \mathcal{E}_p^{\nu(n+1)}(\mathcal{C}^m)$, in inequality we can cross to limit

$$\|y\|_{\ell_\rho} \le 2^{\rho-1} \lim_{n\to\infty} \|y\|_{\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)} = 2^{\rho-1}\|y\|_{\mathcal{C}^m}.$$

By Hölder's inequality

$$\sum_{n=1}^{\infty} \|x_n\|_{\ell_\rho} \le 2^{\rho-1} \sum_{n=1}^{\infty} \|x_n\|_{\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)} \le 2^{\rho-1}\Big(\sum_{n=1}^{\infty} 2^{n(\rho-1)}\|x_n\|^{\rho}_{\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)}\Big)^{1/\rho}$$

the series $x = \sum_{n=1}^{\infty} x_n$ are absolutely convergent in $\ell_\rho[\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)]$. Let's show that the series $\sum_{n=1}^{\infty} x_n$ is convergent to x in $\ell_\rho[\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)]$. Since

$$\sum_{n=1}^{\infty} \|x_n\|_{\mathcal{C}^m} \le \sum_{n=1}^{\infty} \|x_n\|_{\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)} \le \Big(\sum_{n=1}^{\infty} 2^{n(\rho-1)}\|x_n\|^{\rho}_{\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)}\Big)^{1/\rho},$$

that series $x = \sum_{n=1}^{\infty} x_n$ are absolutely convergent in $\mathcal{C}^m$. Thus, for $\varepsilon > 0$ there exists N that

$$\Big\|x - \sum_{n=1}^{N} x_n\Big\|_{\ell_\rho} = \Big\|\sum_{n>N} x_n\Big\|_{\ell_\rho} \le \sum_{n>N} \|x_n\|_{\ell_\rho} \le 2^{\rho-1} \sum_{n>N} \|x_n\|_{\mathcal{C}^m} < \varepsilon,$$

and that is why the space $\ell_\rho[\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)]$ is complete. Since

$$\|x\|_{\mathcal{C}^m} \le \inf_{x=\sum x_n} \sum_{n=1}^{\infty} \|x_n\|_{\mathcal{C}^m} \le \inf_{x=\sum x_n} \sum_{n=1}^{\infty} \|x_n\|_{\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)} \le \|x\|_{\ell_\rho},$$

that $\|x\|_{\mathcal{C}^m} \le \|x\|_{\ell_\rho} \le 2^{\rho-1}\|x\|_{\mathcal{C}^m}$ for $x \in \mathcal{E}_p^{\nu(n)}(\mathcal{C}^m)$. By Corollary 1 we have $\bigcup_{n=1}^{\infty}\mathcal{E}_p^{\nu(n)}(\mathcal{C}^m) = \mathcal{E}(\mathcal{C}^m)$. Therefore,

$$\|x\|_{\mathcal{C}^m} \le \|x\|_{\ell_\rho} \le 2^{\rho-1}\|x\|_{\mathcal{C}^m}, \qquad x \in \mathcal{E}(\mathcal{C}^m).$$

For proof of second equality is sufficiently to use the closure of A^r in $(\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,q}$ and instead of $\mathcal{C}^m$ in the previous reasoning we believe $(\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,q}$. ■

5. Consider in the space $\mathfrak{X} = L_\rho(\Omega)$ $(1 < \rho < \infty)$ on bounded domain $\Omega \subset \mathbb{R}^n$ of class C^∞ with boundary $\partial\Omega$ the regular elliptic operator of $2l$-th order

$$A : \mathcal{C}^1 \ni u \longmapsto \sum_{|\alpha|\le 2l} a_\alpha(t)D^\alpha u(t) \in L_\rho(\Omega), \qquad a_\alpha \in C^\infty(\overline{\Omega}),$$

$$\mathcal{C}^1 := \Big\{u \in W_\rho^{2l}(\Omega) : B_j u(t)_{|\partial\Omega} = 0; \quad j = 1,\dots,l\Big\},$$

and such that $A^{-1} \in \mathfrak{L}[L_\rho(\Omega)]$. We denote the Sobolev space by $W^{2l}_\rho(\Omega)$ and

$$B_j = \sum_{|\alpha|\le k_j} b_{j,\alpha}(t) D^\alpha, \qquad b_{j,\alpha}(t) \in C^\infty(\partial\Omega), \quad (0 \le k_1 < k_2 < \ldots < k_l)$$

is the collection of boundary operators. It is known (§ 5.4.3 [11]) that A has discrete spectrum $\sigma(A) = \{\lambda_n\}_{n\in\mathbb{N}}$, i.e. $\lim\limits_{n\to\infty} |\lambda_n| = \infty$ and root subspaces $\mathcal{R}_n$ of every eigenvalues $\lambda_n \in \mathbb{C}$ are finite-dimensional and $\lim\limits_{n\to\infty} |\lambda_n| = \infty$. From Theorem 5.4.4 [11] there follows

$$\mathcal{C}^m = \left\{ u \in \mathcal{W}^{2lm}_\rho(\Omega) : \, B_j A^k u(t)_{|\partial\Omega} = 0, \; j = 1,\ldots,l; \; k = 1,\ldots,m-1 \right\}.$$

THEOREM 3. *For numbers $1 \le p, p_0, p_1 < \infty$ and $0 < \theta < 1$ the equality*

$$\mathcal{E}^\nu_p \left((\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,p} \right) = \mathcal{E}^\nu_{p_0}(\mathcal{C}^m) \bigcap \mathcal{E}^\nu_{p_1}(\mathcal{C}^{m+r}) =$$
$$Lin\, \{\mathcal{R}_n : \, |\lambda_n| < \min(\nu^{\frac{1}{m+1}}, \, \nu^{\frac{1}{m+r+1}})\},$$

where Lin is linear algebraic hull of vectors, is valid.

Proof. For operators A^{m+r+1} in $\mathcal{C}^{m+r}$ we have $\sigma(A^{m+r+1}) = \{\lambda_n^{m+r+1}\}$ ([3], Theorem VII.10). Using Theorem 1 [8] for linear hulls of finite collections of root subspaces we obtain

$$\mathcal{E}^\nu_1(\mathcal{C}^{m+r}) = Lin\, \{\mathcal{R}_n : \, |\lambda_n|^{m+r+1} < \nu\}, \qquad r \in \mathbb{Z}_+.$$

Therefore, the spaces $\mathcal{E}^\nu_1(\mathcal{C}^{m+r})$ are finite-dimensional. From Proposition 2 for $1 \le p \le \infty$ and $\nu = \nu_0^{1-\theta}(\nu_0 + \varepsilon)^\theta$, $\varepsilon > 0$ we have

$$\mathcal{E}^\nu_p(\mathcal{C}^{m+r}) = (\mathcal{E}^{\nu_0}_1(\mathcal{C}^{m+r}), \, \mathcal{E}^{\nu_0+\varepsilon}_1(\mathcal{C}^{m+r}))_{\theta,p}.$$

Using density the subspace $\mathcal{E}^{\nu_0}_1(\mathcal{C}^{m+r}) \bigcap \mathcal{E}^{\nu_0+\varepsilon}_1(\mathcal{C}^{m+r})$ in the space $(\mathcal{E}^{\nu_0}_1(\mathcal{C}^{m+r}), \, \mathcal{E}^{\nu_0+\varepsilon}_1(\mathcal{C}^{m+r}))_{\theta,p}$ (Theorem 1.6.2 [11]) for sufficiently small $\varepsilon > 0$ we obtain $\mathcal{E}^\nu_1(\mathcal{C}^{m+r}) = \mathcal{E}^\nu_p(\mathcal{C}^{m+r})$ for any $\nu > 0$, $1 \le p < \infty$. From this and Theorem 1 we have

$$\mathcal{E}^\nu_p \left((\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,p} \right) = (\mathcal{E}^\nu_{p_0}(\mathcal{C}^m), \, \mathcal{E}^\nu_{p_1}(\mathcal{C}^{m+r}))_{\theta,p} =$$
$$\mathcal{E}^\nu_{p_0}(\mathcal{C}^m) \bigcap \mathcal{E}^\nu_{p_1}(\mathcal{C}^{m+r}) =$$
$$Lin\, \{\mathcal{R}_n : \, |\lambda_n|^{m+1} < \nu\} \bigcap Lin\, \{\mathcal{R}_n : \, |\lambda_n|^{m+r+1} < \nu\} =$$
$$Lin\, \{\mathcal{R}_n : \, |\lambda_n| < \min(\nu^{\frac{1}{m+1}}, \, \nu^{\frac{1}{m+r+1}})\}. \qquad \blacksquare$$

REMARK 1. Using the result from [10], we have

$$\mathcal{E}^\nu_\infty(\mathcal{C}^m) = Lin\, \{\mathcal{R}_n : |\lambda_n|^{m+1} < \nu\} \bigcup \{x_n : A x_n = \lambda_n x_n; \; |\lambda_n|^{m+1} = \nu\}.$$

Further, suppose that coefficients of equation a_α are constants.

We denote by $Exp\,(\mathbb{C}^n)$ the space of all entire analytical functions of the exponential type on $\mathbb{C}^n$.

THEOREM 4. *For numbers m, $r \in \mathbb{N}$, $\theta \in (0,1)$ and $1 \le p < \infty$ the equalities*

$$\mathcal{E}((\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,p}) = \Big\{u \in Exp\,(\mathbb{C}^n)_{|\,\Omega} : B_j A^k u_{|\,\partial\Omega} = 0; \\ j = 1, \dots, l; \quad k \in \mathbb{Z}_+\Big\} = Lin\,\{\mathcal{R}_n : n \in \mathbb{N}\},$$

where Lin is linear algebraic hull of all root vectors of A, are valid.

Proof. From Theorem 3 and Corollary 4 we have

$$\mathcal{E}((\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,p}) = \bigcup_{\nu} \mathcal{E}_p^{\nu}((\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,p}) =$$

$$\bigcup_{\nu} Lin\,\{\mathcal{R}_n : |\lambda_n| < \min(\nu^{\frac{1}{m+1}}, \nu^{\frac{1}{m+r+1}})\} = Lin\,\{\mathcal{R}_n : n \in \mathbb{N}\}.$$

It remains to use the equality

$$\mathcal{E}(\mathcal{C}^m) = \Big\{u \in Exp\,(\mathbb{C}^n)_{|\,\Omega} : B_j A^k u_{|\,\partial\Omega} = 0; j = 1, \dots, l; \ k \in \mathbb{Z}_+\Big\} \\ = Lin\,\{\mathcal{R}_n : n \in \mathbb{N}\},$$

which established in [7]. ∎

REMARK 2. Under the condition $k_j \ne 2l\theta - 1/\rho$ for all $j = 1, \dots, l$ and $m = 1$ the equality (see [11], § 4.3.3)

$$(L_\rho(\Omega), \mathcal{C}^1)_{\theta,p} = \Big\{u \in \mathcal{B}^{2l\theta}_{\rho,p}(\Omega) : B_j u_{|\,\partial\Omega} = 0; j = 1, \dots, l\Big\},$$

where $\mathcal{B}^{2l\theta}_{\rho,p}(\Omega)$ is Besov space, is valid.

Since the entire functions of the exponential type are dense in $\mathcal{B}^{2l\theta}_{\rho,p}(\Omega)$ ([11], § 2.5.4), that the restriction of A on $(L_\rho(\Omega), \mathcal{C}^1)_{\theta,p}$ has the complete system of root vectors $Lin\,\{\mathcal{R}_n : n \in \mathbb{N}\}$.

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