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SPECTRAL PROPERTIES OF GENERALIZED INVERSES

Abstract. Let $\mathcal{A}$ denote a unital complex Banach algebra. An element $a \in \mathcal{A}$ is said to be relatively regular if $aba = a$ for some $b \in \mathcal{A}$. Then b will be called a generalized inverse of a. In this note we study spectral properties of generalized inverses of a.

1. Relatively regular elements

Throughout this paper let $\mathcal{A}$ denote a complex unital Banach algebra with unit $\mathbf{1}$. If $a \in \mathcal{A}$, then $\sigma(a)$, $\rho(a)$, and $r(a)$ denote the spectrum, the resolvent set and the spectral radius of a, respectively.

An element $a \in \mathcal{A}$ is called *relatively regular* if there is $x \in \mathcal{A}$ such that $axa = a$. In this case x is called a *generalized inverse* or a g_1*-inverse* of a. The set of all relatively regular elements of $\mathcal{A}$ is denoted by $\hat{\mathcal{A}}$. Let $a \in \hat{\mathcal{A}}$. If $b \in \mathcal{A}$ satisfies the two equations

$$aba = a \quad \text{and} \quad bab = b,$$

then b will be called a g_2*-inverse* of a.

PROPOSITION 1. *Let $a \in \hat{\mathcal{A}}$. If $x \in \mathcal{A}$ is a g_1-inverse of a, then $b = xax$ is a g_2-inverse of a. Hence $\hat{\mathcal{A}} = \{a \in \mathcal{A} : \text{there is } b \in \mathcal{A} \text{ with } aba = a \text{ and } bab = b\}$.*

Proof. Simple verification. ∎

By $\mathcal{A}^{-1}$ we denote the set of all invertible elements of $\mathcal{A}$. It is clear that $\mathcal{A}^{-1} \subseteq \hat{\mathcal{A}}$ and if $a \in \mathcal{A}^{-1}$, $b \in \mathcal{A}$, and $aba = a$, then $b = a^{-1}$.

PROPOSITION 2. *If $b \in \mathcal{A}$ is a g_2-inverse of $a \in \hat{\mathcal{A}}$, then the set of all generalized inverses of a consists of all elements of the form*

$$b + u - bauab,$$

where u is arbitrary in $\mathcal{A}$.

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Proof. [4, Theorem 2, § 2.3]. ∎

NOTATIONS. For $a \in \mathcal{A}$ let

$$a^{-1}(0) = \{x \in \mathcal{A} : ax = 0\}.$$

If $a \neq 0$, then the *conorm* $\gamma(a)$ of a is defined by

$$\gamma(a) = \inf\{\|ax\| : d(x, a^{-1}(0)) = 1\},$$

where $d(x, a^{-1}(0))$ denotes the distance of x from $a^{-1}(0)$.

PROPOSITION 3. *Let* $a \in \hat{\mathcal{A}} \setminus \{0\}$. *Then:*

(1) $a\mathcal{A} = \{ax : x \in \mathcal{A}\}$ *is closed;*
(2) $\gamma(a) > 0$;
(3) *if* $x \in \mathcal{A}$ *is a generalized inverse of* a, *then*

$$\frac{1}{\gamma(a)} \leq \|x\|.$$

Proof. Let $x \in \mathcal{A}$ such that $axa = a$. Put $p = ax$ and $q = \mathbf{1} - xa$. Then it is easy to see that $p^2 = p$, $q^2 = q$, $p\mathcal{A} = a\mathcal{A}$ and $a^{-1}(0) = q\mathcal{A}$. Thus we have $a\mathcal{A} = \{y \in \mathcal{A} : py = y\}$. This shows (1). Use (1) and [5, Satz 55.2] to see that (2) holds. Now take $y \in \mathcal{A}$ such that $d(y, a^{-1}(0)) = 1$. Since $ay = a(xay)$, $y - xay \in a^{-1}(0)$, thus

$$1 = d(y, a^{-1}(0)) = d(xay, a^{-1}(0)) \leq \|xay\| \leq \|x\| \, \|ay\|,$$

hence

$$\frac{1}{\|x\|} \leq \|ay\| \text{ for all } y \in \mathcal{A} \text{ with } d(y, a^{-1}(0)) = 1.$$

This gives $\frac{1}{\|x\|} \leq \gamma(a)$. ∎

DEFINITIONS:

(1) An element $a \in \hat{\mathcal{A}}$ is said to be *decomposably regular* if there is $b \in \mathcal{A}^{-1}$ such that $aba = a$.

(2) An element $a \in \hat{\mathcal{A}}$ is called *simply polar* if there is $b \in \mathcal{A}$ with $aba = a$ and $ab = ba$.

EXAMPLE. If X is a complex Banach space and if $\mathcal{A}$ is the Banach algebra of all bounded linear operators on X, it follows from [3, Theorem 3.8.6] that $A \in \mathcal{A}$ is decomposably regular if and only if $A \in \hat{\mathcal{A}}$ and

$$N(A) \simeq X/A(X),$$

where $N(A)$ denotes the kernel and $A(X)$ denotes the range of A. It follows that if $\dim X < \infty$, then each $A \in \mathcal{A}$ is decomposably regular.

The proof of the following theorem can be found in [3, Theorem 7.3.4].

THEOREM 1. *If $\dot{\mathcal{A}} = \{p \in \mathcal{A} : p^2 = p\}$ is the set of idempotents of $\mathcal{A}$, then*

$$\{a \in \hat{\mathcal{A}} : a \text{ is decomposably regular}\} = \mathcal{A}^{-1}\dot{\mathcal{A}} = \dot{\mathcal{A}}\mathcal{A}^{-1} = \hat{\mathcal{A}} \cap \mathrm{cl}(\mathcal{A}^{-1}),$$

where $cl(\mathcal{A}^{-1})$ denotes the closure of $\mathcal{A}^{-1}$.

In what follows we will use the following

NOTATIONS. For $a \in \hat{\mathcal{A}}$ put

$$G_1(a) = \{b \in \mathcal{A} : b \text{ is a } g_1\text{-inverse of } a\},$$
$$G_2(a) = \{b \in \mathcal{A} : b \text{ is a } g_2\text{-inverse of } a\},$$

and

$$\alpha(a) = \inf\{r(b) : b \in G_1(a)\}.$$

PROPOSITION 4. *Let $a \in \hat{\mathcal{A}}$. If $\alpha(a)r(a) < 1$, then a is decomposably regular.*

Proof. If $r(a) = 0$, then a is decomposably regular, since $a \in \mathrm{cl}\ (\mathcal{A}^{-1})$ (Theorem 1). Thus we can assume that $r(a) > 0$. Hence $\alpha(a) < 1/r(a)$. Therefore there is $b \in G_1(a)$ with $r(b) < 1/r(a)$. Now take some $z \in \mathbb{C}$ such that $r(b) < |z| < 1/r(a)$. It follows that $za - \mathbf{1} \in \mathcal{A}^{-1}$ and $z - b \in \mathcal{A}^{-1}$. From

$$(za - \mathbf{1})ba = zaba - ba = za - ba = (z - b)a$$

we get

$$ba = (za - \mathbf{1})^{-1}(z - b)a.$$

Put $c = (za - \mathbf{1})^{-1}(z - b)$. Then $c \in \mathcal{A}^{-1}$, $ca = ba$, and

$$a = a(ba) = aca,$$

thus, a is decomposably regular. ∎

THEOREM 2. *For $a \in \hat{\mathcal{A}}$ the following assertions are equivalent:*

(1) *a is not decomposably regular;*
(2) *$\{z \in \mathbb{C} : |z|r(a) \leq 1\} \subseteq \sigma(b)$ for each $b \in G_1(a)$.*

Proof. (1) ⇒ (2): Proposition 4 shows that $r(a) > 0$. Let $b \in G_1(a)$ and take $z \in \mathbb{C}$ such that $|z|r(a) < 1$. Then $za - \mathbf{1} \in \mathcal{A}^{-1}$. Assume to the contrary that $z \in \rho(b)$. As in the proof of Proposition 4 we then see that a is decomposably regular, a contradiction. Hence

$$\{z \in \mathbb{C} : |z|r(a) < 1\} \subseteq \sigma(b).$$

Since $\sigma(b)$ is closed, it follows that

$$\{z \in \mathbb{C} : |z|r(a) \leq 1\} \subseteq \sigma(b).$$

(2) ⇒ (1): Since the spectrum is always bounded, we see that $r(a) > 0$. Hence $0 \in \{z \in \mathbb{C} : |z| \leq 1/r(a)\} \subseteq \sigma(b)$ for all $b \in G_1(a)$. Thus $b \subseteq \mathcal{A}^{-1}$ for each $b \in G_1(a)$. ∎

As an immediate consequence of Theorem 2 and its proof we get

COROLLARY 1. *If $a \in \hat{\mathcal{A}}$ is not decomposably regular, then $r(a) > 0$ and*

$$\alpha(a)r(a) \geq 1.$$

PROPOSITION 5. *Suppose that $a \in \hat{\mathcal{A}}$ is simply polar.*

(1) *There is a unique $b \in \mathcal{A}$ such that*

$$aba = a, \quad bab = b \quad \text{and} \quad ab = ba.$$

(2) *a is decomposably regular.*

(3) *$r(a) = 0$ if and only if $a = 0$.*

Proof. (1) There is some $c \in \mathcal{A}$ such that $aca = a$ and $ac = ca$. Put $b = cac$. By Proposition 1, b is a g_2-inverse of a. Furthermore we have

$$ab = acac = ac = ca = caca = ba.$$

If $d \in G_2(a)$ and $ad = da$, then

$$\begin{aligned} d &= dad = d^2a = d^2aba = d^2a^2b = d(ada)b \\ &= dab = dabab = da^2b^2 = adab^2 = ab^2 \\ &= bab = b. \end{aligned}$$

(2) Take $b \in \mathcal{A}$ such that $aba = a$, $bab = b$, and $ab = ba$. Put $x = b + (\mathbf{1} - ab)$ and $y = a + (\mathbf{1} - ab)$. Then $axa = aba + a(\mathbf{1} - ab)a = aba = a$. An easy computation gives

$$xy = \mathbf{1} = yx,$$

thus $x \in \mathcal{A}^{-1}$.

(3) Suppose that $r(a) = 0$. Take $c \in G_1(a)$ with $ac = ca$. Then, by [5, Satz 13.11], $r(ac) \leq r(a)r(c) = 0$. Since $ac = (ac)^n$ for all $n \in \mathbb{N}$,

$$\|ac\|^{1/n} = \|(ac)^n\|^{1/n} \to 0 \ (n \to \infty),$$

thus $ac = 0$. Therefore $a = (ac)a = 0$. ∎

COROLLARY 2. *Let $a \in \hat{\mathcal{A}} \setminus \mathcal{A}^{-1}$ and $a \neq 0$. If a is simply polar and if $b \in \mathcal{A}$ is the unique g_2-inverse of a with $ab = ba$, then*

(1) *0 is a simple pole of the resolvent $(z\mathbf{1} - a)^{-1}$;*

(2) *$\sigma(a) \setminus \{0\} \neq \emptyset$;*

(3) *$r(b) = (\operatorname{dist}(0, \sigma(a) \setminus \{0\}))^{-1} \geq r(a)^{-1}$.*

Proof. (1) follows from [8, Proposition 2.7].

(2) Since $a \neq 0$, $\sigma(a) \neq \{0\}$, by Proposition 5.

(3) The first equation follows from Proposition 2.7 in [8]. Since $ab = ba$ and $aba = a$, it follows by induction that $a^n = a^n b^n a^n$ for each $n \in \mathbb{N}$, thus $1 \leq \|a^n\| \, \|b^n\|$ for all $n \in \mathbb{N}$. This gives $1 \leq r(a)r(b)$. ∎

2. Spectra and spectral radii of generalized inverses

Througout this section we assume that $a \in \hat{\mathcal{A}} \setminus \mathcal{A}^{-1}$ *and that* $a \neq 0$. Furthermore let b denote a fixed g_2-inverse of a, thus $aba = a$ and $b = bab$. Put $p = \mathbf{1} - ab$ and $q = \mathbf{1} - ba$. Then $p^2 = p$ and $q^2 = q$. Since $a \notin \mathcal{A}^{-1}$ we have $p \neq 0$ or $q \neq 0$.

Define the function $f : \mathbb{C} \to \mathcal{A}$ by

$$f(z) = \begin{cases} b + zp, & \text{if } p \neq 0, \\ b + zq, & \text{if } p = 0 \end{cases} \qquad (z \in \mathbb{C}).$$

In what follows we only consider the case where $p \neq 0$. The proofs of our results are similar if $p = 0$ (and so $f(z) = b + zq$).

PROPOSITION 6. *For each* $z \in \mathbb{C}$, $f(z) \in G_1(a)$.

Proof. From $pa = 0$ it follows that $f(z)a = ba$. Thus $af(z)a = aba = a$. ∎

THEOREM 3. (1) $z \in \sigma(f(z))$ *for each* $z \in \mathbb{C}$.

(2) $(\sigma(b) \cup \{z\}) \setminus \{0\} \subseteq \sigma(f(z)) \subseteq \sigma(b) \cup \{z\}$ $(z \in \mathbb{C})$.

(3) *If* a *is not decomposably regular, then*

$$\sigma(b) \cup \{z\} = \sigma(f(z)) \ \text{ for all } \ z \in \mathbb{C}.$$

Proof. (1) Since $bp = 0$, $f(z)p = zp^2 = zp$, thus $(z\mathbf{1} - f(z))p = 0$. Since $p \neq 0$, it follows that $z \in \sigma(f(z))$.

(2) We divide the proof in several steps.

(i) We have $0 \in \sigma(b)$. Indeed, if $0 \in \rho(b)$, then $\mathbf{1} = b^{-1}b = b^{-1}(bab) = ab$, thus $p = 0$, a contradiction.

(ii) The equation

$$\frac{1}{\lambda}(\lambda\mathbf{1} - b)(\lambda\mathbf{1} - zp) = \lambda\mathbf{1} - f(z)$$

for $\lambda \in \mathbb{C} \setminus \{0\}$ is easily verified.

(iii) We now show that $\sigma(f(z)) \subseteq \sigma(b) \cup \{z\}$. To this end take $\lambda \in \sigma(f(z))$. If $\lambda = 0$, then $\lambda \in \sigma(b) \cup \{z\}$ by (i). If $\lambda \neq 0$, we see from (ii) that

$$(\lambda\mathbf{1} - b)(\lambda\mathbf{1} - zp) \notin \mathcal{A}^{-1},$$

thus $\lambda \in \sigma(b)$ or $\lambda \in \sigma(zp) = \{z, 0\}$. Hence $\lambda \in \sigma(b) \cup \{z\}$.

(iv) It remains to show that $(\sigma(b) \cup \{z\} \setminus \{0\} \subseteq \sigma(f(z))$. Take $\lambda \in \sigma(b) \cup \{z\}$ with $\lambda \neq 0$. If $\lambda = z$, then $\lambda \in \sigma(f(z))$, by (1). Hence we assume that $\lambda \neq z$ and so $\lambda \in \sigma(b)$. Furthermore we can assume that $z \neq 0$, since $f(0) = b$. Now suppose that $\lambda \in \rho(f(z))$. From (ii) we derive then that $\lambda\mathbf{1} - zp \notin \mathcal{A}^{-1}$. This gives

$$-z\left(\frac{z - \lambda}{z}\mathbf{1} - ba\right) \notin \mathcal{A}^{-1}.$$

Since $ba = (ba)^2$, we get $\frac{z-\lambda}{z} = 0$ or $\frac{z-\lambda}{z} = 1$, thus $z = \lambda$ or $\lambda = 0$, a contradiction.

The proof of (2) is now complete.

(3) follows from (2) and Proposition 6. ∎

COROLLARY 3. $\bigcup_{c \in G_1(a)} \sigma(c) = \mathbb{C}$.

Proof. Use Proposition 6 and Theorem 3(1). ∎

COROLLARY 4. *For $z \in \mathbb{C}$ we have*

$$r(f(z)) = \begin{cases} r(b), & \text{if } |z| \leq r(b) \\ |z|, & \text{if } |z| > r(b). \end{cases}$$

Proof. From Theorem 3(2) we see that $r(b) \leq r(f(z))$ for all $z \in \mathbb{C}$. Now take $z \in \mathbb{C}$ such that $|z| \leq r(b)$ and $\mu \in \sigma(f(z))$ with $|\mu| = r(f(z))$. Theorem 3(2) shows that $\mu \in \sigma(b)$ or $\mu = z$, hence $|\mu| \leq r(b)$ and therefore $r(f(z)) \leq r(b)$. Hence we have shown that $r(f(z)) = r(b)$ if $|z| \leq r(b)$. If $z \in \mathbb{C}$ and $|z| > r(b)$, then it follows from Theorem 3(1), (2) that $r(f(z)) = |z|$. ∎

COROLLARY 5. *For each $c \in G_2(a)$ we have*

$$[r(c), \infty) \subseteq \{r(x) : x \in G_1(a)\}.$$

COROLLARY 6. *Let $\beta(a) = \inf\{r(c) : c \in G_2(a)\}$. Then*

$$(\beta(a), \infty) \subseteq \{r(x) : x \in G_1(a)\}.$$

3. Norms of generalized inverses

THEOREM 4. *If $a \in \hat{\mathcal{A}} \setminus \mathcal{A}^{-1}$, $a \neq 0$ and $b \in G_2(a)$ then*

$$[\|b\|, \infty) \subseteq \{\|x\| : x \in G_1(a)\}.$$

Proof. As above we ssume that $p = \mathbf{1} - ab \neq 0$. Take $\alpha > \|b\|$. Define the function $f : [0, \infty) \to \mathcal{A}$ by $f(t) = b + tp$. Corollary 4 gives

$$r(f(t)) = t \text{ for } t > r(b).$$

Thus $\|f(t)\| \geq t$ for $t > r(b)$, hence

$$(*) \qquad \lim_{t \to \infty} \|f(t)\| = \infty.$$

Since $\|f(0)\| = \|b\| < \alpha$ and since $t \mapsto \|f(t)\|$ is continuous, $(*)$ shows that there is $t_0 > 0$ such that $\|f(t_0)\| = \alpha$. Put $x = f(t_0)$. By Proposition 6, $x \in G_1(a)$. ∎

DEFINITIONS. An element $h \in \mathcal{A}$ is called *hermitian* if $\|\exp(ith)\| = 1$ for all real t.

We say that $a \in \mathcal{A}$ is *Moore-Penrose-invertible* if there exists $x \in \mathcal{A}$ satisfying the following *Moore-Penrose conditions* ([6]):

$$axa = a, \; xax = x, \; ax \text{ is hermitian and } xa \text{ is hermitian.}$$

It follows from [7, Lemma 2.1] that for $a \in \mathcal{A}$ there is at most one $x \in \mathcal{A}$ satisfying the Moore-Penrose conditions. Let

$$\mathcal{A}^\dagger = \{a \in \mathcal{A} : a \text{ is Moore} - \text{Penrose invertible}\}.$$

For $a \in \mathcal{A}^\dagger$ the unique $x \in \mathcal{A}$ satisfying the Moore-Penrose conditions is denoted by $a^\dagger$ and is called the *Moore-Penrose inverse* of a. It is clear that $\mathcal{A}^\dagger \subseteq \hat{\mathcal{A}}$ and that for $a \in \mathcal{A}^\dagger$, $a^\dagger \in G_2(a)$.

THEOREM 5. *Let $a \in \mathcal{A}^\dagger \setminus \mathcal{A}^{-1}$ and $a \neq 0$. Then:*

(1) $\frac{1}{\gamma(a)} = \|a^\dagger\|$;

(2) $[\|a^\dagger, \infty) = \{\|x\| : x \in G_1(a)\}$.

Proof. (1) is shown in [7, Theorem 2.3].

(2) Put $M = \{\|x\| : x \in G_1(a)\}$. Then, by Theorem 4, $[\|a^\dagger\|, \infty) \subseteq M$. Now let $x \in G_1(a)$. From Proposition 3(3) we see that $\frac{1}{\gamma(a)} \leq \|x\|$. Thus, by (1), $\|a^\dagger\| \leq \|x\|$. This shows that $M \subseteq [\|a^\dagger\|, \infty)$. ∎

4. C*-algebras

Througout this section $\mathcal{A}$ denotes a C^*-algebra. It follows from [3, Proposition 12.20] that for $a \in \mathcal{A}$,

$$a \text{ is hermitian } \Leftrightarrow a = a^*.$$

The following important result is shown in [4, Theorem 6].

THEOREM 6. $\hat{\mathcal{A}} = \mathcal{A}^\dagger$.

COROLLARY 7. *Let $a \in \hat{\mathcal{A}} \setminus \mathcal{A}^{-1}$ and $a \neq 0$. Then*

$$[\|a^\dagger\|, \infty) = \{\|x\| : x \in G_1(a)\}.$$

Proof. Theorem 5 and Theorem 6. ∎

NOTATIONS. An element $a \in \mathcal{A}$ is said to be

(i) an *isometry* if $a^*a = \mathbf{1}$,

(ii) a *partial isometry* if $aa^*a = a$,

(iii) *unitary* if $a^*a = \mathbf{1} = aa^*$,

(iv) *normal* if $a^*a = aa^*$.

COROLLARY 8. *Suppose that $a \in \mathcal{A}$ is an isometry, then:*

(1) $\|a\| = \|a^*\| = r(a) = 1$,
(2) $a \in \hat{\mathcal{A}}$ *and* $a^\dagger = a^*$,
(3) $ba = \mathbf{1}$ *for each* $b \in G_1(a)$,
(4) $G_1(a) = G_2(a)$,
(5) *If* $a \notin \mathcal{A}^{-1}$ *then*

$$\{\|b\| : b \in G_1(a)\} = \{r(b) : b \in G_1(a)\} = [1, \infty).$$

Proof. (1)–(4) are clear.

(5) It follows from (1), (2), and Corollary 7 that

$$\{\|b\| : b \in G_1(a)\} = [1, \infty).$$

Put $M = \{r(b) : b \in G_1(a)\}$. Let $b \in G_1(a)$. By (3), $ba = \mathbf{1}$, thus $b^n a^n = \mathbf{1}$ for all $n \in \mathbb{N}$, hence $1 \le \|b^n\|\,\|a^n\| \le \|b^n\|\,\|a\|^n = \|b^n\|$. This gives $r(b) \ge 1$. Thus $M \subseteq [1, \infty)$. Since $r(a^\dagger) = r(a^*) = 1$, $1 \in M$. Now take $\alpha > 1 = r(a^\dagger)$ and put $b = a^\dagger + \alpha(\mathbf{1} - aa^\dagger)$. Since $a \notin \mathcal{A}^{-1}$, $p = 1 - aa^\dagger \neq 0$. Corollary 4 shows now that $r(b) = \alpha$. Therefore $[1, \infty) \subseteq M$. ∎

COROLLARY 9. *Suppose that $a \in \mathcal{A} \setminus \{0\}$ is a non-unitary partial isometry. Then:*

(1) $a \in \hat{\mathcal{A}}$ *and* $a^\dagger = a^*$,
(2) $\{\|b\| : b \in G_1(a)\} = [\|a\|, \infty)$.

Proof. (1) Clear.

(2) Since $\|a^\dagger\| = \|a^*\| = \|a\|$, the result follows from Corollary 7. ∎

PROPOSITION 7. *If $a \in \mathcal{A}$ is normal and if $a \in \widehat{\mathcal{A}}$, then a is simply polar:*

$$aa^\dagger = a^\dagger a.$$

Proof. [4, Theorem 10]. ∎

5. Holomorphically regular elements

If $\mathcal{A}$ is a complex unital Banach algebra, then an element $a \in \mathcal{A}$ is called *holomorphically regular* if there is a neighbourhood $U \subseteq \mathbb{C}$ of 0 and a holomorphic function $f : U \to \mathcal{A}$ such that

$$(a - z\mathbf{1})f(z)(a - z\mathbf{1}) = a - z\mathbf{1} \text{ for all } z \in U.$$

It is clear that in this case $a \in \widehat{\mathcal{A}}$. In [9, Theorem 1.4] we have shown the following result:

PROPOSITION 8. *For $a \in \mathcal{A}$ the following conditions are equivalent:*

(1) $a \in \mathcal{A}$ *and* $a^{-1}(0) \subseteq \bigcap_{n=1}^{\infty} a^n \mathcal{A}$;
(2) *a is holomorphically regular.*

THEOREM 7. *If $a \in \mathcal{A}$ is holomorphically regular, $a \notin \mathcal{A}^{-1}$ and $b \in G_1(a)$, then:*

(1) $r(b) > 0$;
(2) $\{z \in \mathbb{C} : |z| \leq \frac{1}{r(b)}\} \subseteq \sigma(a)$;
(3) $1 \leq r(a)r(b)$.

Proof. Put $U := \{z \in \mathbb{C} : |z|r(b) < 1\}$ and $f(z) = (\mathbf{1} - zb)^{-1}b$ for $z \in U$. It is shown in [9, Corollary 1.5] that

$$(*) \qquad (a - z\mathbf{1})f(z)(a - z\mathbf{1}) = a - z\mathbf{1} \text{ for all } z \in U.$$

Now take $z_0 \in U$ and assume that $z_0 \in \rho(a)$. From $(*)$ we get that $f(z_0) = (a - z_0\mathbf{1})^{-1}$. Thus

$$(\mathbf{1} - z_0b)^{-1}b = b(\mathbf{1} - z_0b)^{-1} = (a - z_0\mathbf{1})^{-1}.$$

Therefore

$$b(a - z_0\mathbf{1}) = (a - z_0\mathbf{1})b = \mathbf{1} - z_0b,$$

hence $ab = ba = \mathbf{1}$, a contradiction, since $a \notin \mathcal{A}^{-1}$. Therefore we have shown that $U \subseteq \sigma(a)$. Since $\sigma(a)$ is bounded, we derive that (1) holds. Furthermore, since $\sigma(a)$ is closed, we get from $U \subseteq \sigma(a)$ that (2) holds. (3) follows from (2). ∎

COROLLARY 10. *If $a \in \mathcal{A}$ is holomorphically regular, then 0 is an interior point of $\sigma(a)$ and $1 \leq \alpha(a)r(a)$.*

PROPOSITION 9. *Suppose that $a \in \mathcal{A}$ is holomorphically regular and $b \in G_1(a)$. Then*

$$a^n b^n a^n = a^n \textit{ for all } n \in \mathbb{N}.$$

Proof. Since $b \in G_1(a)$, $aba = a$. Now suppose that $a^n b^n a^n = a^n$ for some $n \in \mathbb{N}$. Put $p = a^n b^n$ and $q = \mathbf{1} - ba$. Then $p^2 = p, q^2 = q, p\mathcal{A} = a^n\mathcal{A}$ and $q\mathcal{A} = A^{-1}(0)$. Proposition 8 (1) shows then that $q\mathcal{A} \subseteq p\mathcal{A}$, hence $q = pq$. Therefore

$$\mathbf{1} - ba = a^n b^n (\mathbf{1} - ba),$$

thus

$$a^n b^{n+1} a = a^n b^n - \mathbf{1} + ba.$$

We conclude that

$$\begin{aligned} a^{n+1}b^{n+1}a^{n+1} &= a^{n+1}b^n a^n - a^{n+1} + aba^{n+1} \\ &= a(a^n b^n a^n) = a^{n+1}. \end{aligned}$$ ∎

REMARK. From Proposition 9 we get a second proof of Theorem 7(3).

THEOREM 8. *If $\mathcal{A}$ is a C^*-algebra and $a \in A$ is normal, then*

$$a \textit{ is holomorphically regular } \Leftrightarrow a \in \mathcal{A}^{-1}.$$

P r o o f. The implication "$\Leftarrow$" is clear. Now suppose that a is holomorphically regular. Assume to the contrary that $a \notin \mathcal{A}^{-1}$. Proposition 7 shows that a is simply polar, thus, by Corollary 2 (1), 0 is an isolated point of $\sigma(a)$. But this contradicts Corollary 10. ∎

PROPOSITION 10. *Let* $a \in \widehat{\mathcal{A}}$.

(1) *If* $b \in G_2(a)$ *and* $r = \mathbf{1} - ab - ba$, *then*

$$r^{-1}(0) = (a^{-1}(0) \cap a\mathcal{A}) \oplus (b^{-1}(0) \cap b\mathcal{A})$$

and

$$r\mathcal{A} = (a^{-1}(0) + a\mathcal{A}) \cap (b^{-1}(0) + b\mathcal{A}).$$

(2) *If* $\mathcal{A}$ *is a* C^**-algebra, then*

$$(a^\dagger)^{-1}(0) = (a^*)^{-1}(0) \text{ and } a^\dagger\mathcal{A} = a^*\mathcal{A}.$$

P r o o f. (1) If $x \in a^{-1}(0) \cap a\mathcal{A}$, then $x = (\mathbf{1} - ba)x = abx$, hence $rx = 0$. Thus $a^{-1}(0) \cap a\mathcal{A} \subseteq r^{-1}(0)$. A similar argument gives $b^{-1}(0) \cap b\mathcal{A} \subseteq r^{-1}(0)$. Now take $x \in r^{-1}(0)$, thus $x = bax + abx$. It follows that $ax = ax + a(abx)$, hence $abx \in a^{-1}(0) \cap a\mathcal{A}$. From $bx = b(bax) + bx$ we get $bax \in b^{-1}(0) \cap b\mathcal{A}$. Therefore

$$x = abx + bax \in (a^{-1}(0) \cap a\mathcal{A}) + (b^1(0) \cap b\mathcal{A}).$$

Next we show that $(a^{-1}(0) \cap a\mathcal{A}) \cap (b^{-1}(0) \cap b\mathcal{A}) = \{0\}$.

Take $x \in (a^{-1}(0) \cap a\mathcal{A}) \cap (b^{-1}(0) \cap ba)$. Then

$$x = (\mathbf{1} - ba)x = abx = (\mathbf{1} - ab)x = bax,$$

hence $0 = x - bax = x$. The proof of the first assertion is now complete.

If $y \in r\mathcal{A}$, then $y = -abx + (\mathbf{1} - ba)x$ for some $x \in \mathcal{A}$. Hence $y \in a\mathcal{A} + a^{-1}(0)$. A similar argument gives $y \in b\mathcal{A} + b^{-1}(0)$.

Now take $z \in (a^{-1}(0) + a\mathcal{A}) \cap (b^{-1}(0) + b\mathcal{A})$. Then $z = x_1 + x_2 = y_1 + y_2$ with $ax_1 = 0, x_2 = abx_2, by_1 = 0$ and $y_2 = bay_2$. Put $\omega = x_1 - y_2$. Then $\omega = y_1 - x_2$ and $r\omega = \omega - ba\omega - ab\omega = \omega - ba(x_1 - y_2) - ab(y_1 - x_2) = \omega + bay_2 + abx_2 = \omega + y_2 + x_2 = x_1 - y_2 + y_2 + x_2 = x_1 + x_2 = z$. Therefore $z \in r\mathcal{A}$.

(2) From $aa^\dagger a = a$ and $a^\dagger aa^\dagger = a^\dagger$ we derive $a^*(a^\dagger)^*a^* = a^*$ and $(a^\dagger)^* = (a^\dagger)^*a^*(a^\dagger)^*$, thus $a^* \in \widehat{\mathcal{A}}$ and $(a^\dagger)^* \in G_2(a^*)$. Then

$$\begin{aligned}(a^*)^{-1}(0) &= (\mathbf{1} - (a^\dagger)^*a^*)\mathcal{A} = (\mathbf{1} - (aa^\dagger)^*)\mathcal{A} \\ &= (\mathbf{1} - aa^\dagger)\mathcal{A} = (a^\dagger)^{-1}(0)\end{aligned}$$

and

$$a^*\mathcal{A} = a^*(a^\dagger)^*\mathcal{A} = (a^\dagger a)^*\mathcal{A} = a^\dagger a\mathcal{A} = a^\dagger\mathcal{A}.$$ ∎

THEOREM 9. *If* $\mathcal{A}$ *is a* C^**-algebra and* $a \in \mathcal{A}$ *is holomorphically regular, then:*

(1) a^* *is holomorphically regular;*
(2) $\mathcal{A} = a^{-1}(0) \oplus (a^*)^{-1}(0) \oplus (a\mathcal{A} \cap a^*\mathcal{A})$.

Proof. (1) Take any $b \in G_1(a)$ and put $f(z) = (\mathbf{1} - zb)^{-1}b$ for $|z| < r(b)^{-1}$. As in the proof of Theorem 7,

$$(a - z\mathbf{1})f(z)(a - z\mathbf{1}) = a - z\mathbf{1} \text{ for } |z| < r(b)^{-1}.$$

Thus

$$(a^* - \mu\mathbf{1})(\mathbf{1} - \mu b^*)^{-1}b^*(a^* - \mu\mathbf{1}) = a^* - \mu\mathbf{1}$$

for each $\mu \in \mathbb{C}$ with $|\mu| < r(b)^{-1}$.

(2) Put $b = a^\dagger$. Since $a^{-1}(0) = (\mathbf{1} - ba)\mathcal{A} \subseteq a\mathcal{A} = ab\mathcal{A}$, we have $\mathbf{1} - ba = ab(\mathbf{1} - ba) = ab - ab^2a$, hence

$$\mathbf{1} - ba - ab = -ab^2a.$$

By (1), a^* ist holomorphically regular, thus $(a^*)^{-1}(0) \subseteq a^*\mathcal{A}$. Now use Proposition 10 (2) to get $b^{-1}(0) \subseteq b\mathcal{A}$. Therefore $(\mathbf{1} - ab)\mathcal{A} \subseteq ba\mathcal{A}$, thus $\mathbf{1} - ab = ba(\mathbf{1} - ab) = ba - ba^2b$, hence

$$\mathbf{1} - ba - ab = -ba^2b.$$

This gives $ab^2a = ba^2b$. Put $s = ab^2a$. By Proposition 8, $a^2b^2a^2 = a^2$, thus $s^2 = (ba^2b)^2 = ba^2b^2a^2b = ba^2b = s$. Therefore $s \in \dot{\mathcal{A}}$. Since $a^{-1}(0) \subseteq a\mathcal{A}$ and $b^{-1}(0) \subseteq b\mathcal{A}$, Proposition 10 (1) gives

$$s^{-1}(0) = a^{-1}(0) \oplus b^{-1}(0)$$

and

$$a\mathcal{A} = a\mathcal{A} \cap b\mathcal{A}.$$

Now use $\mathcal{A} = s^{-1}(0) \oplus s\mathcal{A}$ and Proposition 10 (2) to get the result. ∎

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